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A FINITENESS CRITERION FOR COHOMOLOGY OF FRÉCHET-MONTEL SHEAVES

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Abstract. The paper deals with a finiteness criterion for the cohomology with compact supports of calculable Fréchet-Montel sheaves over locally compact topological spaces.

1. Result: redaction and commentary

A sheaf of Abelian groups $\mathcal{F}$ over a topological space X is called *calculable* (cf. [3]) if for every integer $k \geq 1$ and any open neighborhood U of an arbitrary point $x \in X$ there exists an open neighborhood $V \subset U$ of x such that the restriction map $H^k(U;\mathcal{F}) \to H^k(V;\mathcal{F})$ has zero image.

A sheaf of Abelian groups $\mathcal{F}$ over a locally compact topological space X is called a *Fréchet-Montel sheaf* (cf. [4]) if the following conditions are satisfied:

(a) for every open set $U \subset X$ the group of sections $\Gamma(U;\mathcal{F})$ is a Fréchet topological vector space;

(b) for any two open sets $V \subset U$ in X the restriction map $\Gamma(U;\mathcal{F}) \to \Gamma(V;\mathcal{F})$ is linear and continuous, and completely continuous when V is relatively compact in U.

Grothendieck proved that for a calculable Fréchet-Montel sheaf $\mathcal{F}$ over a compact topological space X the vector spaces $H^k(X;\mathcal{F})$ $(k = 0, 1, \dots)$ are finite-dimensional (cf. [3], p. 4). This implies the Cartan-Serre finiteness theorem [2].

On the other hand, Andreotti and Kas proved that for a coherent analytic sheaf $\mathcal{F}$ over a complex space X, countable at infinity, the vector space $H^k_c(X;\mathcal{F})$ is finite-dimensional if for some relatively compact open set $U \subset X$ the canonical map $H^k_c(U;\mathcal{F}) \to H^k_c(X;\mathcal{F})$ is surjective (cf. [1], p. 241). This assertion plays an important role in the study of q-pseudoconcave complex spaces.

We prove in the present paper the following theorem:

Theorem. *Let X be a locally compact topological space with a countable base of open sets, and let $\mathcal{F}$ be a calculable Fréchet-Montel sheaf over X. Then the vector space $H^k_c(X;\mathcal{F})$ is finite-dimensional if and only if for some relatively compact open set $U \subset X$ the canonical map $H^k_c(U;\mathcal{F}) \to H^k_c(X;\mathcal{F})$ is surjective.*

This implies the Grothendieck theorem providing the space X has a countable base of open sets and the Andreotti-Kas criterion both cited above.

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2. Auxiliary notions and facts

Let X be a locally compact topological space, countable at infinity, let $\mathfrak{U} = (U_i)_{i\in I}$ be a locally finite covering of X by relatively compact open sets, let J be a finite subset in I, and let $\mathcal{F}$ be an arbitrary sheaf of Abelian groups over X. We set $C^k_J(\mathfrak{U};\mathcal{F}) = \coprod \Gamma(U_{i_0\dots i_k};\mathcal{F})$, where the direct sum is taken over all collections $(i_0,\dots,i_k)$ of indices among which at least one belongs to J. Then $\delta C^k_J(\mathfrak{U};\mathcal{F}) \subset C^{k+1}_J(\mathfrak{U};\mathcal{F})$ and, consequently, a subcomplex $C^{\cdot}_J(\mathfrak{U};\mathcal{F})$ of the cochain complex $C^{\cdot}(\mathfrak{U};\mathcal{F})$ is defined. We consider the cohomology groups $H^k_J(\mathfrak{U};\mathcal{F}) = H^k C^{\cdot}_J(\mathfrak{U};\mathcal{F})$.

Lemma 1. *Let $\mathcal{F}$ be a Godement sheaf over X, i.e., there exists a family of Abelian groups G_x $(x\in X)$ such that $\Gamma(U;\mathcal{F}) = \prod_{x\in U} G_x$ for every open set $U\subset X$. Then $H^k_J(\mathfrak{U};\mathcal{F}) = 0$ for $k\neq 0$.*

Proof. One can assume that $I = \{1,2,\dots\}$ and $J = \{1,\dots,n\}$ for some $n\in I$. Then the sets $S_i = U_i \setminus \bigcup_{j>i} U_j$ $(i = 1,2,\dots)$ cover the space X and do not intersect in pairs. Let $C^0(Z)$ be a Godement sheaf defined by the constant sheaf Z over X, i.e., $\Gamma(U;C^0(Z)) = Z^U$ for every open set $U\subset X$. We set $\lambda_i(x) = 1$ for $x\in S_i$, and $\lambda_i(x) = 0$ for $x\in X\setminus S_i$. Then we get a sequence of the sections $\lambda_i \in \Gamma(X;C^0(Z))$ $(i = 1,2,\dots)$. Let $f = (f_{i_0\dots i_k}) \in C^k_J(\mathfrak{U};\mathcal{F})$ be a cocycle of a degree $k\geq 1$, i.e., $\delta f = 0$. Since the sheaf $\mathcal{F}$ is a module over the sheaf of rings $C^0(Z)$, the sections $g_{i_0\dots i_{k-1}} = \sum_{i\in I}\lambda_i f_{ii_0\dots i_{k-1}}$ of $\mathcal{F}$ over $U_{i_0\dots i_{k-1}}$ are defined. Obviously, the cochain $g = (g_{i_0\dots i_{k-1}})$ belongs to the group $C^{k-1}_J(\mathfrak{U};\mathcal{F})$. Moreover,

$$(\delta g)_{i_0\dots i_k} = \sum_{i=1}^{\infty}\lambda_i\sum_{s=0}^{k}(-1)^s f_{ii_0\dots\hat{i}_s\dots i_k} = f_{i_0\dots i_k},$$

because

$$f_{i_0\dots i_k} - \sum_{s=0}^{k}(-1)^s f_{ii_0\dots\hat{i}_s\dots i_k} = (\delta f)_{ii_0\dots i_k} = 0.$$

In other words, $\delta g = f$. The lemma is proved.

We set $S(J) = X\setminus\bigcup_{i\notin J} U_i$. Then $S(J)$ is a compact set in X and for any sheaf of Abelian groups $\mathcal{F}$ over X there is a canonical isomorphism $H^0_J(\mathfrak{U};\mathcal{F}) = \Gamma_{S(J)}(X;\mathcal{F})$.

If $\mathcal{F}$ is a flabby sheaf of Abelian groups over X, then with the help of Lemma 1 one can show easily that for every $k = 0,1,\dots$ there is a canonical isomorphism $H^k_J(\mathfrak{U};\mathcal{F}) = H^k_{S(J)}(X;\mathcal{F})$ and, consequently, $H^k_J(\mathfrak{U};\mathcal{F}) = 0$ for $k\neq 0$.

Lemma 2. *For any sheaf of Abelian groups $\mathcal{F}$ over X there exists a canonical map $H^k_J(\mathfrak{U};\mathcal{F}) \to H^k_{S(J)}(X;\mathcal{F})$.*

Proof. We make use of the Godement canonical resolution $0 \to \mathcal{F} \to C^0(\mathcal{F}) \to C^1(\mathcal{F}) \to \cdots$ of the sheaf $\mathcal{F}$ and consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma_{S(J)}(X;\mathcal{F}) & \longrightarrow & C^0_J(\mathfrak{U};\mathcal{F}) & \longrightarrow & C^1_J(\mathfrak{U};\mathcal{F}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma_{S(J)}(X;C^0(\mathcal{F})) & \longrightarrow & C^0_J(\mathfrak{U};C^0(\mathcal{F})) & \longrightarrow & C^1_J(\mathfrak{U};C^0(\mathcal{F})) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma_{S(J)}(X;C^1(\mathcal{F})) & \longrightarrow & C^0_J(\mathfrak{U};C^1(\mathcal{F})) & \longrightarrow & C^1_J(\mathfrak{U};C^1(\mathcal{F})) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

in which all rows starting with the second are exact by Lemma 1. From this in the obvious way by means of Weil's diagram chasing (cf. [5]) we get the desired map.

Lemma 3. *Let $\mathcal{F}$ be a calculable sheaf of Abelian groups over X, and let S be a compact set in X. Then for every integer $k \geq 0$ and any sufficiently fine locally finite covering $\mathfrak{U} = (U_i)_{i \in I}$ of X by open sets there exists a finite set $J \subset I$ such that $S \subset S(J)$ and the image of the map $H^k_S(X;\mathcal{F}) \to H^k_{S(J)}(X;\mathcal{F})$ is contained in the image of the map $H^k_J(\mathfrak{U};\mathcal{F}) \to H^k_{S(J)}(X;\mathcal{F})$.*

Proof. First let $\mathfrak{U}$ be an arbitrary locally finite covering of the space X by relatively compact open sets, and let J be a finite subset in I such that $S \subset S(J)$. We consider the commutative diagram used in the proof of Lemma 2 in which all rows starting with the second are exact. Fixing an integer $k \geq 1$ and substituting in succession the covering $\mathfrak{U}$ by more and more fine ones, after a finite number of steps with the help of Weil's diagram chasing (cf. [5]) we get the assertion of the lemma.

By a *Fréchet complex* we mean a complex $E^{\cdot}$ in which E^k are Fréchet spaces and differentials $d : E^k \to E^{k+1}$ are continuous linear maps. By a *morphism* of Fréchet complexes $f : E^{\cdot} \to F^{\cdot}$ we mean a sequence of continuous linear maps $f : E^k \to F^k$ compatible with the differentials.

Lemma 4. *Let $f_n : E^{\cdot}_n \to E^{\cdot}_{n+1}$ $(n = 0, 1, \dots)$ be morphisms of Fréchet complexes, $f_{n,m} = f_{m-1} \circ \cdots \circ f_n$ for $n < m$, and let $E^{\cdot} = \varinjlim E^{\cdot}_n$ be the corresponding inductive limit of complexes. Let us assume that for some k the following conditions are satisfied*:

(a) *the map $f_0 : E^k_0 \to E^k_1$ is completely continuous*;

(b) *the canonical map $H^k E^{\cdot}_0 \to H^k E^{\cdot}$ is surjective.*

Then for every $n \geq 1$ there exists $m > n$ such that the quotient-space $H^k E^{\cdot}_n / \ker f_{n,m}$ (where $\ker f_{n,m}$ is the kernel of the map $H^k E^{\cdot}_n \to H^k E^{\cdot}_m$ induced by the morphism of complexes $f_{n,m}$) is finite-dimensional and separated; *in particular, the vector space $H^k E^{\cdot}$ is finite-dimensional.*

Proof. For fixed $n \geq 1$ and every $m > n$ we set $G_m = \{(x, y) \in Z^k E^{\cdot}_n \times E^{k-1}_m : f_{n,m} x = dy\}$. Then there is an exact sequence $\varinjlim G_m \xrightarrow{p_1} Z^k E^{\cdot}_n \to H^k E^{\cdot}$, where p_1

is the projection of the product $Z^k E_n^{\cdot} \times E_m^{k-1}$ onto the first factor. By condition (b) the continuous linear map $Z^k E_0^{\cdot} \oplus \varinjlim G_m \to Z^k E_n^{\cdot}$ is surjective. Consequently, in view of a theorem of Banach and the Baire theorem, there exists $m > n$ such that the continuous linear map of Fréchet spaces $Z^k E_0^{\cdot} \oplus G_m \to Z^k E_n^{\cdot}$ is surjective. By condition (a) and the Schwartz almost epimorphism theorem the quotient-space $Z^k E_n^{\cdot}/p_1 G_m = H^k E_n^{\cdot}/\ker f_{n,m}$ is finite-dimensional and separated. In particular, since the canonical map $H^k E_n^{\cdot} \to H^k E^{\cdot}$ is surjective, the vector space $H^k E^{\cdot}$ is finite-dimensional.

3. Proof of the Theorem

The necessity of the condition of the Theorem is obvious; we prove the sufficiency. We set $C_c^{\cdot}(X;\mathcal{F}) = \varinjlim C_J^{\cdot}(\mathfrak{U};\mathcal{F})$, where $\mathfrak{U}$ runs through all locally finite coverings of the space X by relatively compact open sets and J runs through finite subsets in corresponding sets of indices. Then $H_c^k(X;\mathcal{F}) = H^k C_c^{\cdot}(X;\mathcal{F})$. Obviously, one can choose sequences of coverings $\mathfrak{U}_n$ and finite sets J_n $(n = 0, 1, \dots)$, respectively, such that $C_c^{\cdot}(X;\mathcal{F}) = \varinjlim C_{J_n}^{\cdot}(\mathfrak{U}_n;\mathcal{F})$. Since $S = \overline{U}$ is a compact set in X, there is the commutative diagram

$$\begin{tikzcd} H_c^k(U;\mathcal{F}) \arrow[rr] \arrow[dr] & & H_c^k(X;\mathcal{F}) \\ & H_S^k(X;\mathcal{F}) \arrow[ur] & \end{tikzcd}$$

in which the right inclined arrow is surjective. By Lemma 3 there exist, respectively, a locally finite covering $\mathfrak{U}$ of X by relatively compact open sets and a finite set of indices J such that the canonical map $H_J^k(\mathfrak{U};\mathcal{F}) \to H_c^k(X;\mathcal{F})$ is surjective. One can assume that in the sequences chosen above $\mathfrak{U}_0 = \mathfrak{U}$ and $J_0 = J$. By Lemma 4 the vector space $H_c^k(X;\mathcal{F})$ is finite-dimensional. The Theorem is proved.

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