A COEFFICIENT OF AN ASYMPTOTIC EXPANSION
OF LOGARITHMS OF DETERMINANTS
FOR CLASSICAL ELLIPTIC
PSEUDODIFFERENTIAL OPERATORS WITH PARAMETERS

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Abstract. For classical elliptic pseudodifferential operators \( A(\lambda) \) of order \( m > 0 \) with parameter \( \lambda \) of weight \( \chi > 0 \), it is known that \( \log \det A(\lambda) \) admits an asymptotic expansion as \( \lambda \to +\infty \). In this paper we show, with some assumptions, that the coefficient of \( \lambda^{-1/\chi} \) can be expressed by the value of a zeta function at 0 for some elliptic \( \psi \)DO on \( M \times S^1 \) multiplied by \( m^2 \).

I. INTRODUCTION

Let \( M \) be a compact oriented Riemannian manifold of dimension \( d \) and let \( E \xrightarrow{\pi} M \) be a vector bundle of rank \( k \). Let \( A(\lambda) \colon C^\infty(E) \to C^\infty(E) \) be a classical elliptic pseudodifferential operator of order \( m > 0 \) with parameter \( \lambda \) of weight \( \chi > 0 \), where \( \lambda \) is a nonnegative real number. That is, the symbol of \( A(\lambda) \) has an asymptotic expansion as in (2.1).

We assume that there is an angle \( \theta \) such that the principal symbol \( a_m(x, \xi, \lambda) \) does not have any eigenvalues on the ray \( \{ z \in \mathbb{C} \mid z = \rho e^{i\theta}, \rho \geq 0 \} \) for \( |\xi| + |\lambda|^\frac{1}{\chi} \neq 0 \) and that \( A(\lambda) \) does not have any eigenvalues in a sector \( L[\theta - \epsilon, \theta + \epsilon] = \{ z \in \mathbb{C} \mid \theta - \epsilon \leq \arg z \leq \theta + \epsilon \} \) for some small \( \epsilon > 0 \). We call this \( \theta \) an Agmon angle. In fact from the compactness of the set \( \{ (x, \xi, \lambda) \mid x \in M, |\xi|^2 + |\lambda|^\frac{2}{\chi} = 1 \} \), we know that \( a_m(x, \xi, \lambda) \) does not have any eigenvalues in a sector \( L[\theta - \delta, \theta + \delta] \) for sufficiently small \( \delta > 0 \).

It is shown in [BFK] that as \( \lambda \to +\infty \), \( \log \det_\theta A(\lambda) \) admits an asymptotic expansion of the form

\[
\log \det_\theta A(\lambda) \sim \pi_d \lambda^\frac{d}{\chi} + \pi_{d+1} \lambda^\frac{d+1}{\chi} + \cdots + \pi_0 + \pi_1 \lambda^{-\frac{1}{\chi}} + \cdots + \sum_{j=0}^{d} q_j \lambda^{\frac{j}{\chi}} \log \lambda
\]

as \( \lambda \to +\infty \). Here each coefficient \( \pi_i \) and \( q_j \) is computable in terms of the asymptotic symbol of \( A(\lambda) \).

Let \( \sum_{j=0}^{\infty} a_{m-j}(x, \xi, \lambda) \) be an asymptotic symbol of \( A(\lambda) \) for some local coordinate \( U \). We also assume that for each \( j \), the function \( a_{m-j}(x, \xi, \lambda) \) is smooth. Then from \( A(\lambda) \) we can construct a classical elliptic pseudodifferential operator \( P \) with the

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Then $\tilde{\zeta}$ symbol

Now for each $j$, $\zeta$ where $\pi$ is the natural projection.

The purpose of this note is to show that $\pi_1 = \frac{m}{2} \zeta_P(0)$. So far we are unable to give a similar interpretation of other coefficients $\pi_i$ ($i \neq 1$) and $q_j$.

II. CONSTRUCTION OF $P$

Suppose that in a local coordinate system $U$, the asymptotic symbol of $A(\lambda)$ is $\sigma(A(\lambda)) \sim \sum_{j=0}^{\infty} a_{m-j}(x, \xi, \lambda)$. Then

$$a_{m-j}(x, \xi, \lambda) : U \times \mathbb{R}^d \times [0, \infty) \to \{k \times k \text{ matrices}\}$$

with

$$a_{m-j}(x, t\xi, t^2\lambda) = t^{m-j}a_{m-j}(x, \xi, \lambda) \quad \text{for} \quad t > 0.$$  

Now for each $j$, we extend $a_{m-j}$ to

$$\tilde{a}_{m-j} : U \times S^1 \times \mathbb{R}^d \times \mathbb{R} \to \{k \times k \text{ matrices}\} \quad \text{by} \quad (x, t, \xi, \lambda) \mapsto a_{m-j}(x, \xi, |\lambda|^2).$$

Then $\tilde{a}_{m-j}$ is smooth by the assumption of the smoothness of $a_{m-j}(x, |\lambda|^2)$.

Choose an elliptic $DO$ $\tilde{A} : C^\infty(p^*E) \to C^\infty(p^*E)$ whose asymptotic symbol is $\sigma(\tilde{A}) \sim \sum_{j=0}^{\infty} \tilde{a}_{m-j}(x, t, \xi, \lambda)$ in a local coordinate $U \times S^1$. Since the principal symbol $a_m(x, \xi, |\lambda|^2)$ of $\tilde{A}$ does not have any eigenvalues in the sector $L_{[\theta - \delta, \theta + \delta]}$ for $|\xi| + |\lambda| \neq 0$, there exists $R > 0$ such that $\text{Spectrum}(\tilde{A}) \cap \{|z| \geq R, \theta - \delta \leq \arg z \leq \theta + \delta\}$ is empty (see [Sh] for details). Since the spectrum of $\tilde{A}$ is discrete, there are only finitely many eigenvalues of $\tilde{A}$ in $\{|z| < R, \theta - \delta \leq \arg z \leq \theta + \delta\}$. Note that each eigenspace corresponding to an eigenvalue is a finite-dimensional vector space by the ellipticity of $\tilde{A}$ of order $m > 0$. Let $Q$ be the span of $\{\text{eigenvectors of } \tilde{A} \text{ whose eigenvalues are } \rho e^{i\theta} \text{ for some } \rho, 0 \leq \rho \leq R\}$. Then $Q$ is a finite-dimensional vector space. Define $\phi : C^\infty(p^*E) \to C^\infty(p^*E)$ be the natural projection onto $Q$.

Define $P = \tilde{A} - Re^{i\theta}$. Then $P$ is injective with an Agmon angle $\theta$ and the asymptotic symbol of $P$ is exactly the same as the asymptotic symbol of $\tilde{A}$. Define $\zeta_P(s) = \sum \lambda_i^{-s}$, where $\lambda_i$’s are the eigenvalues of $P$. Then by [Se] (also see [Wo]), $\zeta_P(s)$ is regular at $0$ with

$$\zeta_P(0) = \frac{e^{i\theta}}{m(2\pi)^{d+1}} \cdot \int_{M \times S^1} d\text{vol}(x, t) \int_{|\xi|^2 + \lambda^2 = 1} d(\xi, \lambda) \int_0^\infty \text{tr} \tilde{r}_{m-d-1}(x, t, \xi, \lambda, e^{i\theta} \mu) d\mu,$$
where \( \tilde{r}_{-m-d-1} \) is the homogeneous part of degree \(-m-d-1\) in the asymptotic symbol of the resolvent \((P - \mu I)^{-1}\). Note that \( \tilde{r}_{-m-d-1}(x, t, \xi, \lambda, e^{i\theta} \mu) \) does not depend on \( t \) in this case.

Recall that \( \log \det_\theta A(\lambda) \sim \pi_\mu \lambda^d + \cdots + \pi_0 + \pi_1 \lambda^{-\frac{1}{2}} + \cdots + \sum_{j=0}^d q_j \lambda^\frac{1}{2} \log \lambda \) as \( \lambda \to +\infty \). Then our goal is to prove the following theorem.

**Theorem.** \( \pi_1 = \frac{m}{2} \zeta_\mu(0) \).

### III. Proof of the Theorem

From the Appendix of [BFK] we can derive the local formula

\[
\pi_1 = \frac{-e^{i\theta}}{(2\pi)^d} \int_M \int_{\mathbb{R}^d} \int_0^\infty \text{tr} \ r_{-m-d-1}(x, \xi, 1, e^{i\theta} \mu) \ d\mu d\xi d \text{vol}(x),
\]

where \( r_{-m-d-1} \) is the homogeneous part of degree \(-m-d-1\) in the asymptotic symbol of the resolvent \((A(\lambda) - \mu I)^{-1}\).

Note the relation \( \tilde{r}_{-m-d-1}(x, t, \xi, \lambda, e^{i\theta} \mu) = r_{-m-d-1}(x, \xi, |\lambda|^d, e^{i\theta} \mu) \). Then

\[
\zeta_\mu(0) = \frac{-2e^{i\theta}}{m(2\pi)^{d+1}} \cdot \int_M d \text{vol}(x) \int_{|\xi|^2 + \lambda^2 = 1} \int_0^\infty \text{tr} \ r_{-m-d-1}(x, \xi, |\lambda|^d, e^{i\theta} \mu) \ dm \]

since the integrand is even in \( \lambda \).

Set

\[
(I) = \int_{|\xi|^2 + \lambda^2 = 1} \text{tr} \ r_{-m-d-1}(x, \xi, |\lambda|^d, e^{i\theta} \mu) \ dm.
\]

Then using the projection from the upper hemisphere to \( \{ \xi \in \mathbb{R}^d ||\xi| < 1 \} \), we obtain

\[
(I) = \int_{|\xi| < 1} \int_0^\infty \text{tr} \ r_{-m-d-1}(x, \xi, (\sqrt{1 - |\xi|^2})^d, e^{i\theta} \mu)(1 - |\xi|^2)^{-\frac{1}{2}} \ dm \ d\xi
\]

\[
= \int_{|\xi| < 1} \int_0^\infty (1 - |\xi|^2)^{-\frac{\mu + d + 2}{2}} \text{tr} \ r_{-m-d-1}
\]

\[
\cdot \left( x, \frac{\xi_1}{\sqrt{1 - |\xi|^2}}, \cdots, \frac{\xi_d}{\sqrt{1 - |\xi|^2}}, 1, e^{i\theta} \frac{\mu}{\sqrt{1 - |\xi|^2}} \right) \ dm \ d\xi,
\]

since the weight of \( \mu \) is \( m \).

Consider a map \( \Phi : \mathbb{R}^d \times (0, \infty) \to \{ \xi \in \mathbb{R}^d ||\xi| < 1 \} \times (0, \infty) \) defined by

\[
(\eta_1, \ldots, \eta_d, \nu) \mapsto \left( \frac{\eta_1}{\sqrt{1 + |\eta|^2}}, \ldots, \frac{\eta_d}{\sqrt{1 + |\eta|^2}}, \frac{\nu}{\sqrt{1 + |\eta|^2}} \right) = (\xi_1, \ldots, \xi_d, \mu).
\]

Then \( \Phi \) is a diffeomorphism with \( \det(J(\Phi)) = (1 + |\eta|^2)^{-\frac{\mu + d + 2}{2}} = (1 - |\xi|^2)^{\frac{\mu + d + 2}{2}} \).

Hence we can get \( I = \int_{\mathbb{R}^d} \int_0^\infty \text{tr} \ r_{-m-d-1}(x, \eta, 1, e^{i\theta} \nu) \ d\nu d\eta. \)

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