THE CLASSICAL BANACH SPACES $\ell_\varphi/h_\varphi$

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Abstract. In this paper we study some structural and geometric properties of the quotient Banach spaces $\ell_\varphi(I)/h_\varphi(S)$, where $I$ is an arbitrary set, $\varphi$ is an Orlicz function, $\ell_\varphi(I)$ is the corresponding Orlicz space on $I$ and $h_\varphi(S) = \{x \in \ell_\varphi(I) : \forall \lambda > 0, \exists s \in S \text{ such that } I_\varphi(x/\lambda) < \infty\}$, $S$ being the ideal of elements with finite support. The results we obtain here extend and complete the ones obtained by Leonard and Whitfield (Rocky Mountain J. Math. 13 (1983), 531–539). We show that $\ell_\varphi(I)/h_\varphi(S)$ is not a dual space, that $Ext(B_{\ell_\varphi(I)}(h_\varphi(S))) = \emptyset$, if $\varphi(t) > 0$ for every $t > 0$, that $S_{\ell_\varphi(I)/h_\varphi(S)}$ has no smooth points, that it cannot be renormed equivalently with a strictly convex or smooth norm, that $\ell_\varphi(I)/h_\varphi(S)$ is a Grothendieck space, etc.

1. Notation and preliminaries

Let $\varphi : \mathbb{R} \to [0, +\infty]$ denote an Orlicz function, i.e. a function which is even, nondecreasing, left continuous for $x \geq 0$, $\varphi(0) = 0$ and $\varphi(x) \to \infty$ as $x \to \infty$. Define $a(\varphi) = \sup\{t \geq 0 : \varphi(t) = 0\}$, $\tau(\varphi) = \sup\{t \geq 0 : \varphi(t) < \infty\}$ and assume that $\tau(\varphi) > 0$. Fix an arbitrary set $I$ and, for $x \in \mathbb{R}^I$, define $I_\varphi(x) = \sum_{i \in I} \varphi(x_i)$. Let $\ell_\varphi(I)$ be the corresponding Orlicz space, i.e. $\ell_\varphi(I) = \{x \in \mathbb{R}^I : \exists \lambda > 0 \text{ such that } I_\varphi(x/\lambda) < \infty\}$. Consider in $\ell_\varphi(I)$ the F-norm $|x|_\varphi := \inf\{\lambda > 0 : I_\varphi(x/\lambda) \leq \lambda\}$, $\forall x \in \ell_\varphi(I)$, and the associated distance $d(x, y) = |x - y|_\varphi$. It is known that $(\ell_\varphi(I), d)$ is a complete F-space.

Let $S \subseteq \ell_\varphi(I)$ be the ideal of elements of finite support. Define $h_\varphi(S)$ by:

$$h_\varphi(S) = \{x \in \ell_\varphi(I) : \forall \lambda > 0, \exists s \in S \text{ such that } I_\varphi(\frac{x - s}{\lambda}) < \infty\},$$

and $\delta(x)$ by:

$$\delta(x) = \inf\{\lambda > 0 : \exists s \in S \text{ such that } I_\varphi(\frac{x - s}{\lambda}) < \infty\}, x \in \ell_\varphi(I).$$

Clearly, $h_\varphi(S)$ is a closed ideal of $\ell_\varphi(I)$ such that $h_\varphi(S) = \{x \in \ell_\varphi(I) : \forall \lambda > 0, I_\varphi(\lambda x) < \infty\}$, if $\varphi$ is finite, and $\overline{S} = h_\varphi(S)$, where $\overline{S}$ is the closure of $S$ in $\ell_\varphi(I)$.

We are interested in the quotient space $\ell_\varphi(I)/h_\varphi(S)$. Hence we must impose the condition $\ell_\varphi(I) \neq h_\varphi(S)$. Note that this happens if and only if $I$ is infinite and $\varphi \notin \Delta_2^0$, i.e. $\varphi$ doesn't satisfy the $\Delta_2$ condition at 0.

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3777
If \( \varphi \) is convex we can consider the Luxemburg norm \( \| \cdot \|_L \) and the Luxemburg distance \( d_L \):

\[
\| x \|_L = \inf \{ \lambda > 0 : I_\varphi(x/\lambda) \leq 1 \}, \quad d_L(x, y) = \| x - y \|_L, \quad x, y \in \ell_\varphi(I),
\]
as well as the Amemiya-Orlicz norm \( \| \cdot \|_o \) and the Amemiya-Orlicz distance \( d_o \):

\[
\| x \|_o = \inf_{k>0} \{ \frac{1}{k} (1 + I_\varphi(kx)) \}, \quad d_o(x, y) = \| x - y \|_o, \quad x, y \in \ell_\varphi(I).
\]

It is known that, \( \forall x \in \ell_\varphi(I) \), \( \| x \|_L \leq \| x \|_o \leq 2 \| x \|_L \) and that these norms define on \( \ell_\varphi(I) \) the same topology as \( \| \cdot \|_\varphi \). Denote by \( B^+_L \) (resp. \( B^+_o \)) and \( S^+_L \) (resp. \( S^+_o \)) the closed unit ball and unit sphere of \( (\ell_\varphi(I), \| \cdot \|_L) \) (resp. \( (\ell_\varphi(I), \| \cdot \|_o) \)). Recall that a Banach M-space is a Banach lattice \((X, \| \cdot \|)\) such that \( \| x \lor y \| = \| x \| \lor \| y \| \), whenever \( x, y \in X^+ \).

**Proposition 1.1.** Let \( I \) be an infinite set and \( \varphi \) an Orlicz function such that \( \ell_\varphi(I) \neq h_\varphi(S) \). Then:

1. For each \( x \in \ell_\varphi(I) \) we have \( \delta(x) = d(x, h_\varphi(S)) \) and, if \( \varphi \) is convex, also \( \delta(x) = d_L(x, h_\varphi(S)) = d_o(x, h_\varphi(S)) \).
2. \( \delta \) is a monotone seminorm on \( \ell_\varphi(I) \) such that \( \ker(\delta) = h_\varphi(S) \).
3. Let \( \| \cdot \| \) be the quotient F-norm on \( \ell_\varphi(I)/h_\varphi(S) \). Then \( (\ell_\varphi(I)/h_\varphi(S), \| \cdot \|) \) is a Banach M-space.
4. If \( \varphi \) is convex, the space \( \ell_\varphi(I)/h_\varphi(S) \) equipped with the quotient norms corresponding to the Luxemburg norm as well as to the Orlicz norm is order isomorphic and isometric to \( (\ell_\varphi(I)/h_\varphi(S), \| \cdot \|) \).

**Proof.** (1) Let \( x \in \ell_\varphi(I) \) and fix \( \epsilon > 0 \). Then \( \exists z \in S \) such that \( I_\varphi \left( \frac{x - s - y_\alpha + z_\alpha}{\delta(x) + \epsilon} \right) < +\infty \) and \( 0 \leq s^+ \leq x^+, 0 \leq s^- \leq x^- \). Pick \( \{ y_\alpha \}_{\alpha \in A}, \{ z_\alpha \}_{\alpha \in A} \) in \( h_\varphi(S)^+ \) with \( y_\alpha \uparrow x^+, z_\alpha \uparrow x^- \). Since \( I_\varphi \) is \( \alpha \)-continuous, we get:

\[
I_\varphi \left( \frac{x - s - y_\alpha + z_\alpha}{\delta(x) + \epsilon} \right) = I_\varphi \left( \frac{x^+ - s^+ - y_\alpha + x^- - s^- - z_\alpha}{\delta(x) + \epsilon} \right) \to 0
\]

with respect to (for short, wrt) \( \alpha \in A \). Hence \( d(x, h_\varphi(S)) \leq \delta(x) \), since \( \epsilon > 0 \) is arbitrary. If \( \varphi \) is convex, the above also proves that \( d_L(x, h_\varphi(S)) \leq \delta(x) \).

Concerning the Amemiya-Orlicz norm, since \( I_\varphi \left( \frac{x - s - y_\alpha + z_\alpha}{\delta(x) + \epsilon} \right) \to 0 \) wrt \( \alpha \in A \), we have:

\[
\| x - s - y_\alpha + z_\alpha \|_o \leq (\delta(x) + \epsilon) \left[ 1 + I_\varphi \left( \frac{x - s - y_\alpha + z_\alpha}{\delta(x) + \epsilon} \right) \right] \to \delta(x) + \epsilon \text{ wrt } \alpha \in A,
\]

whence, \( \epsilon \) being arbitrary, it follows that \( d_o(x, h_\varphi(S)) \leq \delta(x) \).

For the contrary inequality, if \( \delta(x) = 0 \), the above proves that \( 0 = \delta(x) = d(x, h_\varphi(S)) = d_L(x, h_\varphi(S)) = d_o(x, h_\varphi(S)) \). Assume that \( \delta(x) > 0 \) and pick a fixed \( y \in h_\varphi(S) \). Suppose that there exists \( 0 < \lambda < \delta(x) \) such that \( I_\varphi \left( \frac{x - y}{\lambda} \right) < +\infty \). Take \( \lambda < t < \delta(x) \) and denote \( r = \lambda/t \). Then \( 0 < r < 1 \) and \( \exists \beta \in S \) such that \( I_\varphi \left( \frac{x - y}{(1-r)\lambda} \right) < +\infty \). Since \( \frac{x - y}{t} = r \frac{x - y}{rt} + (1 - r) \frac{y - s}{(1-r)t} \), we have:

\[
I_\varphi \left( \frac{x - s}{t} \right) \leq I_\varphi \left( \frac{x - y}{\lambda} \right) + I_\varphi \left( \frac{y - s}{(1-r)t} \right) < +\infty,
\]
a contradiction. Hence $0 < \lambda < \delta(x)$, $\forall y \in h_\varphi(S)$, $I_\varphi \left( \frac{y - x}{\delta(x)} \right) = +\infty$, which implies $d(x, h_\varphi(S)) = \delta(x) \leq d_L(x, h_\varphi(S))$. As $\| \cdot \|_o \geq \| \cdot \|_L$, we also get $d_L(x, h_\varphi(S)) \geq \delta(x)$.

(2) and (3) were proved in [15] and (4) follows easily from the above.

In the sequel $\ell_\varphi(I)/h_\varphi(S)$ will be the Banach $M$-space $(\ell_\varphi(I)/h_\varphi(S), \| \cdot \|)$ and $Q$ the quotient map $Q : \ell_\varphi(I) \to \ell_\varphi(I)/h_\varphi(S)$. Let $\beta I$ denote the Stone-Weierstrass compactification of $I$, when we consider in $I$ the discrete topology. Denote by $\mathcal{F}(I)$ the class of finite subsets of $I$. If $x \in \mathbb{R}^k$ and $A \subseteq I$, define $x_A = x \cdot 1_A$ and $x^A = x \cdot 1_{I \setminus A}$.

**Proposition 1.2.** Let $I$ be an infinite set and $\varphi$ an Orlicz function such that $\ell_\varphi(I) \neq h_\varphi(S)$. If $a(\varphi) > 0$, then

$$\ell_\varphi(I)/h_\varphi(S) \cong (\ell_\infty(I)/c_o(I), \| \cdot \|_{\infty}) \cong (C(\beta I \setminus I), \| \cdot \|_{\infty})$$

(order isomorphism and isometry).

**Proof.** First of all, it is clear that $\ell_\varphi(I) = \ell_\infty(I)$ and $h_\varphi(S) = c_o(I)$, as sets and algebraically. Consider the map $i : \ell_\infty(I) \to \ell_\varphi(I)$ such that $i(x) = a(\varphi) \cdot x$ and the quotient map $q : \ell_\infty(I) \to \ell_\infty(I)/c_o(I)$. Note that $\|i(x)\|_{\varphi} \leq \|x\|_{\infty}$ and that:

$$\forall x \in \ell_\infty(I), \quad \|q(x)\| = \inf_{A \in \mathcal{F}(I)} \|x^A\|_{\infty},$$

$$\|Q(i(x))\| = d(i(x), h_\varphi(S)) = \inf_{A \in \mathcal{F}(I)} |i(x^A)|_{\varphi}.$$

Clearly, $\|Q(i(x))\| \leq \|q(x)\|$, whence, if $\|q(x)\| = 0$, we get $\|Q(i(x))\| = \|q(x)\| = 0$. Assume that $\|q(x)\| =: a > 0$ and take $0 < \epsilon < a$. Find sequences, $\{A_n\}_{n \geq 1}$ in $\mathcal{F}(I)$ and $\{n_i\}_{n \geq 1}$ in $I$, such that $A_n \subseteq A_{n+1}$, $i_n \in A_{n+1} \setminus A_n$ and $|x_{i_n}| > a - \epsilon/2$. Then:

$$\forall n \geq 1, \quad I_\varphi \left( \frac{i(x^A_n)}{a - \epsilon} \right) = I_\varphi \left( \frac{a(\varphi) \cdot x^A_n}{a - \epsilon} \right) \geq \sum_{k > n} \varphi \left( \frac{a(\varphi) \cdot x_{i_k}}{a - \epsilon} \right) = +\infty,$$

which implies $|i(x^A_n)|_{\varphi} \geq a - \epsilon$, $\forall n \geq 1$, whence $\|Q(i(x))\| \geq a - \epsilon$. Since $\epsilon > 0$ is arbitrary, we get $\|Q(i(x))\| \geq a$ and finally $\|Q(i(x))\| = a$.  

\[ \square \]

2. **Proximinality**

Let $(X, D)$ be a metric linear space with a distance $D$ and $M \subseteq X$ a subspace of $X$. Consider the distance $D(x, M) = \inf \{D(x, m) : m \in M\}$, $x \in X$, and say that $x \in X$ is $M$-approximable if $\exists m \in M$ such that $D(x, M) = D(x, m)$. Denote by $Ap(M, X)$ the subset of $M$-approximable elements of $X$. If $Ap(M, X) = X$, $M$ is said to be proximinal in $X$. If $M$ is proximinal in $X$ then, obviously, $M$ is closed in $X$.

Let $(X, \| \cdot \|)$ be a normed space and $M \subseteq X$ a closed subspace. Denote by $B_X$, $S_X$ its closed unit ball and unit sphere, respectively, and by $X^*$ its topological dual. Define $Top(M, X) = \{x \in S_X : \text{distance } (x, M) = 1\}$. Clearly, $Top(M, X) \subseteq Ap(M, X) \setminus M$ and $x \in Top(M, X)$ iff $x \in S_X$ and $q(x) \in S_{X/M}$, where $q$ is the canonical quotient map $q : X \to X/M$. In normed spaces, the proximinality has been characterized by Godini as follows:
Theorem 2.1 (Godini). If $X$ is a normed space and $M \subseteq X$ a closed subspace, then the following are equivalent: (1) $q(B_X) = B_{X/M}$; (2) $q(B_X)$ is closed in $X/M$; (3) $M$ is proximinal in $X$.

Proof. See [7].

Proposition 2.2. Let $I$ be an infinite set and $\varphi$ an Orlicz function such that $\ell_\varphi(I) \neq h_\varphi(S)$. Then:

(a) $h_\varphi(S)$ is proximinal in $(\ell_\varphi(I), |\cdot|)$ and, if $\varphi$ is convex, also in $(\ell_\varphi(I), \|\cdot\|_L)$.

(b) Assume that $\varphi$ is convex. Then:

(1) $x \in \text{Top}(h_\varphi(S), (\ell_\varphi(I), \|\cdot\|_z))$ if and only if $x \in \text{Top}(h_\varphi(S), (\ell_\varphi(I), \|\cdot\|_z))$, for $z = L$ or $z = 0$.

(2) $\text{Top}(h_\varphi(S), (\ell_\varphi(I), \|\cdot\|_o)) = \text{Top}(h_\varphi(S), (\ell_\varphi(I), \|\cdot\|_L)) \cap S^o_\varphi$.

(3) $\text{Top}(h_\varphi(S), (\ell_\varphi(I), \|\cdot\|_L)) = \{ x \in \ell_\varphi(I) : I_\varphi(x) \leq 1, I_\varphi(\lambda x^a) = \infty, \forall \lambda > 1, \forall A \in \mathfrak{A}(I) \}$.

(4) If $a(\varphi) = 0$, then $\text{Top}(h_\varphi(S), (\ell_\varphi(I), \|\cdot\|_o)) = \emptyset$.

If $a(\varphi) > 0$, then

$\text{Top}(h_\varphi(S), (\ell_\varphi(I), \|\cdot\|_o)) = \{ x \in \ell_\varphi(I) : |x| \leq a(\varphi), \forall i \in I \}$.

and $\forall \epsilon > 0$, $\text{card}\{i \in I : |x_i| \geq a(\varphi) - \epsilon \} = \infty$.

(5) $h_\varphi(S)$ is proximinal in $(\ell_\varphi(I), \|\cdot\|_o)$ if $a(\varphi) > 0$.

Proof. (a) Pick $x \in \ell_\varphi(I)$. If $\delta(x) = 0$, by Proposition 1.1 we get that $d(x, h_\varphi(S)) = 0$. Hence $x \in h_\varphi(S)$ since $h_\varphi(S)$ is closed in $(\ell_\varphi(I), |\cdot|)$. Assume that $\delta(x) > 0$ and $x \geq 0$. Let $\epsilon_k \downarrow 1$ be such that $1 - \frac{1}{\epsilon_k} =: \eta_k \leq 2^{-k}, k \geq 1$. Since $I_\varphi \left( \frac{x}{\delta(x)x_k} \right) < \infty$ and $I_\varphi$ is o-continuous, there exists a finite subset $A_1 \subseteq I$ such that $I_\varphi \left( \frac{x - u_k}{\delta(x)x_k} \right) \leq 2^{-2a}$, where $u_1 := x \cdot 1_{A_1}$ and $0 < a \leq \inf \{ 1, \delta(x) \}$ is arbitrary. Let $x_2 := x - u_1$. Then there exists a finite subset $A_2 \subseteq I \setminus A_1$ such that $I_\varphi \left( \frac{x - u_k}{\delta(x)x_k} \right) \leq 2^{-3a}$, where $u_2 := x \cdot 1_{A_2}$. By reiteration we obtain a family of pairwise disjoint elements $\{ u_n \}_{n \geq 1}$ in $S^+$ such that, if $x_n = x - \sum_{k=0}^{n-1} u_k, n \geq 1, u_0 = 0$, then $u_n \leq x_n$ and $I_\varphi \left( \frac{x - u_n}{\delta(x)x_n} \right) \leq 2^{-n-1}a$.

Let $g_r = \sum_{k=0}^{r} \eta_k u_{k+1}, \eta_0 = 1$. We claim that $\{ g_r \}_{r \geq 0}$ is a Cauchy sequence in $(\ell_\varphi(I), |\cdot|)$. Indeed, fix $\epsilon > 0$ and take $r_0 \in \mathbb{N}$ such that, $\forall r > r_0, \eta_r/\epsilon \leq \frac{1}{\delta(x)x_r}$ and $\sum_{k \geq r_0} 2^{-(k+1)} \leq \epsilon/a$. Then, $\forall s \geq r > r_0$, we have:

$$I_\varphi \left( \frac{g_s - g_r}{\epsilon} \right) = \sum_{k=r+1}^{s} I_\varphi \left( \frac{\eta_k u_{k+1}}{\epsilon} \right) \leq \sum_{k=r+1}^{s} I_\varphi \left( \frac{u_{k+1}}{\delta(x)x_k} \right) \leq (\epsilon/a)a = \epsilon.$$ 

Hence $\sum_{k \geq 0} \eta_k u_{k+1} =: g \in h_\varphi(S)$. Note also that $\sum_{k \geq 0} u_{k+1} =: f \in \ell_\varphi(I)$, because $\ell_\varphi(I)$ is $\sigma$-o-complete and $0 \leq f \leq x$. Let $z = x - f$. Then $f \wedge z = 0$ and $0 \leq z \leq x_{k+1}, \forall k \geq 0$. So $I_\varphi \left( \frac{z}{\delta(x)x_k} \right) \leq 2^{-(k+1)}a, \forall k \geq 1$. Since $I_\varphi$ is
left-continuous, we get \( I_\varphi \left( \frac{x-g}{\delta(x)} \right) = 0 \). Hence:

\[
I_\varphi \left( \frac{x-g}{\delta(x)} \right) = I_\varphi \left( \frac{x-z-g+z}{\delta(x)} \right) = I_\varphi \left( \sum_{k \geq 0} \left( 1 - \eta_k \right) u_{k+1} + z \right)
\]

\[
= \left[ \sum_{k \geq 0} I_\varphi \left( \frac{u_{k+1}}{\delta(x) \epsilon_k} \right) + I_\varphi \left( \frac{z}{\delta(x)} \right) \right] \leq a \sum_{k \geq 0} 2^{-(k+1)} \leq a.
\]

Thus \( D(x, g) \leq \delta(x) \) with \( D = d \) or \( D = d_L \) and \( d_L(x, y) = \| x - y \|_L \). Since

\[
D(x, g) \geq \delta(x)
\]

we get \( D(x, g) = \delta(x) \).

In the general case (i.e. \( x^+ > 0, x^- > 0 \)), if \( \delta(x) > 0 \) (i.e. \( x \notin h_\varphi(S) \)), by the above it is possible to find \( g_1, g_2 \in h_\varphi(S) \) such that \( 0 \leq g_1 \leq x^+, 0 \leq g_2 \leq x^- \)

\[
\text{and } I_\varphi \left( \frac{x-g_1}{\delta(x)} \right) \leq \frac{2}{3} \leq I_\varphi \left( \frac{x-g_2}{\delta(x)} \right).
\]

Thus, if \( g = g_1 - g_2 \), we get \( I_\varphi \left( \frac{x-g}{\delta(x)} \right) = \left[ I_\varphi \left( \frac{x-g_1}{\delta(x)} \right) + I_\varphi \left( \frac{x-g_2}{\delta(x)} \right) \right] \leq a \). Hence \( D(x, g) = \delta(x) \).

(b)(1) Observe that, for \( z = L \) or \( z = a \), we have \( \| x \|_z = \| x \|_z \) and \( d_z(x, h_\varphi(S)) = \inf \{ \| x - y \|_z : y \in h_\varphi(S) \} = \inf \{ \| x - y \|_z : y \in h_\varphi(S) \} = d_z([x], h_\varphi(S)) \).

(b)(2) If \( f \in \text{Top}(h_\varphi(S), (\ell_\varphi(I), \| \cdot \|_o)) \), then \( 1 = d_o(f, h_\varphi(S)) = d_L(f, h_\varphi(S)) \leq \| f \|_L \leq \| f \|_o = 1 \). Hence \( f \in \text{Top}(h_\varphi(S), (\ell_\varphi(I), \| \cdot \|_L)) \cap S^o_\varphi \).

If \( f \in \text{Top}(h_\varphi(S), (\ell_\varphi(I), \| \cdot \|_L)) \cap S^o_\varphi \), then \( 1 = d_L(f, h_\varphi(S)) = d_o(f, h_\varphi(S)) \leq \| f \|_o = 1 \). Hence \( f \in \text{Top}(h_\varphi(S), (\ell_\varphi(I), \| \cdot \|_o)) \).

(b)(3) It is enough to remark that \( x \in \text{Top}(h_\varphi(S), (\ell_\varphi(I), \| \cdot \|_o)) \) iff \( \| x \|_L \leq 1 \) and \( \delta(x) \geq 1 \). But these conditions are equivalent to \( I_\varphi(x) \leq 1 \) and, \( \forall \lambda > 1, \forall A \in \mathfrak{F}(I), I_\varphi(\lambda x^A) = \infty \).

(b)(4) First of all, note that if \( x \in \text{Top}(h_\varphi(S), (\ell_\varphi(I), \| \cdot \|_o)) \), then \( |x_i| \in [0, a(\varphi)], \forall i \in I. \) Indeed, we have that \( \delta(x) \geq 1 \), i.e.:

\[
(*) \quad \forall \lambda > 1, \forall A \in \mathfrak{F}(I), I_\varphi(\lambda x^A) = \infty.
\]

Since \( 1 = \| x \|_o = \inf_{k>0} \{ \frac{1}{k} (1 + I_\varphi(kx)) \} \), we get that \( 1 = 1 + I_\varphi(x) \), whence \( I_\varphi(x) = 0 \) and \( |x_i| \in [0, a(\varphi)], \forall i \in I. \)

Therefore, if \( a(\varphi) = 0 \), it is clear that \( \text{Top}(h_\varphi(S), (\ell_\varphi(I), \| \cdot \|_o)) = \emptyset \). Assume that \( a(\varphi) > 0 \) and that \( x \in \text{Top}(h_\varphi(S), (\ell_\varphi(I), \| \cdot \|_o)) \). Then, by the above, \( |x_i| \leq a(\varphi), \forall i \in I. \) By (*) it follows that \( \forall \epsilon > 0 \), card \{ \( i \in I : |x_i| \geq a(\varphi) - \epsilon \} = \infty. \) Finally if \( x \in \ell_\varphi(I) \) satisfies \( |x_i| \leq a(\varphi), \forall i \in I, \) and card \{ \( i \in I : |x_i| \geq a(\varphi) - \epsilon \} = \infty, \forall \epsilon > 0 \), we easily conclude that \( \| x \|_o = \inf_{k>0} \{ \frac{1}{k} (1 + I_\varphi(kx)) \} = 1 \) and that \( \delta(x) \geq 1 \), i.e. \( x \in \text{Top}(h_\varphi(S), (\ell_\varphi(I), \| \cdot \|_o)) \).

(b)(5) If \( a(\varphi) = 0 \) it is clear, by the above, that \( h_\varphi(S) \) is not proximinal in \( (\ell_\varphi(I), \| \cdot \|_o) \). Assume that \( a(\varphi) > 0 \). By Proposition 2.1, it is enough to prove that, if \( x \in \text{Top}(h_\varphi(S), (\ell_\varphi(I), \| \cdot \|_L)) \), then there exists \( f \in h_\varphi(S), 0 < f \leq x \), such that \( \| x - f \|_o = 1 \). Denote \( h = (x - a(\varphi)) \vee 0 \) and observe that \( h \in h_\varphi(S) \) (because, \( \forall \lambda > 0, \) card \{ \( i \in I : \lambda h_i > a(\varphi) \} < N_0 \). Clearly \( I_\varphi(x-h) = 0 \) and, \( \forall \lambda > 1, I_\varphi(\lambda(x-h)) = \infty \) (because \( d_L(x, h_\varphi(S)) = d_L(x-h, h_\varphi(S)) = 1 \)). Hence:

\[
\| x-h \|_o = \inf_{k>0} \{ \frac{1}{k} (1+ I_\varphi(k(x-h))) \} = 1 + I_\varphi(x-h) = 1.
\]
3. Extremal structures

Denote by $\text{Ext}(C)$ the set of extreme points of a convex set $C$. If $a(\varphi) > 0$, we have, by Proposition 1.2 and [10, Theorem 4.1], that the ball $B_{\ell_p(I)/h_\varphi(S)}$ has an abundance of extreme points. In fact, we get

$$\text{Ext}(B_{\ell_p(I)/h_\varphi(S)}) = \text{Ext}(B_{\ell_\infty(I)/c_0(I)}) = q(\text{Ext}(B_{\ell_\infty(I)}))$$

and

$$B_{\ell_p(I)/h_\varphi(S)} = \overline{\text{co}}(\text{Ext}(B_{\ell_p(I)/h_\varphi(S)})).$$

If $a(\varphi) = 0$ the situation is completely different.

**Proposition 3.1.** Let $I$ be an infinite set and $\varphi$ an Orlicz function such that $\ell_\varphi(I) \neq h_\varphi(S)$ and $a(\varphi) = 0$. Then $\text{Ext}(B_{\ell_p(I)/h_\varphi(S)}) = \emptyset$.

**Proof.** Assume that $e \in \text{Ext}(B_{\ell_p(I)/h_\varphi(S)})$. Pick $w \in \ell_\varphi(I)$ such that $Q(w) = e$. Then $d(w, h_\varphi(S)) = 1$ and there exists $g \in h_\varphi(S)$ such that $1 = d(w, h_\varphi(S)) = d(w, g) = d(w - g, 0)$, whence, $\forall \lambda > 1$, $I_\varphi\left(\frac{w-g}{\lambda}\right) \leq \lambda$. By the left-continuity of $I_\varphi$ we get that $I_\varphi(w - g) \leq \lambda$, $\forall \lambda > 1$, i.e. $I_\varphi(w - g) \leq 1$. Let $u = w - g$ and suppose, without loss of generality, that $I_\varphi(u) \leq 1/2$ (if not, put $u_i = 0$ for $i \in A$ and some $A \in \mathcal{G}(I)$). Since $a(\varphi) = 0$, we can choose a countable subset $B = \{i_n\}_{n \geq 1}$ of $I$ such that $u_{i_n} \to 0$, as $n \to \infty$, and, if $h = u \cdot 1_B$, then $h \in h_\varphi(S)$ and $Q(u - h) = e$. Since $a(\varphi) = 0$ we have that $\text{card}(\text{supp}(u)) = 80$. Let $\text{supp}(u) = \{j_r\}_{r \geq 1}$ and define $x, y \in \ell_\varphi(I)$ as follows:

$$x_i = \begin{cases} u_{i_1}, & \text{if } i \neq B \\ u_{i_k}, & \text{if } i = i_k, \ k \geq 1 \end{cases}, \quad y_i = \begin{cases} u_{i_1}, & \text{if } i \neq B \\ -u_{i_k}, & \text{if } i = i_k, \ k \geq 1 \end{cases}$$

Then $Q(x) = Q(y)$ (because $x - y \notin h_\varphi(S)$), $Q(x), Q(y) \in B_{\ell_p(I)/h_\varphi(S)}$ (because $I_\varphi(x), I_\varphi(y) \leq 1$) and $\frac{1}{2}(Q(x) + Q(y)) = Q(u - h) = e$, a contradiction. Hence $\text{Ext}(B_{\ell_p(I)/h_\varphi(S)}) = \emptyset$. \hfill $\square$

If $X$ is a normed space and $x \in S_X$, denote $\text{Grad}(x) = \{x^* \in S_{X^*} : x^*(x) = 1\}$. We say that $x \in S_X$ is smooth iff $\text{card}(\text{Grad}(x)) = 1$.

**Proposition 3.2.** Let $I$ be an infinite set and $\varphi$ an Orlicz function such that $h_\varphi(S) \neq \ell_\varphi(I)$. Then $S_{\ell_p(I)/h_\varphi(S)}$ has no smooth points.

**Proof.** Let $e \in S_{\ell_p(I)/h_\varphi(S)}$ such that $I_\varphi(x) \leq 1$ and $Q(e) = e$. Then $I_\varphi(\lambda x) = \infty$, $\forall \lambda > 1$. We claim that there exists $C \subseteq I$ such that, if $y = x_C$ and $z = x^C$, then $Q(y), Q(z) \in S_{\ell_{\varphi}(I)/h_{\varphi}(S)}$. Indeed, since $I_\varphi((1 + 2^{-n})x) = \infty$, we can choose two sequences of nonempty and finite subsets $\{A_n\}_{n \geq 1}$, $\{B_n\}_{n \geq 1}$ of $I$ such that: (i) $\sum_{i \in A_n} \varphi((1 + 2^{-n})x_i) \geq 2^n \leq \sum_{i \in B_n} \varphi((1 + 2^{-n})x_i)$; (ii) $A_n \cap B_n = \emptyset = (A_n \cup B_n) \cap (A_m \cup B_m)$, $n \neq m$. Now, take $C = \cup_{n \geq 1} A_n$. Note that $I_\varphi(y + z) = I_\varphi(x) \leq 1$, $Q(y + z) \in S_{\ell_\varphi(I)/h_\varphi(S)}$ and $y + z = x$.

There exists $y^* \in \text{Grad}(Q(y))$ and $z^* \in \text{Grad}(Q(z))$ such that:

$$1 \geq y^*(Q(y) + Q(z)) = y^*(Q(y)) + y^*(Q(z)) = 1 \pm y^*(Q(z)),$$

whence we get $y^*(Q(z)) = 0$. In a similar way, we get $z^*(Q(y)) = 0$. This means that $y^* \neq z^*$. We have:

$$y^*(Q(x)) = y^*(Q(y) + Q(z)) = y^*(Q(y)) + y^*(Q(z)) = 1 + 0 = 1,$$

$$z^*(Q(x)) = z^*(Q(y) + Q(z)) = z^*(Q(y)) + z^*(Q(z)) = 0 + 1 = 1,$$

which means that $y^*, z^* \in \text{Grad}(e)$, so $e$ is not smooth. \hfill $\square$
4. Order completeness and order continuity

In [15] it is proved that every \( x \in (\ell_\varphi(I)/h_\varphi(S)) \setminus \{0\} \) is \( \sigma\)-o-continuous and not \( \sigma\)-o-complete. Recall that a vector \( x \) of a Banach lattice \( X \) is: (i) \( \sigma\)-o-continuous if for every decreasing sequence \( \{x_n\}_{n \geq 1} \) in \( X^+ \) such that \( x_n \leq |x| \) and \( \inf_{n \geq 1} x_n = 0 \), we have \( \|x_n\| \downarrow 0 \); (ii) \( \sigma\)-o-complete if for every increasing sequence \( \{x_n\}_{n \geq 1} \) in \( X^+ \) such that \( x_n \leq |x| \), there exists \( \sup_{n \geq 1} x_n \). In particular, an increasing sequence in \( \ell_\varphi(I)/h_\varphi(S) \) has supremum if and only if it is a Cauchy sequence.

As a consequence, we get the following known fact: if \( I \) is an infinite set and \( \{A_n\}_{n \geq 1} \) a sequence of closed-and-open (clopen) subsets of \( \beta I \setminus I \) such that \( A_n \subseteq A_{n+1} \) and \( A_n \neq A_{n+1} \), then \( \overline{A} \) is not open in \( \beta I \setminus I \), with \( A := \bigcup_{n \geq 1} A_n \). Indeed, let \( \varphi \) be the convex Orlicz function such that \( \varphi(t) = 0 \) if \( |t| \leq 1 \), but \( \varphi(t) = \infty \) whenever \( |t| > 1 \). Then \( \ell_\varphi(I)/h_\varphi(S) \cong (C(\beta I \setminus I), \| \cdot \|_\infty) \) (order isomorphism and isometry). Consider in \( \ell_\varphi(I)/h_\varphi(S) \) the sequence \( \{A_n\}_{n \geq 1} \), which is increasing and bounded by \( 1_{\beta I \setminus I} \). Since \( \|1_{A_{n+1} \setminus A_n} \| = 1 \), we get that \( \{A_n\}_{n \geq 1} \) is not Cauchy, whence this sequence has no supremum. But, if \( \overline{A} \) were open, \( 1_{\overline{A}} \) should be the supremum of this sequence. Hence \( \overline{A} \) is not open and \( \beta I \setminus I \) is not basically disconnected. Recall that a compact Hausdorff space \( K \) is basically disconnected if the closure of every open \( F_\sigma \)-set (i.e. a countable union of closed sets) in \( K \) is open (see [9, pg.4]).

5. Rotundity and smoothness

Proposition 5.1. If \( I \) is an infinite set and \( \varphi \) is an Orlicz function such that \( \ell_\varphi(I) \neq h_\varphi(S) \), then there exists an order isomorphic isometric copy of \( C(\beta \mathbb{N}\setminus \mathbb{N}) \) in \( \ell_\varphi(I)/h_\varphi(S) \).

Proof. Pick \( x \in \ell_\varphi(I)^+ \) such that \( I_\varphi(x) \leq 1 \), \( Q(x) \in S_{\ell_\varphi(I)/h_\varphi(S)} \) and, if \( A := \text{supp}(x) \), then \( \text{card}(A) = \aleph_0 \). Let \( \{\lambda_n\}_{n \geq 1} \) be a sequence in \( \mathbb{R}^+ \) such that \( \lambda_n \downarrow 1 \).

Note that \( I_\varphi(\lambda_n(x-s)) = \infty \), \( \forall n \geq 1 \), \( \forall s \in S \). Choose a sequence \( \{A_n\}_{n \geq 1} \) of pairwise disjoint finite subsets of \( A \) such that \( A = \bigcup_{n \geq 1} A_n \) and \( I_\varphi(\lambda_n : x \cdot 1_{A_n}) > 1 \), \( n \geq 1 \). If \( a = (a_n)_{n \geq 1} \in \ell_\varphi(S) \), put \( a^k = (0, \ldots, 0, a_{k+1}, a_{k+2}, \ldots) \) and define \( T : \ell_\infty \to \ell_\varphi(I) \) by \( T(a) = \sum_{n \geq 1} a_n x \cdot 1_{A_n} \). Clearly, \( T \) is continuous and we have \( \frac{1}{\lambda_n} \|a^k\|_\infty \leq \|T a^k\|_L \leq \|a^k\|_\infty \). Observe that, if \( a = (a_1, a_2, \ldots, 0, 0, \ldots) \), then \( T(a) = h_\varphi(S) \), whence, by \( h_\varphi(S) \) being closed in \( \ell_\varphi(I) \), we get that \( T(c_0) \subseteq h_\varphi(S) \).

Hence, if \( q \) is the quotient map \( q : \ell_\infty \to \ell_\infty/c_0 \), we have the map \( i : \ell_\infty/c_0 \to \ell_\varphi(I)/h_\varphi(S) \) such that \( i(q(a)) = QT(a), \forall a \in \ell_\infty \). Clearly, this map preserves the order and satisfies \( \|q(a)\| = \lim_{k \to \infty} \|a^k\|_\infty = \lim_{k \to \infty} \|T a^k\|_L = \|QT(a)\| \).

Therefore \( i \) is an order isomorphic isometry between \( \ell_\infty/c_0 \) and \( i(\ell_\infty/c_0) \).

Corollary 5.2. Let \( I \) be an infinite set and \( \varphi \) an Orlicz function such that \( \ell_\varphi(I) \neq h_\varphi(S) \). Then:

1. \( \ell_\varphi(I)/h_\varphi(S) \) is not realcompact and cannot be renormed equivalently in order to be rotund or smooth.
2. \( \ell_\varphi(I)/h_\varphi(S) \) does not have property \( (C) \), it is not \( w\)-Lindelöf and \( (\ell_\varphi(I)/h_\varphi(S))^* = h_\varphi(S)^\perp \) is not \( w^\ast\)-angelic.

Proof: (1) This follows from the fact that \( C(\beta \mathbb{N}\setminus \mathbb{N}) \) is not realcompact (see [13, p. 146], [3]) and cannot be renormed in order to be rotund or smooth (see [2], [10]).

(2) This is a consequence of (1) (see [6]).
6. \( \ell_\varphi(I)/h_\varphi(S) \) is not a dual space

Let \( I \) be an infinite set, \( m = \text{card}(I) \) and \( P_\varphi(I) = \{ A \subseteq I : \text{card}(A) = \aleph_0 \} \). Then, clearly, \( \text{card}(P_\varphi(I)) = m^{\aleph_0} = n \). Note that \( n \geq c \), where \( c = \text{card} (\mathbb{R}) \).

Also there exists a family \( \{ A_t \}_{t \in \mathbb{N}} \) in \( P_\varphi(I) \) such that \( \text{card}(A_t \cap A_s) < \aleph_0 \), for \( t \neq s \). Indeed, let \( \{ I_t \}_{t \in \mathbb{N}} \) be a family of pairwise disjoint subsets of \( I \) such that \( \text{card}(I_t) = m, \forall t \in \mathbb{N} \). Pick \( i_t \in I_t, t \in \mathbb{N} \), and choose a pairwise disjoint family \( \{ I_{ts} \}_{t,s \in \mathbb{N}} \) of subsets of \( I_t \setminus \{ i_t \} \) such that \( \text{card}(I_{ts}) = m, s \in \mathbb{N} \). Pick \( i_{ts} \in I_{ts} \) and choose a pairwise disjoint family \( \{ I_{tsr} \}_{r \in \mathbb{N}} \) of subsets of \( I_{ts} \setminus \{ i_{ts} \} \) such that \( \text{card}(I_{tsr}) = m, r \in \mathbb{N} \). By reiteration we obtain families of elements \( \{ i_t \}_{t \in \mathbb{N}}, \{ i_{ts} \}_{t,s \in \mathbb{N}}, \) etc., of \( I \). Now, consider the family \( \mathcal{F} \) of sequences of the form \( (i_{t_1}, i_{t_1 t_2}, i_{t_1 t_2 t_3}, \ldots), t_j \in \mathbb{N}, j \geq 1 \). It is clear that \( \text{card}(\mathcal{F}) = m^{\aleph_0} = n, \) \( \text{card}(T) = \aleph_0, \forall T \in \mathcal{F} \), and that, if \( T,S \in \mathcal{F}, T \neq S \), then \( \text{card}(T \cap S) < \aleph_0 \).

**Lemma 6.1.** Let \( I \) be an infinite set and \( \varphi \) an Orlicz function such that \( \ell_\varphi(I) \neq h_\varphi(S) \). If \( n = m^{\aleph_0} \) and \( m = \text{card}(I) \), there exists an order isomorphic isometric copy of \( (c_0(n); \| \cdot \|_\infty) \) in \( \ell_\varphi(I)/h_\varphi(S) \).

**Proof.** Let \( \{ A_t \}_{t \in \mathbb{N}} \) be a family of subsets of \( I \) such that \( \text{card}(A_t) = \aleph_0 \) and \( \text{card}(A_t \cap A_s) < \aleph_0 \), when \( t \neq s \). Pick \( x \in \ell_\varphi(I)^+ \) such that \( \ell_\varphi(x) \leq 1, Q(x) \in S_{\ell_\varphi(I)/h_\varphi(S)} \) and \( \text{card}(\text{supp}(x)) = \aleph_0 \). Let \( \text{supp}(x) = \{ j_r \}_{r \geq 1} \). If \( t \in \mathbb{N} \) and \( A_t = \{ i_k \}_{k \geq 1} \), define \( e^t \) such that \( \forall i \in I, e^t_i = 0, \) if \( i \notin A_t \), and \( e^t_i = x_i, \) if \( i = i_r, r \geq 1 \). Then clearly, \( \forall t_1, t_2, \ldots, t_n \in \mathbb{N}, \forall a_1, \ldots, a_n \in \mathbb{R}, \) we have \( \| \sum_{k=1}^n a_k Q(e^{t_k}) \| = \sup \{ |a_k| : k = 1, \ldots, n \} \). i.e. \( \{ Q(e^t) \}_{t \in \mathbb{N}} \) is order isomorphically and isometrically equivalent to the unit basis of \( c_0(n) \).

**Proposition 6.2.** If \( I \) is an infinite set and \( \varphi \) an Orlicz function such that \( \ell_\varphi(I) \neq h_\varphi(S) \), then \( \ell_\varphi(I)/h_\varphi(S) \) is not a dual space.

**Proof.** If \( a(\varphi) > 0 \), we have by Proposition 1.2 that \( \ell_\varphi(I)/h_\varphi(S) \cong C(\beta I \setminus I) \). Grothendieck (see [8]) has shown that, for a compact Hausdorff space \( T \), \( T \) must be hyperstionian in order for \( C(T) \) to be a dual space (see [11, p. 95]). But \( \beta I \setminus I \) is not hyperstionian because it is not basically disconnected.

Assume that \( a(\varphi) = 0 \). Then \( \text{card}(\text{supp}(x)) \leq \aleph_0 \) for each \( x \in \ell_\varphi(I) \). Hence \( \text{card}(\ell_\varphi(I)) \leq n := m^{\aleph_0} \), with \( m = \text{card}(I) \). By Lemma 6.1, there exists a copy of \( c_0(n) \) in \( \ell_\varphi(I)/h_\varphi(S) \) and, by a classical Rosenthal’s result ([12, Cor. 1.2]), if \( \ell_\varphi(I)/h_\varphi(S) \) are a dual space, it should contain a copy of \( \ell_\varphi(\aleph_0) \). But this is a contradiction because \( \text{card}(\ell_\varphi(\aleph_0)) = 2^n > n \geq \text{card}(\ell_\varphi(I)/h_\varphi(S)) \).

7. \( \ell_\varphi(I)/h_\varphi(S) \) is a Grothendieck space

If \( I \) is an infinite set, denote by \( \mathcal{M}(I) \) the Banach lattice of finitely additive signed measures on \( I \) (see [14]). It is known that this space is order isomorphic and isometric to \( C(\beta I)^* \) (i.e. the space of Radon measures on \( \beta I \)). Let \( T \) be this isometry. Then:

1. If \( \nu \in \mathcal{M}(I) \) and \( T(\nu) = \mu \in C(\beta I)^* \), we have, \( \forall A \subseteq I, \nu(A) = \mu(\overline{A}) \), where \( \overline{A} \) is the closure of \( A \) in \( \beta I \).
2. \( T(\{ \nu \in \mathcal{M}(I) : \nu(\{ i \}) = 0, \forall i \in I \}) = C(\beta I \setminus I)^* \) (=Radon measures of \( C(\beta I)^* \) supported on \( \beta I \setminus I \)).

If \( a(\varphi) > 0 \), let \( M = \{ \nu \in \mathcal{M}(I) : \nu(\{ i \}) = 0, \forall i \in I \} = T^{-1}(C(\beta I \setminus I)^*) \). If \( a(\varphi) = 0 \), define \( M \subseteq \mathcal{M}(I) \) as the subspace such that \( \nu \in M \iff \nu(\{ i \}) = 0, \forall i \in I \), and there exists a sequence \( \{ G_k \}_{k \geq 1} \) of pairwise disjoint subsets of \( I \) satisfying:
\( (1) \ |\nu|(I \setminus \bigcup_{k \geq 1} G_k) = 0; \\
(2) \sum_{k \geq 1} \varphi(1/k) \cdot |G_k| < \infty, \text{ where } |G_k| = \text{card}(G_k); \\
(3) \sum_{k \geq 1} \varphi \left( \frac{1}{k} \left[ 1 + \frac{1}{n} \right] \right) \cdot |G_k \cap E| = \infty, \ \forall n \geq 1, \ \forall E \subseteq I \text{ such that } |\nu|(E) > 0. \)

**Proposition 7.1.** Let \( I \) be an infinite set and \( \varphi \) an Orlicz function such that \( \ell_{\varphi}(I) \neq h_{\varphi}(S) \). Then \( (\ell_{\varphi}(I)/h_{\varphi}(S))^* \) is order isomorphic and isometric to \( M \) and \( M \) is 1-complemented in \( C(\beta I)^* \).

**Proof.** The proof is essentially the one given by Ando [1]. Let \( X = \ell_{\varphi}(I)/h_{\varphi}(S) \) and pick \( x^* \in X^{*+} \). If \( E \subseteq I \), define \( x^*_E \) as \( x^*_E(Q(h)) = x^*(Q(h_E)), \forall h \in \ell_{\varphi}(I) \), with \( h_E = h \cdot 1_E \). Then \( x^*_E \in X^{*+} \) and for disjoint subsets \( E, F \) of \( I \) we have \( x^*_E \cdot x^*_F \in X^{*+} \) and \( \|x^*_E \cdot x^*_F\| = \|x^*_E\| + \|x^*_F\| \). So, we can define the measure \( \nu_{x^*} \in \mathfrak{M}(I)^+ \) as follows: \( \forall E \subseteq I, \quad \nu_{x^*}(E) = \|x^*_E\| \). Note that this map \( X^{*+} \ni x^* \to \nu_{x^*} \in \mathfrak{M}(I)^+ \) is linear, monotone (i.e. \( x^* \geq y^* \geq 0 \) implies \( \nu_{x^*} \geq \nu_{y^*} \)) and \( \|\nu_{x^*}\| = \|x^*\| \) (see Lemmas 2 and 3 of [1]).

We claim that \( \nu_{x^*} \in M^+ \). Clearly, \( \nu_{x^*}(\{i\}) = 0 \), \( \forall i \in I \), whence, if \( a(\varphi) > 0 \), we get \( \nu_{x^*} \in M^+ \). Assume that \( a(\varphi) > 0 \) and pick \( f \in \ell_{\varphi}(I)^+ \) such that \( I_\varphi(f) \leq 1 \) and \( \|x^*_E\| = x^*(Q(f_E)), \forall E \subseteq I \) (see Lemma 2 of [1]). Define \( G_1 = \{i \in I : |f_i| \geq 1\}, \ G_k = \{i \in I : \frac{1}{k} \leq |f_i| < \frac{1}{k-1}\}, \ k \geq 2, \) and observe that \( |G_k| < \infty, \ k \geq 1, \) because we suppose that \( a(\varphi) = 0 \). We have:

(a) \( \nu_{x^*}(I \setminus \bigcup_{k \geq 1} G_k) = \|x^*_E(Q I_\varphi(f_i))\| = x^*(Q(f_{E \cap I_\varphi(f_i)})) = x^*(0) = 0. \)

(b) \( \sum_{k \geq 1} \varphi\left(\frac{1}{k}\right) \cdot |G_k| \leq I(\varphi(f) < \infty. \)

(c) Let \( E \subseteq I \) be such that \( \nu_{x^*}(E) > 0 \). Then:

\[
0 < \nu_{x^*}(E) = \|x^*_E\| = x^*(Q(f_E)) = x^*_E(Q(f_E)) \leq \|Q(f_E)\| \cdot \|x^*_E\|, \]

whence we get \( 1 \leq \|Q(f_E)\| \), i.e., \( d(f_E; h_{\varphi}(S)) \geq 1 \). Hence, \( \forall \lambda > 1, \forall g \in h_{\varphi}(S), \) we have \( I_{\varphi}(\lambda(f_E - g)) = \infty. \) Pick \( n \in \mathbb{N} \) and choose \( k_0 \in \mathbb{N} \) such that \( \forall k > k_0 \), \( (1 + \frac{1}{n} \frac{1}{k}) \geq (1 + \frac{1}{n} \frac{1}{k_0}) \). Then, since \( f_{E \cap I_\varphi(f_i), G_1}, \) we have:

\[
\sum_{k \geq 1} \varphi\left(\frac{1}{k} \right) \cdot |G_k \cap E| \geq \sum_{k \geq k_0} \varphi\left(\frac{1}{k} \right) \cdot |G_k \cap E| \geq I_{\varphi}\left(\frac{1}{k} \right) \cdot | \bigcup_{k \geq k_0} G_k \cap E| = \infty, \]

and this completes the proof of the claim.

If \( \nu \in \mathfrak{M}(I)^+ \), define \( x^*_{\nu} : X^{*} \to \mathbb{R} \) as follows:

\[
\forall h \in \ell_{\varphi}(I)^+, \ x^*_{\nu}(Q(h)) = \inf \sum_{k=1}^{n} \delta(h_{E_k}) \cdot \nu(E_k),
\]

where the infimum is taken over all finite pairwise disjoint partitions \( \{E_k\}_{k=1}^{n} \) of \( I \).

By Lemmas 4, 5 and 6 of [1] and defining

\[
\forall h \in \ell_{\varphi}(I), \ x^*_{\nu}(Q(h)) = x^*_h(Q(h^+)) - x^*_h(Q(h^-)),
\]

we have that \( x^*_{\nu} \in X^{*+} \) and \( |x^*_{\nu}| \leq |\nu| = \nu(I) \). In addition, if \( \nu \in M^+ \) and \( x^* \in X^{*+} \) (see [1, Theorems 2 and 3]), then: (i) \( \|x^*_E\| = \nu(E), \forall E \subseteq I; \) (ii) \( x^*_{\nu_{x^*}} = x^*, \nu_{x^*} = \nu \). Hence the positive cones \( M^+ \) and \( X^{*+} \) are order isomorphic and isometric. If \( \nu \in \mathfrak{M}(I) \) and \( x^* \in X^{*} \), define \( \nu_{x^*} = \nu_{x^*} - \nu_{x^*}, \ x^*_{\nu} = x^*_{\nu} - x^*_{\nu} - \nu_{x^*}. \) With this extension we obtain an order isomorphism and isometry between \( X^{*} \) and \( M \). The projection \( P : \mathfrak{M}(I) \to M \) is defined as \( P(\nu) = \nu_{x^*}, \forall \nu \in \mathfrak{M}(I). \quad \square \)
Proposition 7.2. Let $I$ be an infinite set, $\varphi$ an Orlicz function such that $\ell_{\varphi}(I) \neq h_{\varphi}(S)$, $\{x_n^*\}_{n \geq 1}$ a sequence in $(\ell_{\varphi}(I)/h_{\varphi}(S))^*$ and $\epsilon > 0$. Then there exists $f \in \ell_{\varphi}(I)^+$ such that $I_{\varphi}(f) \leq \epsilon$ and:

1. $\nu_{x_n^*}(E) = x_n^*(Q(f_E))$, $\forall n \geq 1$, $\forall E \subseteq I$;
2. $\nu_{x_n^*}(g) = x_n^*(Q(gf))$, $\forall n \geq 1$, $\forall g \in \ell_{\infty}(I)$.

Proof. (A) If $x^* \in (\ell_{\varphi}(I)/h_{\varphi}(S))^*$, by Lemma 2 of [1], there exists $f \in \ell_{\varphi}(I)^+$ such that $I_{\varphi}(f) \leq \epsilon$ and $\nu_{x^*}(E) = x^+(Q(f_E))$, $\nu_{x^*^-}(E) = x^-(Q(f_E))$, $\forall E \subseteq I$.

Hence:

$\forall E \subseteq I$, $\nu_{x^*}(E) = \nu_{x^*+}(E) - \nu_{x^*^-}(E) = x^+(Q(f_E)) - x^-\nu(E) = x^*(Q(f_E))$.

So, considering $\nu_{x^*}$ as a member of $C(\beta I)^*$, we get that $\nu_{x^*}(g) = x^*(Q(gf))$, $\forall g \in \ell_{\infty}(I)$.

(B) For each $x_n^*$, take $f_n \in \ell_{\varphi}(I)^+$ satisfying (A) and such that $I_{\varphi}(f_n) \leq \epsilon/2^n$. Let $f = \sup_{n \geq 1} f_n$. Then we have $I_{\varphi}(f) \leq \epsilon$ (see Lemma 1 of [1]) and (1), (2) are fulfilled, $\forall n \geq 1$.

A Banach space is said to be a Grothendieck space (see [4]) if for each sequence $\{x_n^*\}_{n \geq 0}$ in $X^*$ such that $x_n^* \rightharpoonup x_0^*$ in the w*-topology, we have that $x_n^* \rightarrow x_0^*$ in the w-topology of $X^*$.

Proposition 7.3. Let $I$ be an infinite set and $\varphi$ an Orlicz function such that $\ell_{\varphi}(I) \neq h_{\varphi}(S)$. Then $\ell_{\varphi}(I)/h_{\varphi}(S)$ is a Grothendieck space.

Proof. Let $\{x_n^*\}_{n \geq 0}$ be a sequence in $(\ell_{\varphi}(I)/h_{\varphi}(S))^*$ such that $x_n^* \rightarrow x_0^*$ in the w*-topology. By Proposition 7.2 there exists $f \in \ell_{\varphi}(I)^+$ such that, $\forall g \in \ell_{\infty}(I)$, $\forall n \geq 0$, $\nu_{x_n^*}(g) = x_n^*(Q(gf))$. Since $Q(gf) \in \ell_{\varphi}(I)/h_{\varphi}(S)$, we have

$\lim_{n \rightarrow \infty} x_n^*(Q(gf)) = x_0^*(Q(gf))$.

Hence $\nu_{x_n^*} \rightarrow \nu_{x_0^*}$ in the w*-topology as members of $C(\beta I)^*$. Since $C(\beta I)$ is Grothendieck, we get $\nu_{x_n^*} \rightarrow \nu_{x_0^*}$ in the w-topology of $C(\beta I)^*$. Therefore $x_n^* \rightarrow x_0^*$ in the w-topology, because $(\ell_{\varphi}(I)/h_{\varphi}(S))^*$ is a subspace of $C(\beta I)^*$.

Remarks. Since $\ell_{\varphi}(I)/h_{\varphi}(S)$ has the Dunford-Pettis property ($M$-spaces have the Dunford-Pettis property because they are $L_1$-preduals) and is a Grothendieck space, we obtain that $\ell_{\varphi}(I)/h_{\varphi}(S)$ has no infinite dimensional complemented subspaces $Y$ with $Y$ - w*-sequentially compact. Also from Proposition 7.3 we get again that $\ell_{\varphi}(I)/h_{\varphi}(S)$ cannot be renormed in order to be smooth, because a Grothendieck smooth space is reflexive ([4, p. 215]) and $\ell_{\varphi}(I)/h_{\varphi}(S)$ is not, containing a copy of $C(\beta \mathbb{N} \setminus \mathbb{N})$.

Question. Is $\ell_{\varphi}(I)/h_{\varphi}(S)$ primary? Recall that Drewnowski and Roberts proved, under CH, that $\ell_{\infty}/c_0$ is primary (see [5]).

References

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