CLASSIFYING SPACES AND HOMOTOPY
SETS OF AXES OF PAIRINGS

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Dedicated to Professor Teiichi Kobayashi on his 60th birthday

Abstract. We consider the maps between classifying spaces of the form $BK \times BL \to BG$. The main theorem shows that if the restriction map on $BL$ is a weak epimorphism, then the restriction on $BK$ should factor through the classifying spaces of the center of the compact Lie group $G$. An application implies that $BG$ is an H–space (Hopf space) if and only if $G$ is abelian.

For a map $f : Y \to Z$, the set of the homotopy classes of axes, denoted by $f^\perp(X, Z)$, consists of all homotopy classes of maps $\alpha : X \to Z$ such that there is a map (called a pairing) $\mu : X \times Y \to Z$ with restrictions (axes) $\mu|_X \simeq \alpha$ and $\mu|_Y \simeq f$, [16]. If $Z$ is an H–space, then $f^\perp(X, Z) = [X, Z]$ : for the H–multiplication $m : Z \times Z \to Z$, a pairing of $f$ and $\alpha$ is given by the composite map $m \circ (\alpha \times f)$. It follows that, for example, if $G, K,$ and $L$ are compact Lie groups, then $f^\perp(K, G) = [K, G]$ with $f : L \to G$. In this paper we will study the classifying space version, namely $f^\perp(BK, BG)$ with $f : BL \to BG$. As group theoretical analog indicates, our results will show that few maps in $[BK, BG]$ belong to $f^\perp(BK, BG)$, in general. Other forms of axial maps (H–pairing) are studied in [6].

The main theorem deals with the case that the map $f : BL \to BG$ is a weak epimorphism studied in [8]. (We recall the definition in §2.) An obvious example of a weak epimorphism is given by a map $f = B\rho$ induced by a group epimorphism $\rho$. The unstable Adams operations $\{\psi^k\}$ are also weak epimorphisms.

Theorem 1. Let $L$ and $G$ be connected compact Lie groups and let $K$ be a compact (not necessarily connected) Lie group. If $f : BL \to BG$ is a weak epimorphism, the following hold:

1. If $\alpha \in f^\perp(BK, BG)$, then the map $\alpha$ factors through $BZ(G)$ up to homotopy, where $Z(G)$ denotes the center of $G$.

2. Moreover, we have $f^\perp(BK, BG) = \text{Hom}(K, Z(G))$.

Taking $L = G$ and $f = \text{id}$ (the identity map), our problem asks about possible $BK$–actions on $BG$. In fact, this work was motivated by the following result of G. Dula and D. Gottlieb.
Theorem 2 ([2]). Let \( \alpha : X \to Z \) be a map, and \( X \) an \( H \)-space. Then the following are equivalent.

(a) There exists a space \( Y \) and a homotopy equivalence \( X \times Y \to Z \) such that the orbit map \( X \to X \times Y \to Z \) is homotopy equivalent to \( \alpha \).
(b) The map \( \alpha \) is the orbit of an action of \( X \) on \( Z \) and \( \alpha^\#: [Z, X] \to [X, X] \) is onto.

For instance, let \( X = BS^1 \) and \( Z = BU(n) \). A map \( \alpha : BS^1 \to BU(n) \) is induced by a homomorphism \( \rho : S^1 \to U(n) \) so that \( \alpha = B\rho \). Since \( BU(n) \) is indecomposable, [7], Theorem 2 implies that the \( BS^1 \)-action on \( BU(n) \) under the map \( \alpha \) does not exist if \( \alpha \) is induced by the inclusion \( S^1 = U(1) \hookrightarrow U(n) \). Theorem 1 says that the \( BS^1 \)-action on \( BU(n) \) under the map \( \alpha \) exists if and only if \( \rho(S^1) \) is central in \( U(n) \). Generally, a consequence of Theorem 1 shows that \( BK \)-actions on \( BG \) under \( \alpha \) exist if and only if the map \( \alpha : BK \to BG \) is induced by a homomorphism \( \rho : K \to G \) such that \( \rho(K) \) is central in \( G \). In particular, if \( K \) is connected and \( G \) is semi-simple, there are no non-trivial \( BK \)-actions on \( BG \).

Furthermore, if we take \( K = L = G \) and \( f = \alpha = id \), the problem now asks whether \( BG \) is an \( H \)-space. Corollary 2.4 implies that \( BG \) is an \( H \)-space if and only if \( G \) is abelian. \( (G \) need not be connected.\) Related results were obtained in [12] and [1].

I would like to thank Giora Dula for bringing this problem to my attention. I would also like to thank my colleague, Nobuyuki Oda for his help.

1. Mapping spaces and centralizers of \( p \)-toral groups

We give a necessary and sufficient condition that a map \( \alpha : X \to Z \) be contained in \( f^\perp(X, Z) \) in terms of mapping spaces. Then a special case of classifying space version is discussed.

Proposition 1.1. Suppose \( X, Y \) and \( Z \) are pointed connected spaces. For a map \( f : Y \to Z \), a map \( \alpha : X \to Z \) is contained in \( f^\perp(X, Z) \) if and only if the map \( f \) factors through \( map(X, Z)_\alpha \), the connected component of the mapping space containing \( \alpha \), under the evaluation map \( map(X, Z)_\alpha \overset{ev}{\longrightarrow} Z \).

Remark. It is easy to see that \( \alpha \in f^\perp(X, Z) \) if and only if \( f \in \alpha^\perp(Y, Z) \). Consequently \( \alpha \) must factor through \( map(Y, Z)_f \).

Proof of Proposition 1.1. If \( \alpha \in f^\perp(X, Z) \), there is a pairing \( \mu : X \times Y \to Z \) with \( \mu|_X \simeq \alpha \) and \( \mu|_Y \simeq f \). Let \( \overline{\mu} : Y \to map(X, Z)_\alpha \) be the adjoint map of \( \mu \). Then we see that the map \( f \) is expressed as the composite of the adjoint map and the evaluation map:

\[
    f : Y \xrightarrow{\overline{\mu}} map(X, Z)_\alpha \xrightarrow{ev} Z
\]

since \( ev \circ \overline{\mu}(y) = \overline{\mu}(y)(\ast) = \mu(\ast, y) = f(y) \).

Conversely, suppose \( f \) factors through \( map(X, Z)_\alpha \) so that \( f \simeq ev \circ \overline{\mu} \) for some map \( \overline{\mu} : Y \to map(X, Z)_\alpha \). Let \( \epsilon : X \times map(X, Z)_\alpha \to Z \) be the canonical map with \( \epsilon(x, \beta) = \beta(x) \) for \( x \in X \) and \( \beta \in map(X, Z)_\alpha \). Then a pairing is constructed as the composite

\[
    \mu : X \times Y \xrightarrow{1_X \times \overline{\mu}} X \times map(X, Z)_\alpha \xrightarrow{\epsilon} Z,
\]
where 1\_\text{X} is the identity map of X, since \(\mu(x, y) = \epsilon \circ (1\_\text{X} \times \mathbb{P})(x, y) = \epsilon(x, \mathbb{P}(y)) = \mathbb{P}(y)(x)\) so that \(\mu(x, \ast) = \mathbb{P}(\ast)(x) = \alpha(x)\) and \(\mu(\ast, y) = \epsilon(\ast, \mathbb{P}(y)) = \epsilon \circ \mathbb{P}(y) \simeq f(y)\).

We next consider the “BG”–version. It is worth to recall some property of homomorphisms. Suppose \(\rho : L \to G\) and \(\alpha : K \to G\) are homomorphisms. If there is a pairing homomorphism \(\mu : K \times L \to G\) with \(\mu|_K = \alpha\) and \(\mu|_L = \rho\), then the image \(\rho(L)\) must be contained in the centralizer of \(\alpha\) in \(G\), denoted by \(C_G(\alpha)\). The following is a “BG”–analog at a prime \(p\). If a map \(\alpha : BK \to BG\) is induced by a homomorphism, let \(C_G(\alpha)\) denote the centralizer of the homomorphism. For a p–toral group \(K\) (a group extension of a torus by a finite \(p\)-group), it is known, [5] and [14], that any map \(\alpha : BK \to BG\) (at \(p\)) has the form \(\alpha = B\eta\) \((\alpha = (B\eta)_p)\) for some homomorphism \(\eta\). Let \(BG^\wedge_p\) denote the \(p\)-completion of \(BG\). Since \(map(BK, BG)_\alpha\) is \(p\)-equivalent to \(BC_G(\alpha)\), the following is immediate from Proposition 1.1.

**Corollary 1.2.** Suppose \(K\) is \(p\)-toral. Then \(\alpha \in f^\wedge(BK^\wedge_p, BG^\wedge_p)\) if and only if the map \(f\) factors through \(BC_G(\alpha)^\wedge_p\) up to homotopy, under the map \(BC_G(\alpha)^\wedge_p \to BG^\wedge_p\) induced by the inclusion.

As the above indicates, if the mapping space is computable, then the set of the homotopy classes of axes \(f^\wedge(X, Z)\) would be determined. It is, however, hard to compute \(map(X, Z)_\alpha\) or \(map(Y, Z)_f\) in general. Thus our work is to consider the problem, with or without calculation of mapping spaces.

### 2. Weak epimorphisms and the proof of Theorem 1

We will begin with some lemmas to prove Theorem 1.

**Lemma 2.1.** Let \(R\) and \(S\) be rings of polynomials in \(n\) indeterminates over a field. If there is an epimorphism \(\varphi : R \to S\), then \(\varphi\) is an isomorphism.

**Proof.** Let \(p = Ker \varphi\) so that \(R/p \cong S\). We need to show that \(p = 0\). The ideal \(p\) is prime, since \(R/p\) is an integral domain. Consequently, if \(dim(R)\) denotes the Krull dimension of \(R\), then

\[
height(p) + dim(R/p) = dim(R).
\]

It follows that

\[
dim(S) = dim(R/p) = dim(R) - height(p).
\]

Since \(dim(S) = dim(R) = n\), we see that \(height(p) = 0\). In a domain, this means \(p = 0\). \(\square\)

Recall that, for any compact connected Lie group \(G\), there is a covering \(\gamma_G \to \tilde{G} \to G\) such that \(\tilde{G} = G_s \times T\) is a product of a simply–connected Lie group and a torus, and that \(\gamma_G\) is a finite central subgroup of \(\tilde{G}\). Such a covering is called a universal finite covering.

**Lemma 2.2.** Suppose \(G\) is a connected compact Lie group and \(H\) is a connected closed subgroup with inclusion \(\iota : H \to G\). If the \(p\)-completed map \((BH)^\wedge_p : (BH)^\wedge_p \to (BG)^\wedge_p\) is rationally equivalent, then \(H = G\).
Proof. Since $H$ and $G$ are connected, we can find universal finite coverings $\tilde{H}$ and $\tilde{G}$ so that we have the commutative diagram

$$
\begin{array}{ccc}
\tilde{H} & \xrightarrow{i} & \tilde{G} \\
\downarrow & & \downarrow \\
H & \xrightarrow{i} & G
\end{array}
$$

Note that $\tilde{H}$ and $\tilde{G}$ are products of simply-connected simple Lie groups and a torus, $\tilde{H} = \prod_i \tilde{H}_i$ and $\tilde{G} = \prod_j \tilde{G}_j$ respectively. Since the map $\left(\tilde{B}\iota\right)^\wedge_p : \left(\tilde{B}H\right)^\wedge_p \to \left(\tilde{B}G\right)^\wedge_p$ is rationally equivalent, for each $\tilde{G}_j$, by [8], we can find $\tilde{H}_i$ such that the restricted map $\left(\tilde{B}\tilde{H}_i\right)^\wedge_p \to \left(\tilde{B}\tilde{G}_j\right)^\wedge_p$ is a rational equivalence, and the map $\left(\tilde{B}\iota\right)^\wedge_p$ is diagonal.

Using the classification of compact simply-connected simple Lie groups and their maximal rank subgroups, one can show that $i(\tilde{H}_i) = \tilde{G}_j$. Hence $i : \tilde{H} \to \tilde{G}$ is onto. Consequently the inclusion $i : H \to G$ is an epimorphism, and therefore $H = G$. \hfill $\Box$

Lemma 2.3. Let $G$ be a compact Lie group with center $Z(G)$. Then the map induced by the inclusion $i : Z(G) \to G$

$$(Bi)_\# : [BK, BZ(G)] \to [BK, BG]$$

is one–to–one for any compact Lie group $K$.

Proof. Notice that $[BK, BZ(G)] = Hom(K, Z(G))$, since $Z(G)$ is abelian, [11]. For $\rho_1, \rho_2 \in Hom(K, Z(G))$, suppose $(Bi)_\#(B\rho_1) = (Bi)_\#(B\rho_2)$. Thus, for each $p$–toral subgroup $K_q$ of $K$, we have $\rho_1(x) = \rho_2(x)$ for any $x \in K_q$, since the images of $\rho_1$ and $\rho_2$ are contained in the center $Z(G)$. Define $\rho \in Hom(K, Z(G))$ by $\rho(k) = \rho_1(k)\rho_2(k)^{-1}$ for any $k \in K$ so that $\rho|_{K_q} = 0$ for any $p$–toral subgroup $K_q$. Since $[BK_p, BG_p] = [\text{holim} (BK_q)^\wedge_p, BG_p] = \text{holim} [(BK_q)^\wedge_p, BG_p]$, it follows that $\rho = 0$. Consequently $\rho_1 = \rho_2$, and therefore $B\rho_1 = B\rho_2$. \hfill $\Box$

Remark. For $i : H \to G$ the map $[BK, BH] \to [BK, BG]$ need not be one–to–one. A counterexample is given by $[B\mathbb{Z}/2, BT^2] \not\to [B\mathbb{Z}/2, BU(2)]$ where $T^2$ is a maximal torus of $U(2)$.

We consider $f^\wedge(BK, BG)$ when $f : BL \to BG$ is a weak epimorphism, defined as follows. Suppose $L$ and $G$ are connected. A map $BL \to BG$ or $BL^\wedge_p \to BG^\wedge_p$ is called a weak epimorphism, [8], if there exists a fibration $Z \to BL \to BG$ or $Z \to BL^\wedge_p \to BG^\wedge_p$ such that $H^*(\Omega Z ; \mathbb{Q})$ is a finite dimensional $\mathbb{Q}$–module or that $H^*(\Omega Z ; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$ is a finite dimensional $\mathbb{Q}_p^\wedge$–module, respectively. Some examples were given in the introduction.

Proof of Theorem 1. First we show Part (1). Fix a prime $p$. Suppose $K_q$ is a $p$–toral subgroup of $K$ and $\alpha_q = \alpha^\wedge_p |_{(BK_q)^\wedge_p}$. Corollary 1.2 shows that the $p$–completed map $f^\wedge_p : BL^\wedge_p \to BG^\wedge_p$ factors through $BG_G(\alpha_q)^\wedge_p$ up to homotopy. Since $K_q$ is $p$–toral, $\pi_0(G(\alpha_q))$ is a finite $p$–group. Since $BL^\wedge_p$ is 1–connected, the map $f^\wedge_p$ factors through the $p$–completed classifying space of the connected component of the centralizer $C_G(\alpha_q)$. Since $f$ is a weak epimorphism, the map $f^\wedge_p$ is also a weak epimorphism. Consequently there is a fibration $Z \to BL^\wedge_p \to BG^\wedge_p$ satisfying
certain finiteness conditions mentioned above. Let $H$ be the connected component of $C_G(\alpha_q)$ with inclusion $\iota : H \to G$. Then we have the commutative diagram

$$
\begin{array}{ccc}
Y & \longrightarrow & BL_p^\wedge \\
\downarrow & & \downarrow \\
Z & \longrightarrow & BG_p^\wedge
\end{array}
$$

Here $Y$ is a homotopy fibre. Since $f_p^\wedge$ is a weak epimorphism, we see that $H^*(BL_p^\wedge; \mathbb{Q}) \cong H^*(BG_p^\wedge; \mathbb{Q}) \otimes H^*(Z; \mathbb{Q})$. Thus, for the induced homomorphism $(f_p^\wedge)^* : H^*(BG_p^\wedge; \mathbb{Q}) \to H^*(BL_p^\wedge; \mathbb{Q})$ there is a homomorphism (left inverse) $r : H^*(BL_p^\wedge; \mathbb{Q}) \to H^*(BG_p^\wedge; \mathbb{Q})$ with $r \circ (f_p^\wedge)^* = 1$. Consequently $r \circ (f_p^\wedge)^* \circ ((Bi_p\wedge)^* = 1$. Let $\varphi$ be the endomorphism $((Bi_p\wedge)^* \circ r \circ (f_p^\wedge)^*$ of the polynomial ring $H^*(BH_p^\wedge; \mathbb{Q})$. Since $r \circ (f_p^\wedge)^*$ is onto and $(Bi_p\wedge)^*$ is one-to-one, the Krull dimension of the image of $\varphi$ is equal to $\text{rank}(G)$. Hence the endomorphism $\varphi$ is an isomorphism and therefore $(Bi_p\wedge)^*$ is onto. Lemma 2.1 implies that $BH_p^\wedge$ is rationally equivalent to $BG_p^\wedge$. From Lemma 2.2, we see that $C_G(\alpha_q) = G$. This implies that the map $\alpha_q$ is induced by a homomorphism into the center $Z(G)$. Hence $\alpha_q(BK_q)^\wedge$ factors through $BZ(G)^\wedge_p$ for any $p$-toral subgroup $K_q$.

Since $BK_p^\wedge \simeq \text{holim}(BK_q)^\wedge_p$, [9], the map $\alpha_p^\wedge : BK_p^\wedge \to BG_p^\wedge$ factors through $BZ(G)^\wedge_p$ for any $p$. Let $K_0$ denote the connected component of $K$ containing the identity so that there is an exact sequence $K_0 \to K \xrightarrow{\pi} \pi_0 K$. Consider the canonical fibration $BZ(G) \to BG \xrightarrow{Bq} B(G/Z(G))$. Since $(Bq \cdot \alpha|_{BK_0})^\wedge$ is null homotopic, the map $Bq \cdot \alpha|_{BK_0}$ is rationally null homotopic. Since $K_0$ is connected, then $Bq \cdot \alpha|_{BK_0} = 0$. This induces a map $\pi : B\pi_0 K \to B(G/Z(G))$ with $Bq \cdot \alpha = \pi \cdot B\pi$. Since $(Bq \cdot \alpha|_{BK_0})^\wedge = 0$ for any $p$ and the homomorphism $\pi$ is onto, the restriction of $\pi$ on each $p$-Sylow subgroup of the finite group $\pi_0 K$ is null homotopic. This implies $\pi = 0$. Consequently $\alpha$ factors through $BZ(G)$.

Next we show Part (2). The canonical map $\{Bi\}_\#: [BK, BZ(G)] \to [BK, BG]$ factors through $f^\wedge(BK, BG)$. Let $\Phi$ be the map $[BK, BZ(G)] \to f^\wedge(BK, BG)$ and let $\Psi$ be the map $f^\wedge(BK, BG) \to [BK, BG]$ so that $\{Bi\}_\# = \Psi \circ \Phi$. Part (1) shows $\Phi$ is onto. Lemma 2.3 shows $\{Bi\}_\#$ is one-to-one. This implies $\Phi$ is one-to-one. Thus $\Phi$ is bijective. Since $[BK, BZ(G)] = Hom(K, Z(G))$, this completes the proof.

Remark. As shown in §1, there is a strong relationship between the set of the homotopy classes of axes $f^\wedge(BK, BG)$ and $map(BL, BG)_f$. Theorem 1 seems to indicate that $map(BL, BG)_f$ can be homotopy equivalent to $BZ(G)$ when $f : BL \to BG$ is a weak epimorphism. One can show, however, that $map(BS^3, BS^3)_{v^k} \not\simeq BZ(S^3)$ (without $p$-completion), from the following result of Dwyer–Milsen [3]: The space of pointed maps $map(BS^3, BS^3)_{v^k}$ is homotopy equivalent to Sullivan’s profinite completion $SO(3)^\wedge$. What to ask is the $p$-equivalence between $map(BL, BG)_f$ and $BZ(G)$. A result of Jackowski–McClure–Oliver [9] and Notbohm [15] shows that if $f : BG \to BG$ is a self-equivalence, the map $BZ(G) \to map(BG, BG)_f$ is a mod $p$ equivalence for any prime $p$. A related result can be found in [4].

Corollary 2.4. Suppose $G$ is a compact (not necessarily connected) Lie group. If $BG$ is an $H$-space, then $G$ is abelian.
Proof. If $BG$ is an H–space, then $(id)^\perp (BG, BG) = \{BG, BG\}$. Suppose first that $G$ is connected. Taking $\alpha = id$ in Theorem 1, we see that the identity map of $BG$ factors through $BZ(G)$. Consequently $G = Z(G)$, and therefore $G$ is a torus. In general, there is the exact sequence $G_0 \to G \to \pi_0 G$. If $BG$ is an H–space, from the fibration $BG_0 \to BG \to B\pi_0 G$ it follows that both $BG_0$ and $B\pi_0 G$ are also H–spaces. This implies that the groups $G_0$ and $\pi_0 G$ are abelian, since $\pi_0 G$ is a finite group so that $B\pi_0 G = K(\pi_0 G, 1)$. Let $\pi = \pi_0 G$ and let $\pi_p$ denote a $p$–Sylow subgroup of the finite group $\pi$. Suppose $G_p$ is the subgroup of $G$ over $\pi_p$:

$$
\begin{array}{ccc}
G_0 & \longrightarrow & G \\
\downarrow & & \downarrow \\
G_0 & \longrightarrow & G_p \\
\end{array}
$$

Since $BG$ is an H–space, the space $BG_p$ is also an H–space. Notice that $G_p$ is $p$–toral. Consequently the H–multiplication on $BG_p$ is induced by a homomorphism $G_p \times G_p \to G_p$. This implies that $G_p$ is abelian for any prime $p$, and therefore $G(= G_0 \times \pi_0 G)$ is abelian.

Remark. The $p$–completed version of Corollary 2.4 does not hold. For instance, the symmetric group $\Sigma_3$ is non-abelian, but the 2–completed classifying space $(B\Sigma_3)^\wedge_2 = B\mathbb{Z}/2$ is an H–space.

References


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