NOTE ON THE BRADLEY AND RAMANUJAN SUMMATION

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Abstract. The hypergeometric series of Bradley and Ramanujan is evaluated by means of the binomial convolutions of Hagen and Rothe, which presents, alternatively, a short proof of the recent result of Bradley about Ramanujan’s enigmatic claim.

For complex numbers $\alpha$, $\beta$, $\gamma$ and integer $\delta$, define the sum of Ramanujan type by

\[
S_\delta(\alpha, \beta, \gamma; z) = \gamma \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} \frac{\Gamma(\beta + zk) \Gamma(k + \gamma + zk)}{\Gamma(\delta + k + \alpha + \beta + zk) \Gamma(1 + \gamma + zk)}.
\]

It reduces, under parameter replacements $\delta \to 0$, $\beta \to 1 + \beta$ and $\gamma \to m$, to the sum of Bradley [2], who has recently presented a most plausible interpretation for Ramanujan’s enigmatic claim, which may be restated in terms of $S$-sum as “the difference between $\Gamma(1 + \beta - m)/\Gamma(1 + \alpha + \beta - m)$ and $S_0(\alpha, 1 + \beta, m; z) \cdots$” (cf. Bradley [2]).

Theorem. With the $S$-function defined as above, we have the following evaluations:

A: Bradley [2, 1994]. For $\text{Re} (\delta + \beta - \gamma) > 0$,

\[
S_\delta(\alpha, \beta, \gamma; 0) = \frac{\Gamma(\beta) \Gamma(\gamma) \Gamma(\delta + \alpha + \beta - \gamma)}{\Gamma(\delta + \beta) \Gamma(\delta + \alpha + \beta - \gamma)}.
\]

B. For $\text{Re} (1 - \alpha - \beta + \gamma) > 0$,

\[
S_\delta(\alpha, \beta, \gamma; -1) = \frac{\Gamma(\beta) \Gamma(1 - \beta) \Gamma(1 - \alpha + \beta - \gamma)}{\Gamma(\delta + \alpha + \beta) \Gamma(1 - \alpha - \beta) \Gamma(1 - \beta + \gamma)}.
\]

C: Bradley [2, 1994]. When $\alpha$ is a non-positive integer,

\[
S_0(\alpha, \beta, \gamma; z) = \frac{\Gamma(\beta - \gamma)}{\Gamma(\alpha + \beta - \gamma)}.
\]

D. When $\alpha$ is a non-positive integer,

\[
S_1(\alpha, \beta, \gamma; z) = \frac{\alpha z - \beta + \gamma}{\alpha z - \beta} \frac{\Gamma(\beta - \gamma)}{\Gamma(1 + \alpha + \beta - \gamma)}.
\]
Proof. For $z = 0$ and $-1$, we can rewrite

$$S_{\delta}(\alpha, \beta, \gamma; 0) = \frac{\Gamma(\beta)}{\Gamma(\delta + \alpha + \beta)} \times 2F_1 \left[ \begin{array}{c} \alpha, \gamma \\ \delta + \alpha + \beta \end{array} \right],$$

$$S_{\delta}(\alpha, \beta, \gamma; -1) = \frac{\Gamma(\beta)}{\Gamma(\delta + \alpha + \beta)} \times 2F_1 \left[ \begin{array}{c} \alpha, -\gamma \\ 1 - \beta \end{array} \right],$$

which yield (2a) and (2b), respectively, in view of the Gauss theorem [1] (see also [3])

$$2F_1 \left[ \begin{array}{c} a, b \\ c \end{array} \right] = \frac{\Gamma(c - a) \Gamma(c - b)}{\Gamma(c) \Gamma(c - a - b)}, \quad \text{Re}(c - a - b) > 0.$$  

When $\alpha = -n$, a non-positive integer, the $S$-function defined in (1) may be reformulated as

$$S_{\delta}(-n, \beta, \gamma; z) = \gamma \sum_{k=0}^{n} (-1)^{\delta+n} \binom{n}{k} \frac{(\gamma + z k)_{k}}{\gamma + z k} (1 - \beta - z k)_{n - k - \delta}$$

$$= \sum_{k=0}^{n} \frac{\gamma}{\gamma + z k} \binom{-\gamma - z k}{k} \frac{n!}{(\beta + z k)_{\delta}} \binom{\delta - 1 + \beta + z k}{n - k},$$

which reduce, respectively for $\delta = 0$ and $1$, to

$$S_0(-n, \beta, \gamma; z) = n! \binom{\beta - \gamma - 1}{n},$$

$$S_1(-n, \beta, \gamma; z) = \frac{n!}{\beta + zn} \binom{\beta - \gamma + zn}{\beta - \gamma} \binom{\beta - \gamma}{n},$$

by means of the Hagen-Rothe [5] (see also [3, 4]) formulae

$$\sum_{k=0}^{n} \frac{a}{a + bk} \binom{a + bk}{k} \frac{c - bk}{n - k} = \binom{a + c}{n},$$

$$\sum_{k=0}^{n} \frac{a}{a + bk} \binom{a + bk}{k} \frac{c - bn}{c - bk} \binom{c - bk}{n - k} = \frac{a + c - bn}{a + c} \binom{a + c}{n}.$$  

It is obvious that (2c) and (2d) are respectively the reformulations of (4a) and (4b).

\[\square\]

References