ALMOST EVERYWHERE CONVERGENCE OF LACUNARY PARTIAL SUMS OF VILENKIN-FOURIER SERIES

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Abstract. We prove that if $f \in L^p$, $p > 1$, and $\{n_k\}$ is any lacunary sequence of positive integers, then the sequence of $n_k$th partial sums of Vilenkin-Fourier series of $f$ converges almost everywhere to $f$.

1. Introduction

Let $G = \prod_{i=0}^\infty Z_{p_i}$ be the direct product of cyclic groups of order $p_i$, where $\{p_i\}_{i \geq 0}$ is a sequence of integers with $p_i \geq 2$. Let $\mu$ be the Haar measure on $G$ normalized by $\mu(G) = 1$. For $x = \{x_k\} \in G$, let $\phi_k(x) = \exp(2\pi i x_k/p_k)$, $k = 0, 1, 2, \ldots$. The Vilenkin system $\{\chi_n\}$ is the set of all finite products of $\{\phi_k\}$, and is enumerated in the following manner. Let $m_0 = 1$, $m_k = \prod_{i=0}^{k-1} p_i$, $k = 1, 2, \ldots$. We express each nonnegative integer $n$ as a finite sum in the form $n = \sum_{k=0}^\infty \alpha_k m_k$, where $0 \leq \alpha_k < p_k$, and define $\chi_n = \prod_{k=0}^\infty \phi_k^{\alpha_k}$. The functions $\{\chi_n\}$ are the characters of $G$, and they form a complete orthonormal system on $G$. If $p_i = 2$, $i = 0, 1, 2, \ldots$, then $\{\phi_k\}$ are the Rademacher functions and $\{\chi_n\}$ are the Walsh functions. In this paper there is no restriction on the orders $\{p_i\}$.

We consider Fourier series with respect to $\{\chi_n\}$. For $f \in L^1$, let $\hat{f}(j) = \int_G f(t)\chi_j(t) d\mu(t)$, $j = 0, 1, 2, \ldots$, and $S_n f = \sum_{j=0}^{n-1} \hat{f}(j)\chi_j$, $n = 1, 2, \ldots$. Not much is known about the almost everywhere convergence of partial sums when there is no restriction on the orders $\{p_i\}$. We have the following result.

Theorem. Let $1 < p < \infty$ and $\{n_k\}_{k \geq 1}$ be a lacunary sequence of positive integers, i.e., there is $q > 1$ such that $n_{k+1}/n_k \geq q$, $k = 1, 2, \ldots$. Then there is a constant $C_p$ such that

$$\|\sup_k |S_{n_k} f|\|_p \leq C_p \|f\|_p, \quad f \in L^p.$$ (1.1)

It follows that $\lim_{k \to \infty} S_{n_k} f(x) = f(x)$ a.e. for all $f \in L^p$.
In the proof that follows, $C$ and $C_p$ will denote absolute constants which may vary from line to line.

2. Proof of the Theorem

We first prove (1.1). For $k = 0, 1, \ldots$, let $L_k$ be the positive integer such that $2^{L_k} \leq p_k < 2^{L_k+1}$. Since every lacunary sequence can be decomposed into a finite number of lacunary subsequences with ratio $q = 2$, we may assume, by adding more terms to the sequence if necessary, that \( \{n_j\} \) can be relabelled \( \{n_k, \ell\} = 0, 1, \ldots, \ell = 0, 1, \ldots, L_k \), such that for $k = 0, 1, \ldots, 2^\ell m_k \leq n_k, \ell < 2^\ell+1 m_k$ if $\ell = 0, 1, \ldots, L_k - 1$, and $2^\ell m_k \leq n_k, L_k < m_k + 1$. (There is no $n_k, L_k$ term if $2^L_k m_k = m_k + 1$.) Also, it is sufficient to show that there is a constant $C_p$ such that

\[
(2.1) \quad \sup_{k=0, \ldots, N-1} |S_{n_k, \ell}f| \leq C_p \|f\|_p
\]

for all $f \in L^p$, $N = 1, 2, \ldots$.

Let $f_k = S_{m_{k+1}}f - S_{m_k}f$, $k = 0, 1, \ldots$, and $f_{-1} = S_1f$. We observe that

\[
\sup_{k=0, \ldots, N-1} |S_{n_k, \ell}f| \leq \sup_{\ell=0, \ldots, L_k} |S_{n_k, \ell}f_k| + \sup_{k=0, \ldots, N-1} |S_{m_k}f|.
\]

Since \( \{S_{n_k}f\} \) is a martingale (see, e.g., [5]), it follows from Doob’s inequality (\( \|\sup_{k \geq 0} |S_{n_k}f| \|_p \leq C_p \|f\|_p \), $f \in L^p$) that (2.1) will be proved if we have

\[
(2.2) \quad \sup_{k=0, \ldots, N-1} |S_{n_k, \ell}f_k| \leq C_p \|f\|_p,
\]

for all $f \in L^p$, $N = 1, 2, \ldots$.

Now,

\[
\sup_{k=0, \ldots, N-1} |S_{n_k, \ell}f_k| \leq \left( \sum_{k=0}^{N-1} |S_{n_k, \ell}f_k|^2 \right)^{1/2} + \sup_{\ell=0, \ldots, L_k-2} |S_{n_k, \ell}f_k|
\]

\[
+ \left( \sum_{k=0}^{N-1} |S_{n_k, L_k-1}f_k|^2 \right)^{1/2} + \left( \sum_{k=0}^{N-1} |S_{n_k, L_k}f_k|^2 \right)^{1/2}.
\]

For each of $\ell = 0, L_k - 1$ and $L_k$, we apply [5, Theorem 2] and Burkholder’s result for martingales [1, Theorem 3.2] to get

\[
\left\| \left( \sum_{k=0}^{N-1} |S_{n_k, \ell}f_k|^2 \right)^{1/2} \right\|_p \leq C_p \left\| \left( \sum_{k=0}^{N-1} |f_k|^2 \right)^{1/2} \right\|_p
\]

\[
\leq C_p \|f\|_p.
\]

To prove (2.2) it remains to show

\[
(2.3) \quad \sup_{\ell=0, \ldots, L_k-2} |S_{n_k, \ell}f_k| \leq C_p \|f\|_p
\]

for all $f \in L^p$, $N = 1, 2, \ldots$. 
We shall use the following operators. Let $k = 0, 1, \ldots$. If $L_k > 2$, define, for $\ell = 1, \ldots, L_k - 2$, the sequence $\{a_{k,\ell}(n)\}_{n \geq 0}$ by
\[
a_{k,\ell}(n) = \begin{cases} 
1 & \text{if } 2^\ell m_k \leq n < 2^{\ell+1} m_k, \\
\frac{j}{2^\ell - 1} & \text{if } (2^{\ell-1} + j)m_k \leq n < (2^{\ell-1} + j + 1)m_k, \\
1 - \frac{j + 1}{2^\ell - 1} & \text{if } (2^{\ell+1} + j)m_k \leq n < (2^{\ell+1} + j + 1)m_k, \\
0 & \text{otherwise},
\end{cases}
\]
and set
\[
A_{k,\ell}f = \sum_{n=0}^{\infty} a_{k,\ell}(n) \hat{f}(n) \chi_n.
\]
It is proved in [6, pp. 665-666] that
\[
\left\| \left( \sum_{k=0}^{N-1} \sum_{\ell=1}^{L_k - 2} |A_{k,\ell}f|^2 \right)^{1/2} \right\|_p \leq C_p \|f\|_p, \quad f \in L^p, \quad N = 1, 2, \ldots.
\]
(We interpret a sum $\sum_{j=k}^{\ell}$ with $\ell < k$ as zero.)

The second operator is defined as follows. Let $k = 0, 1, \ldots$. If $L_k \geq 2$, define, for $\ell = 1, \ldots, L_k - 1$, the sequence $\{b_{k,\ell}(n)\}_{n \geq 0}$ by
\[
b_{k,\ell}(n) = \begin{cases} 
1 & \text{if } m_k \leq n < 2^{\ell-1} m_k \text{ or } m_{k+1} - (2^{\ell-1} - 1)m_k \leq n < m_{k+1}, \\
1 - \frac{j}{2^\ell - 1} & \text{if } (2^{\ell-1} + j)m_k \leq n < (2^{\ell-1} + j + 1)m_k \text{ or } \\
m_{k+1} - (2^{\ell-1} + j)m_k \leq n < m_{k+1} - (2^{\ell-1} + j - 1)m_k, \\
0 & \text{otherwise}.
\end{cases}
\]
For $f \in L^1$, set
\[
B_{k,\ell}f = \sum_{n=0}^{\infty} b_{k,\ell}(n) \hat{f}(n) \chi_n.
\]
To estimate $B_{k,\ell}f$, we need the following definitions. Let $\{G_k\}$ be the sequence of subgroups of $G$ defined by
\[
G_0 = G, \quad G_k = \prod_{i=0}^{k-1} \{0\} \times \prod_{i=k}^{\infty} \mathbb{Z}_{p_i}, \quad k = 1, 2, \ldots.
\]
We shall identify $G$ with the unit interval $(0, 1)$ by associating with each $\{x_i\} \in G$, $0 \leq x_i < p_i$, the point $\sum_{i=0}^{\infty} x_i m_i^{-1} \in (0, 1)$. If we disregard the countable set of $p_i$-rationals, this mapping is one-to-one, onto and measure preserving. On the
interval \((0, 1)\), cosets of \(G_k\) are intervals of the form \((jm_k^{-1}, (j + 1)m_k^{-1})\), \(j = 0, 1, \ldots, m_k - 1\). A set \(I\) is called a generalized interval if \(I \subset x + G_k\) for some \(x \in G\), \(k = 0, 1, \ldots, I\) is a union of cosets of \(G_{k+1}\), and \(I\) is an interval if we consider \(x + G_k\) as a circle.

Let

\[
Mf(x) = \sup_{I \text{ generalized interval}} \frac{1}{\mu(I)} \int_I |f| \, d\mu
\]

be the Hardy-Littlewood maximal function for the Vilenkin system. The following pointwise estimate will be obtained in \(\S 3\).

**Lemma.** There is a constant \(C\) such that

\[
\sup_{\ell = 1, \ldots, L_k - 1} \sup_{k=0,1,\ldots} |B_{k,\ell}f(x)| \leq CMf(x), \quad f \in L^1, \ x \in G.
\]

To estimate \(S_{n_k,\ell}f_k\), \(\ell = 1, \ldots, L_k - 2\), \(k = 0, 1, \ldots\), we shall use the part of \(b_{k,\ell}(n)\) with \(n \in [m_k, 2^\ell m_k)\). In order to get rid of the remaining part of \(b_{k,\ell}(n)\), we define

\[
H^N f = \sum_{k=0}^{N-1} \left( S_{2^k,1}f - S_{m_k}f \right).
\]

Since \(H^N f = S_{m,N}f - \sum_{k=0}^{N-1} \left( S_{m_{k+1}}f - S_{2^k - 1, m_k}f \right)\), and it is proved in [4, Theorem 1*] that

\[
\left\| \sum_{k=0}^{N-1} \left( S_{m_{k+1}}f - S_{2^k - 1, m_k}f \right) \right\|_p \leq C_p \|f\|_p
\]

for all \(f \in L^p\), \(N = 1, 2, \ldots\), we have

\[
(2.5) \quad \|H^N f\|_p \leq C_p \|f\|_p, \quad f \in L^p, \ N = 1, 2, \ldots.
\]

(See also [2].)

We are now ready to prove (2.3). We observe that for \(k = 0, \ldots, N - 1, \ell = 1, \ldots, L_k - 2\),

\[
S_{n_k,\ell}f_k = B_{k,\ell}(H^N f) + S_{n_k,\ell}(A_{k,\ell}f).
\]

Hence

\[
(2.6) \quad \sup_{\ell = 1, \ldots, L_k - 2} \sup_{k=0,1,\ldots} |S_{n_k,\ell}f_k| \leq \sup_{\ell = 1, \ldots, L_k - 2} B_{k,\ell}(H^N f) + \left( \sum_{k=0}^{N-1} \sum_{\ell=1}^{L_k-2} |S_{n_k,\ell}(A_{k,\ell}f)|^2 \right)^{1/2}.
\]

To estimate the first term on the right, we apply the lemma, the fact that the Hardy-Littlewood maximal operator \(M\) is bounded in \(L^p\) (see [3]) and (2.5). We have

\[
\left\| \sup_{\ell = 1, \ldots, L_k - 2} |B_{k,\ell}(H^N f)| \right\|_p \leq C \|M(H^N f)\|_p
\]

\[
\leq C_p \|H^N f\|_p \leq C_p \|f\|_p
\]
for all $f \in L^p$, $N = 1, 2, \ldots$. For the last term in (2.6) we use [5, Theorem 2] and (2.4). We obtain

$$\left\| \sum_{k=0}^{N-1} \sum_{\ell=1}^{L_k-2} |S_{n_k,\ell} (A_{k,\ell} f)|^2 \right\|_p^{1/2} \leq C_p \left\| \sum_{k=0}^{N-1} \sum_{\ell=1}^{L_k-2} |A_{k,\ell} f|^2 \right\|_p^{1/2},$$

for all $f \in L^p$, $N = 1, 2, \ldots$. This proves (2.3). The proof of (1.1) will be complete once we prove the lemma.

Finally, if $f \in L^p$, $\lim_{k \to \infty} \|S_{m_k} f - f\|_p = 0$ since $S_{m_k} f$ is the average of $f$ over the cosets of $G_k$. As a consequence of this and (1.1), we have $\lim_{k \to \infty} S_{m_k} f(x) = f(x)$ a.e.

### 3. Proof of the Lemma

We shall use the following notation. For each generalized interval $I$, we define the generalized interval $3I$ as follows. If $I = G$, let $3I = G$. Suppose $I$ is a proper subset of $x + G_k$, $x \in G$, $k = 0, 1, 2, \ldots$, and is the union of cosets of $G_{k+1}$. If $\mu(I) \geq \mu(G_k)/3$, let $3I = x + G_k$. If $\mu(I) < \mu(G_k)/3$, consider $x + G_k$ as a circle and define $3I$ to be the interval in this circle which has the same center as $I$ and has measure $\mu(3I) = 3\mu(I)$. For all cases, we have $\mu(3I) \leq 3\mu(I)$.

For $k = 0, 1, \ldots, \ell = 1, \ldots, L_k - 1$ and $f \in L^1$, we have

$$B_{k,\ell} f(x) = \int_G f(t) \left[ \sum_{n=0}^{\infty} b_{k,\ell}(n) \chi_n(x - t) \right] d\mu(t).$$

Since $b_{k,\ell}(n)$ vanishes for $n \notin [m_k, m_{k+1})$ and is constant for $n \in [\alpha m_k, (\alpha + 1)m_k)$, $\alpha = 0, 1, \ldots$, we get

$$\sum_{n=0}^{\infty} b_{k,\ell}(n) \chi_n = \sum_{\alpha=1}^{p_k-1} b_{k,\ell}(\alpha m_k) \sum_{n=\alpha m_k}^{(\alpha+1)m_k-1} \chi_n = \sum_{\alpha=1}^{p_k-1} b_{k,\ell}(\alpha m_k) \phi_k^\alpha D_m,$$

where $D_n = \sum_{j=0}^{n-1} \chi_j$, $n = 1, 2, \ldots$, denotes the $n$th Dirichlet kernel. Since $D_m = \mu(G_k)^{-1} \chi_{G_k}$, we have

$$(3.1) \quad B_{k,\ell} f(x) = \frac{1}{\mu(G_k)} \int_{x + G_k} f(t) M_{k,\ell}(x - t) \, d\mu(t),$$

where

$$M_{k,\ell} = \sum_{\alpha=1}^{p_k-1} b_{k,\ell}(\alpha m_k) \phi_k^\alpha$$

$$= \sum_{\alpha=1}^{2^\ell-1} b_{k,\ell}(\alpha m_k) \phi_k^\alpha + \phi_k^0 + \sum_{\alpha=-2^\ell}^{1} b_{k,\ell}(p_k + \alpha m_k) \phi_k^\alpha - \phi_k^0.$$
Let
\[ D_{k,j} = \sum_{a=-j}^{j} \phi_{k}^{a}, \quad j = 0, 1, \ldots, 2^{L_k} - 1, \]
and
\[ K_{k,n} = \frac{1}{n} \sum_{j=0}^{n-1} D_{k,j}, \quad n = 1, \ldots, 2^{L_k} - 1. \]

Then
\[ M_{k,\ell} = \frac{1}{2^{\ell-1}} \left( D_{k,2^{\ell-1}} + D_{k,2^{\ell-1}+1} + \cdots + D_{k,2^\ell-1} \right) - \phi_{k}^{0} \]
\[ = 2K_{k,2^{\ell}} - K_{k,2^{\ell}-1} - \phi_{k}^{0}. \]

For \( n = 1, \ldots, 2^{L_k} - 1 \), let
\[ \sigma_{k,n} f(x) = \frac{1}{\mu(G_k)} \int_{x+G_k} f(t)K_{k,n}(x-t) \, d\mu(t). \]

From (3.1) and (3.2) we obtain
\[ |B_{k,\ell} f(x)| \leq 2|\sigma_{k,2^{\ell}} f(x)| + |\sigma_{k,2^{\ell}-1} f(x)| + M f(x). \]

The lemma will be proved if we show
\[ |\sigma_{k,n} f(x)| \leq C M f(x), \]
for all \( k = 0, 1, \ldots, n = 1, \ldots, 2^{L_k} - 1, \quad x \in G. \)

By a direct computation we have
\[ K_{k,n}(x) = \begin{cases} \frac{1}{n} \left( \frac{\sin^{2} \frac{\pi x}{p_{k}}}{\sin^{2} \frac{\pi x}{p_{k}}} \right) & \text{if } x_k \neq 0, \\ n & \text{if } x_k = 0, \end{cases} \]
and hence
\[ 0 \leq K_{k,n}(x) \leq \min \left\{ \left( \frac{n \sin^{2} \frac{\pi x}{p_{k}}} {p_{k}} \right)^{-1}, n \right\}. \]

Let \( I \) be a generalized interval containing \( x \) such that \( I \subset x + G_k \), \( I \) is a union of cosets of \( G_{k+1} \) and
\[ \frac{p_{k}}{n} - 1 < \frac{\mu(I)}{\mu(G_{k+1})} \leq \frac{p_{k}}{n}. \]
(Note that \( p_{k}/n \geq 2 \) since \( n \leq 2^{L_k} - 1 \).) For \( j = 1, 2, \ldots, \), let \( 3^{j+1} I = 3(3^j I) \), and \( J = \min\{ j \geq 1 : 3^j I = x + G_k \} \). Then
\[ |\sigma_{k,n} f(x)| \leq \frac{1}{\mu(G_k)} \int_{3I} |f(t)|K_{k,n}(x-t) \, d\mu(t) \]
\[ + \sum_{j=1}^{J-1} \frac{1}{\mu(G_k)} \int_{3^{j+1} I \setminus 3^j I} |f(t)|K_{k,n}(x-t) \, d\mu(t). \]
By (3.4),
\[
\frac{1}{\mu(G_k)} \int_{3I} |f(t)|K_{k,n}(x-t) \, d\mu(t) \leq \frac{3\mu(I)}{\mu(I)} \frac{1}{\mu(3I)} \int_{3I} |f(t)| \, d\mu(t) \leq CMf(x).
\]
To estimate the last term in (3.5), we apply the other estimate in (3.4). We observe that when \( x \in I, t \notin 3^j I, j = 1, \ldots, J - 1 \),
\[
\left| \sin \frac{\pi(x_k - t_k)}{p_k} \right| \geq \frac{C\mu(3^{j-1}I)}{\mu(G_k)}.
\]
Hence
\[
\frac{1}{\mu(G_k)} \int_{3^{j+1}I \setminus 3^j I} |f(t)|K_{k,n}(x-t) \, d\mu(t)
\leq \frac{\mu(3^{j+1}I)}{\mu(G_k)} \frac{1}{\mu(3^{j+1}I)} \int_{3^{j+1}I \setminus 3^j I} |f(t)| \left( \frac{n \sin^2 \frac{\pi(x_k - t_k)}{p_k}}{p_k} \right)^{-1} \, d\mu(t)
\leq C3^{-j} \frac{\mu(G_k)}{n\mu(I)} \int_{3^{j+1}I} |f(t)| \, d\mu(t)
\leq C3^{-j} Mf(x).
\]
Substituting these estimates into (3.5) and summing over \( j \), we obtain (3.3). The lemma follows.

This completes the proof of the theorem.

REFERENCES


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