THE MULTIPLIER OPERATORS ON THE WEIGHTED PRODUCT SPACES

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Abstract. In this paper, we proved the boundedness of multiplier operators on the weighted $L^p$ product spaces.

Let $m(\xi)$ be a function on $\mathbb{R}^n$ and let $f$ be a smooth function on $\mathbb{R}^n$. Suppose

$$T_m f(\xi) = m(\xi) \hat{f}(\xi).$$

Then $T_m f$ is called a multiplier operator. It is well known that a multiplier operator is bounded on the weighted $L^p$, $1 < p < \infty$, spaces for some suitable weights if the function $m(\xi)$ satisfies Hörmander’s condition

$$\int_{s\leq|\xi|\leq2s} |\partial_\xi^\alpha m(\xi)|^2 d\xi \leq s^{n-2|\alpha|}$$

for $|\alpha| \leq [n/2] + 1$ (see [3, page 418]). The keys to proving the boundedness of multiplier operator in the weighted $L^p$ spaces are basically

(i) the Hardy-Littlewood maximal operator is bounded by the sharp function, more precisely,

$$\int (M f(x))^p W(x) dx \leq C \int (f^\#(x))^p W(x) dx$$

where $W \in A_p$, $1 < p \leq \infty$, and

$$f^\#(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

$f_Q$ being the average of $f$ over cube $Q$ in $\mathbb{R}^n$;

(ii) an estimate,

$$(T_m f^\#(x) \leq C (M |f|^q(x))^{1/q}$$

for some $q > 1$.

The purpose of this paper is to study the boundedness of multiplier operators on the weighted $L^p$ product spaces.

Denote

$$\text{osc}_R f = \inf_{f_1, f_2} \left( \frac{1}{|R|} \int_R |f(x_1, x_2) - f_1(x_1) - f_2(x_2)|^2 dx_1 dx_2 \right)^{1/2}$$

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where $R$ is any rectangle in $R^{n_1} \times R^{n_2}$ and the inf is taken over all functions $f_1$ and $f_2$ depending on the variables $x_1$ and $x_2$ respectively.

As in the “one dimensional case”, if one defines the sharp function by

$$f^\#(x) = \sup_{x \in R} \text{osc}_R f,$$

then one might expect to show that the strong maximal operator

$$M_s f(x) = \sup_{t_1, t_2 > 0} \frac{1}{t_1 t_2^{n_1 + n_2}} \int_{|y_1| \leq t_1} \int_{|y_2| \leq t_2} |f(x_1 - y_1, x_2 - y_2)| dy_1 dy_2$$

is bounded by the sharp function. Unfortunately, such an observation is not true due to Carleson’s counterexample [1]. To remove the difficulty on the sharp function in order to obtain an inequality similar to (1), R. Fefferman considered a sharp operator (see [2]) defined as follows.

**Definition.** Let $T$ be an $L^2$ bounded linear operator. Suppose there exists an operator $T^\#$ defined on positive locally square integrable functions which is monotone, i.e.

$$T^\# f(x) \leq T^\# g(x)$$

if $f(x) \leq g(x)$ for all $x \in R^{n_1} \times R^{n_2}$ such that $\text{osc}_R (Tf) \leq \gamma^{-\sigma} T^\# f(x)$ for all $x \in R$, $R$ a rectangle on $R^{n_1} \times R^{n_2}$, and for some $\sigma > 0$, where $f$ is supported outside of the $\gamma$-fold dilation of $R$, $\gamma \geq 2$.

Based on this definition of a sharp operator, R. Fefferman [2] obtained the following inequality:

$$\int \int S^2(Tf)(x) \phi(x) dx \leq C \int \int [(I + T^\#)(|f|)^2] M_s(M_s(M_s(\phi)))(x) dx$$

where $I$ denotes the identity operator and $S$ is the area function defined on the product spaces. Using this inequality, he obtained the following theorem, which we will apply in this paper.

**Theorem A ([2]).** If $T$ is a bounded linear operator on $L^2(R^{n_1} \times R^{n_2})$ whose sharp operator is

$$T^\# f = M_s(f^2)^{1/2},$$

then for $p > 2$

$$\int_{R^{n_1} \times R^{n_2}} |Tf|^p W \leq C \int_{R^{n_1} \times R^{n_2}} |f|^p W$$

whenever $W \in A_{p/2}(R^{n_1} \times R^{n_2})$.

Proof. See [2, page 123].

In this paper, we will prove the following Theorem.
Theorem. Let \( m(\xi_1, \xi_2) \) be a function and \( m \in C^{p_1}(\mathbb{R}^{n_1} \setminus \{0\}) \times C^{p_2}(\mathbb{R}^{n_2} \setminus \{0\}) \), where \( p_1 = \lceil n_1/2 \rceil + 1 \), and \( p_2 = \lceil n_2/2 \rceil + 1 \). Suppose

\[
\int_{|\xi_1| \approx s_1} \int_{|\xi_2| \approx s_2} |\partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} m(\xi_1, \xi_2)|^2 \, d\xi_1 \, d\xi_2 \leq C s_1^{-2|\alpha_1|+n_1} s_2^{-2|\alpha_2|+n_2};
\]

(2)

\[
\sup_{\xi_2} \int_{|\xi_1| \approx s_1} |\partial_{\xi_1}^{\alpha_1} m(\xi_1, \xi_2)|^2 \, d\xi_1 \, d\xi_2 \leq C s_1^{-2|\alpha_1|+n_1};
\]

\[
\sup_{\xi_1} \int_{|\xi_2| \approx s_2} |\partial_{\xi_2}^{\alpha_2} m(\xi_1, \xi_2)|^2 \, d\xi_1 \, d\xi_2 \leq C s_2^{-2|\alpha_2|+n_2}
\]

for every \( |\alpha_1| \leq p_1 \) and \( |\alpha_2| \leq p_2 \), where \( |\xi_i| \approx s_i \) signifies that \( s_i \leq |\xi_i| \leq 2s_i \).

Then i) for \( 2 < p < \infty \),

\[
\int |T_m f|^p W \leq C \int |f|^p W
\]

whenever \( W \in A_{p/2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \); and ii) for \( 1 < p < 2 \),

\[
\int |T_m f|^p W \leq C \int |f|^p W
\]

whenever \( W^{1/p} \in A_{p/p}^{1/p}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \).

Remark. The weighted norm inequality for \( p = 2 \) can be obtained by using the interpolation theorem.

Proof. From Theorem A, we need to show that the sharp operator \( T^# f = M_s(f^2)^{1/2} \). Then (i) of our Theorem follows from Theorem A.

Let us take a smooth function \( \phi \) on \( \mathbb{R} \) whose Fourier transform \( \hat{\phi}(t) \) has compact support \( \{ 1/2 < |t| < 2 \} \) such that \( \sum \hat{\phi}(2^{-j}|t|) = 1 \) for all \( t \neq 0 \). Let

\[
m_{i,j}(\xi_1, \xi_2) = m(\xi_1, \xi_2) \hat{\phi}(2^{-j}|\xi_1|) \hat{\phi}(2^{-j}|\xi_2|)
\]

and

\[
\tilde{T}_{i,j} f(\xi_1, \xi_2) = m_{i,j}(\xi_1, \xi_2) \tilde{f}(\xi_1, \xi_2) \equiv (\hat{k}_{i,j} * f)(\xi_1, \xi_2).
\]

It is clear that \( T f = \sum_{i,j} T_{i,j} f \).

To prove \( T^# f(x) = (M_s f^2(x))^{1/2} \), one needs to estimate, for every rectangle \( R \),

(3)

\[
osc_R(T f)(x) \leq C \gamma^{-\delta} (M_s f^2(x))^{1/2}
\]

for every \( x \in R \) where \( f \) is supported outside of the \( \gamma \)-fold dilation of the rectangle \( R, \gamma \geq 2 \), i.e. \( \text{supp} \ f \subset \overset{\circ}{{\gamma R}} \). By the homogeneity of multiplier operators, it suffices to assume \( R \) is the unit square. Since the estimates are translation invariant, we may assume the center of \( R \) is at the origin. Let us write a function \( f \), \( \text{supp} \ f \subset \overset{\circ}{{\gamma R}} \), as the sum of the functions \( g + h + G \) where

- support of \( g \subset \overset{\circ}{{\gamma^2 R}} \equiv \{ |y_1| > \gamma, |y_2| \leq \gamma \};
- support of \( h \subset \overset{\circ}{{\gamma^2 R}} \equiv \{ |y_1| > \gamma, |y_2| > \gamma \};
- support of \( G \subset \overset{\circ}{{\gamma^2 R}} \equiv \{ |y_1| \leq \gamma, |y_2| > \gamma \}.\)
We will estimate that the first two terms are dominated by \( \gamma \) since \( \int osc_{i,j} = \sum_{i,j} \int R_{i,j} \leq \sum_{i,j} \int R_{i,j} \).

Without loss of generality it suffices to show (3) for a function \( f = g + h \) where

\[
\text{supp } g \subset \hat{c}R_1 \subset \{ |y_1| > \gamma, |y_2| \leq 2 \};
\]

\[
\text{supp } h \subset \hat{c}R_2 \subset \{ |y_1| > \gamma, |y_2| > 2 \}.
\]

We are going to estimate

\[
osc_R(Tg)(x) \leq C \gamma^{-\sigma}(M_g g^2(x))^{1/2} \quad \text{and} \quad osc_R(Th)(x) \leq C \gamma^{-\sigma}(M_h h^2(x))^{1/2},
\]

since \( osc_Tf \leq osc_R Tg + osc_R Th \). Let us write

\[
osc_R(Tf) \leq \sum_{i \geq 0} osc_R(\sum_j T_{i,j} g) + \sum_{i < 0} osc_R(\sum_j T_{i,j} g)
\]

\[
+ \sum_{i \geq 0} \sum_{j \geq 0} osc_R T_{i,j} h + \sum_{i \geq 0} \sum_{j \leq 0} osc_R T_{i,j} h
\]

\[
+ \sum_{i < 0} \sum_{j \geq 0} osc_R T_{i,j} h + \sum_{i < 0} \sum_{j < 0} osc_R T_{i,j} h
\]

\[
\equiv I + II + III + IV + V + VI.
\]

We will estimate that the first two terms are dominated by \( \gamma^{-\sigma}(M_g g^2(x))^{1/2} \) and the last four terms are dominated by \( \gamma^{-\sigma}(M_h h^2(x))^{1/2} \) for every \( x \in R \). Denote \( \sum_i T_{i,j} g = T_ig \) and write

\[
I = \sum_{i \geq 0} osc_R(T_ig)
\]

\[
\leq \sum_{i \geq 0} \left( \frac{1}{|R|} \int_R \left| T_ig(x_1, x_2) \right|^2 dx_1 dx_2 \right)^{1/2}
\]

\[
= \sum_{i \geq 0} \left( \int |x_1| \leq 1 \int |x_2| \leq 1 \int |y_2| < 2 \int |y_1| > \gamma K_i(x_1 - y_1, x_2 - y_2) g(y_1, y_2)
\]

\[
\cdot dy_1 dy_2^2 dx_2 dx_1 \right)^{1/2}
\]

\[
\leq \sum_{i \geq 0} \sum_{2^{k_1} \geq \gamma/2} \left( \int |x_1| \leq 1 \int |x_2| \leq 1 \int |y_2| < 2 \int |y_1| = 2^{k_1}
\]

\[
\cdot K_i(x_1 - y_1, x_2 - y_2) g(y_1, y_2) dy_1 dy_2 dy_1^2 dx_2 dx_1 \right)^{1/2}.
\]

Since

\[
\int |y_1| = 2^{k_1}, \int |y_2| < 2 K_i(x_1 - y_1, x_2 - y_2) g(y_1, y_2) dy_2 dy_1
\]

is a convolution operator in the variable \( y_2 \), applying Plancherel’s Theorem for the variable \( x_2 \) on (4), one has

\[
I \leq \sum_{i \geq 0} \sum_{2^{k_1} \geq \gamma/2} \left( \int |x_1| \leq 1 \int |\xi_2| \int |y_1| = 2^{k_1} \tilde{K}_i^2(x_1 - y_1, \xi_2) \tilde{g}^2(y_1, \xi_2) dy_1 |d\xi_2| dx_1 \right)^{1/2}
\]
where \( \wedge \) denotes the Fourier transform on the second variable. For \( |x_1| \leq 1, |y_1| \geq \gamma, \gamma \geq 2, |y_1| \approx 2^{k_1} \) then \( |x_1 - y_1| \approx 2^{k_1} \). Hence

\[
I \leq \sum_{i \geq 0} \sum_{2^{k_1} \geq \gamma/2} 2^{-k_1 |p_1|} \left( \int_{|x_1| \leq 1} \int_{|\xi_2| = 2^{k_1}} |x_1 - y_1|^{p_1} \hat{K}_i^2(x_1 - y_1, \xi_2) \cdot \hat{g}^2(y_1, \xi_2) dy_1 d\xi_2 d\xi_1 \right)^{1/2}.
\]

By Hölder’s inequality and changing variable \( y_1 \),

\[
I \leq \sum_{i \geq 0} \sum_{2^{k_1} \geq \gamma/2} 2^{-k_1 |p_1|} \left( \left( \sup_{\xi_2} \int \left| \int_{|y_1| \approx 2^{k_1}} \hat{g}^2(y_1, \xi_2) dy_1 d\xi_2 \right|^{1/2} \right)^{1/2} \right)
\]

\[
\leq \sum_{i \geq 0} \sum_{2^{k_1} \geq \gamma/2} 2^{k_1(-p_1 + n_1/2)} \left( \left( \sup_{\xi_2} \sum_{|\alpha_1| = p_1} \int |\partial_{\xi_1}^{\alpha_1} m_1(\xi_1, \xi_2)|^2 d\xi_1 \right)^{1/2} \right)
\]

\[
\leq \sum_{i \geq 0} \sum_{2^{k_1} \geq \gamma/2} \left( \frac{1}{2^{k_1 n_1}} \right) \left( \int_{|y_2| \leq 2} \int_{|y_1| \approx 2^{k_1}} |g(y_1, y_2)|^2 dy_1 dy_2 \right)^{1/2}
\]

where the last inequality is obtained by applying Plancherel’s Theorem to both integrals and the support of \( g \) is contained by \( \{|y_1| > \gamma, |y_2| \leq 2\} \). Hence, by the hypothesis (2) and \(-p_1 + n_1/2 < 0\),

\[
I \leq C \sum_{i \geq 0} \gamma^{-\sigma} (M_s g^2(0))^{1/2} 2^{i(-p_1 + n_1/2)} \leq C \gamma^{-\sigma} (M_s g^2(0))^{1/2}.
\]

For estimating \( II \), we write

\[
II = \sum_{i \geq 0} oscR(\sum_j T_{i,j} g) = \sum_{i < 0} oscR(T_i g)
\]

\[
\leq \sum_{i < 0} \left( \frac{1}{|R|} \int_R |T_i g(x_1, x_2) - T_i g(0, x_2)|^2 dx_1 dx_2 \right)^{1/2}
\]

\[
= C \sum_{i < 0} \left( \int_R \int_{-R_1} (K_i(x_1 - y_1, x_2 - y_2) - K_i(0 - y_1, x_2 - y_2)) \cdot g(y_1, y_2) dy_1 dy_2 |^2 dx_2 dx_1 \right)^{1/2}
\]

\[
\leq C \sum_{i \geq 0} \sum_{2^{k_1} \geq \gamma/2} \left( \int_{|y_1| \approx 2^{k_1}} \int_{|y_2| \leq 2} (K_i(x_1 - y_1, x_2 - y_2) - K_i(0 - y_1, x_2 - y_2)) \cdot g(y_1, y_2) dy_1 dy_2 |^2 dx_2 dx_1 \right)^{1/2}.
\]
Here, we follow the same procedures as we did in proving I, applying Plancherel’s Theorem for the variable $x_2$.

$$ II \leq C \sum_{i < 0} \sum_{2^k \geq \gamma/2} \left( \int_{|x_1| \leq 1} \int_{|y_1| \geq 2^k} \left( \hat{K}_i^2(x_1 - y_1, \xi_2) - \hat{K}_i^2(0 - y_1, \xi_2) \right) \cdot \hat{g}^2(y_1, \xi_2) dy_1 ||^2 d\xi_2 dx_1 \right)^{1/2} $$

$$ = C \sum_{i < 0} \sum_{2^k \geq \gamma/2} \left( \int_{|x_1| \leq 1} \int_{|y_1| \geq 2^k} \int_0^1 x_1 \partial_{y_1} \hat{K}_i^2(x_1 s_1 - y_1, \xi_2) \cdot \hat{g}^2(y_1, \xi_2) ds_1 dy_1 ||^2 d\xi_2 dx_1 \right)^{1/2} $$

$$ \leq C \sum_{i < 0} \sum_{2^k \geq \gamma/2} \left( \int_0^1 \int_{|x_1| \leq 1} \int_{|y_1| \geq 2^k} \left| \partial_{y_1} \hat{K}_i^2(x_1 s_1 - y_1, \xi_2) \right|^2 dy_1 \right) \cdot \left( \int_{|y_1| \geq 2^k} \left| \hat{g}^2(y_1, \xi_2) \right|^2 d\xi_2 dx_1 \right)^{1/2} $$

$$ \leq C \sum_{i < 0} \sum_{2^k \geq \gamma/2} 2^{k_i(-p_1 + \epsilon_1 + n_1/2)} \left[ \int_0^1 \int_{|x_1| \leq 1} \int_{|y_1| \geq 2^k} \sup_{\xi_2} \left| x_1 s_1 - y_1 \right|^{p_1 - \epsilon_1} \left| \partial_{y_1} \hat{K}_i^2(x_1 s_1 - y_1, \xi_2) \right|^2 dy_1 \right] \cdot \left( \int_{|y_1| \geq 2^k} \left| \hat{g}^2(y_1, \xi_2) \right|^2 d\xi_2 dx_1 \right)^{1/2} \cdot \left( \frac{1}{2^{k_1 n_1}} \int \int_{|y_1| \geq 2^k} \left| \hat{g}^2(y_1, \xi_2) \right|^2 d\xi_2 dx_1 \right)^{1/2}. $$

Taking a very small $\epsilon_1 > 0$ such that $-p_1 + \epsilon_1 + n_1/2 < 0$, changing variable (i.e. $x_1 s_1 - y_1 \rightarrow y_1$) in the integral in the first parentheses and applying Plancherel’s Theorem for the integral in the second parentheses, one has

$$ II \leq C \gamma^{-\sigma} \sum_{i < 0} \sup_{\xi_2} \left( \int \left| y_1 \right|^{-\epsilon_1} \left| y_1 \right|^{p_1} \left| \partial_{y_1} \hat{K}_i^2(y_1, \xi_2) \right|^2 dy_1 \right)^{1/2} \left( M_\gamma g^2(0) \right)^{1/2} $$

$$ \leq C \gamma^{-\sigma} \left( M_\gamma g^2(0) \right)^{1/2} \sum_{i < 0} \sup_{\xi_2} \sum_{|\alpha_1| = p_1} \left( \int \left| \xi_1 \right|^{-n_1 + \epsilon_1} \left| \partial_{\xi_1}^\alpha \left( \xi_1 m_i(\xi_1, \xi_2) \right) \right|^2 d\xi_1 \right)^{1/2} $$

for some $\sigma > 0$, where $*^1$ is the convolution operator on the first variable. By fractional integration,

$$ II \leq C \gamma^{-\sigma} \left( M_\gamma g^2(0) \right)^{1/2} \sum_{i < 0} \sum_{|\alpha_1| = p_1} \sup_{\xi_2} \left( \int \left| \partial_{\xi_1}^\alpha \left( \xi_1 m_i(\xi_1, \xi_2) \right) \right|^q d\xi_1 \right)^{1/q} $$
where $1/q = 1/2 + \epsilon_1/n_1$ (clearly $q < 2$). By Hölder's inequality,

$$II \leq C\gamma^{-\sigma}(M_s g^2(0))^{1/2} \sum_{i < 0} \sum_{|\alpha_1| = p_1} 2^{in_1(1/q - 1/2)} \sup_{\xi_2} \left( \int |\partial_{\xi_1}^{\alpha_1} (\xi_1 m_1(\xi_1, \xi_2))|^2 d\xi_1 \right)^{1/2}$$

$$\leq C\gamma^{-\sigma}(M_s g^2(0))^{1/2} \sum_{i < 0} 2^{i(-p_1 + 1 + n_1/q)}$$

$$\leq C\gamma^{-\sigma}(M_s g^2(0))^{1/2}$$

(since $\epsilon_1 > 0$ then $-p_1 + 1 + n_1/q > 0$).

For estimating III, we write

$$|T_{i,j}h(x_1, x_2)|$$

$$\leq \sum_{k_2 \geq 1} \sum_{2^{k_1} \geq \gamma/2} \int_{|y_1| \approx 2^{k_1}} \int_{|y_2| \approx 2^{k_2}} |K_{i,j}(x_1 - y_1, x_2 - y_2) h(y_1, y_2)| dy_1 dy_2$$

$$\leq \sum_{k_2 \geq 1} \sum_{2^{k_1} \geq \gamma/2} \left( \frac{1}{2^{k_1} 2^{k_2}} \int_{|y_1| \approx 2^{k_1}} \int_{|y_2| \approx 2^{k_2}} |h|^2 dy_1 dy_2 \right)^{1/2}$$

$$\leq C(M_s h^2(0))^{1/2} \sum_{k_2 \geq 1} 2^{k_1(-p_1 + n_1/2)} 2^{k_2(-p_2 + n_2/2)}$$

$$\cdot \left( \int \int |x_1 - y_1|^{p_1} |x_2 - y_2|^{p_2} K_{i,j}(x_1 - y_1, x_2 - y_2)^2 dy_1 dy_2 \right)^{1/2} .$$

As before, applying a change of variable and Plancherel's Theorem,

$$|T_{i,j}h(x_1, x_2)|$$

$$\leq C\gamma^{-\sigma}(M_s h^2(0))^{1/2} \sum_{|\alpha_2| = p_1} \sum_{|\alpha_2| = p_2} \left( \int \int |\partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} m_{i,j}(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2 \right)^{1/2}$$

$$\leq C\gamma^{-\sigma}(M_s h^2(0))^{1/2} 2^{i(-p_1 + n_1/2)} 2^{j(-p_2 + n_2/2)}.$$
Since the estimates for IV and V are similar, we estimate only term V. First let us write

\[ |T_{i,j}h(x_1, x_2) - T_{i,j}h(0, x_2)| \]

\[ \leq \sum_{k_2 \geq 1} \sum_{2^k_1 \geq \gamma/2} \int_{|y_1| \approx 2^{k_1}} \int_{|y_2| \approx 2^{k_2}} |K_{i,j}(x_1 - y_1, x_2 - y_2) - K_{i,j}(0 - y_1, x_2 - y_2)| \]

\[ \cdot |h(y_1, y_2)| dy_1 dy_2 \]

\[ \leq \sum_{k_2 \geq 1} \sum_{2^k_1 \geq \gamma/2} 2^{k_1 \alpha_1/2} 2^{k_2 \alpha_2/2} \int_{|y_1| \approx 2^{k_1}} \int_{|y_2| \approx 2^{k_2}} |K_{i,j}(x_1 - y_1, x_2 - y_2) - K_{i,j}(0 - y_1, x_2 - y_2)|^2 dy_1 dy_2 \]

\[ \leq \sum_{k_2 \geq 1} \sum_{2^k_1 \geq \gamma/2} 2^{k_1 \alpha_1/2} 2^{k_2 \alpha_2/2} (M_s h^2(0))^{1/2} \left( \int_0^1 \int \int |\partial_{y_1} K_{i,j}(x_1 s_1 - y_1, x_2 - y_2)|^2 dy_1 dy_2 ds \right)^{1/2} \]

\[ \approx \sum_{k_2 \geq 1} \sum_{2^k_1 \geq \gamma/2} 2^{k_1(-p_1 + \epsilon_1 + n_1/2)} 2^{k_2(-p_2 + n_2/2)} (M_s h^2(0))^{1/2} \left( \int_0^1 \int \int \right. \]

\[ \left. \cdot |x_1 s_1 - y_1|^{p_2 \alpha_2} |x_2 - y_2|^{p_2} \partial_{y_1} K_{i,j}(x_1 s_1 - y_1, x_2 - y_2)|^2 dy_1 dy_2 ds \right)^{1/2} \]

Taking a positive small \( \epsilon_1 \), changing variables, applying Plancherel’s Theorem and fractional integration, we get

\[ |T_{i,j}h(x_1, x_2) - T_{i,j}h(0, x_2)| \]

\[ \leq C \gamma^{-\sigma} (M_s h^2(0))^{1/2} \]

\[ \cdot \sum_{|\alpha_1| = p_1, |\alpha_2| = p_2} \left( \int \int \|\xi_1|^{-n_1 + \epsilon_1} \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} (\xi_1 m_{i,j}(\xi_1, \xi_2))|^2 d\xi_1 d\xi_2 \right)^{1/2} \]

\[ \leq C \gamma^{-\sigma} (M_s h^2(0))^{1/2} \]

\[ \cdot \sum_{|\alpha_1| = p_1, |\alpha_2| = p_2} \left( \int \left( \int |\partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} (\xi_1 m_{i,j}(\xi_1, \xi_2))|^q d\xi_1 \right)^{2/q} d\xi_2 \right)^{1/2} \]

where \( 1/q = 1/2 + \epsilon_1/n_1 \). By Hölder’s inequality and the hypothesis (2),

\[ |T_{i,j}h(x_1, x_2) - T_{i,j}h(0, x_2)| \]

\[ \leq C \gamma^{-\sigma} (M_s h^2(0))^{1/2} 2^{n_1(1/q - 1/2)} \left( \int \int |\partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} (\xi_1 m_{i,j}(\xi_1, \xi_2))|^2 d\xi_1 d\xi_2 \right)^{1/2} \]

\[ \leq C \gamma^{-\sigma} (M_s h^2(0))^{1/2} 2^{(n_1/n_1 - 1/2)} (-p_1 + n_1/2)_2^{(-p_2 + n_2/2)} \cdot \]

Since \(-p_1 + n_1/q > 0\) for positive small \( \epsilon_1 \) and \(-p_2 + n_2/2 < 0\), we have

\[ V = \sum_{i < 0} \sum_{j \geq 0} \operatorname{osc}_R T_{i,j} h \leq C \gamma^{-\sigma} (M_s h^2(0))^{1/2}. \]
By the Taylor formula and Hölder’s inequality,

\[
|T_{i,j}h(x_1, x_2) - T_{i,j}h(0, x_2) - T_{i,j}h(x_1, 0) + T_{i,j}h(0, 0)| \\
\leq \sum_{k_2 \geq 1} \sum_{2^{k_1} \geq \gamma/2} \left( \frac{2^{k_1} \alpha_{k_2}}{2^{k_1} \alpha_{k_2} n_2} \int_{|y_1| \approx 2^{k_1}} \int_{|y_2| \approx 2^{k_2}} (K_{i,j}(x_1 - y_1, x_2 - y_2) - K_{i,j}(-y_1, x_2 - y_2) - K_{i,j}(x_2 - y_1, -y_2) + K_{i,j}(-y_1, -y_2))h(y_1, y_2)d_1 dy_2 \right)^{1/2} \\
\leq \sum_{k_2 \geq 1} \sum_{2^{k_1} \geq \gamma/2} \left( \frac{2^{k_1} \alpha_{k_2}}{2^{k_1} \alpha_{k_2} n_2} \int_{|y_1| \approx 2^{k_1}} \int_{|y_2| \approx 2^{k_2}} |K_{i,j}(x_1 - y_1, x_2 - y_2) - K_{i,j}(-y_1, x_2 - y_2) - K_{i,j}(x_2 - y_1, -y_2) + K_{i,j}(-y_1, -y_2)|^2 d_1 dy_2 \right)^{1/2} \\
\cdot \left( \int_{|y_1| \approx 2^{k_1}} \int_{|y_2| \approx 2^{k_2}} |h|^2 d_1 dy_2 \right)^{1/2}.
\]

By the Taylor formula and Hölder’s inequality,

\[
|T_{i,j}h(x_1, x_2) - T_{i,j}h(0, x_2) - T_{i,j}h(x_1, 0) + T_{i,j}h(0, 0)| \\
\leq \sum_{k_2 \geq 1} \sum_{2^{k_1} \geq \gamma/2} \left( \frac{2^{k_1} \alpha_{k_2}}{2^{k_1} \alpha_{k_2} n_2} \int_{|y_1| \approx 2^{k_1}} \int_{|y_2| \approx 2^{k_2}} |\partial_{y_1} \partial_{y_2} K_{i,j}(x_1 s_1 - y_1, x_2 s_2 - y_2)|^2 d_1 dy_2 ds_1 ds_2 \right)^{1/2} \\
\approx C(M_h^2(0))^{1/2} \sum_{k_2 \geq 1} \sum_{2^{k_1} \geq \gamma/2} \left( \frac{2^{k_1} \alpha_{k_2}}{2^{k_1} \alpha_{k_2} n_2} \int_{|y_1| \approx 2^{k_1}} \int_{|y_2| \approx 2^{k_2}} ||x_1 s_1 - y_1|^{p_1 - \epsilon_1} |x_2 s_2 - y_2|^{p_2 - \epsilon_2} \right)^{1/2} \\
\cdot \left( \int_{|y_1| \approx 2^{k_1}} \int_{|y_2| \approx 2^{k_2}} ||x_1 s_1 - y_1|^{p_1 - \epsilon_1} |x_2 s_2 - y_2|^{p_2 - \epsilon_2} \right)^{1/2} \\
\cdot \left( \int_{|y_1| \approx 2^{k_1}} \int_{|y_2| \approx 2^{k_2}} \partial_{y_1} \partial_{y_2} K_{i,j}(x_1 s_1 - y_1, x_2 s_2 - y_2)|^2 d_1 dy_2 ds_1 ds_2 \right)^{1/2} \\
\leq C \gamma^{-\sigma} (M_h^2(0))^{1/2} \int \int ||y_1|^{-\epsilon_1} |y_2|^{-\epsilon_2} |y_1|^{p_1} |y_2|^{p_2} \\
\cdot \partial_{y_1} \partial_{y_2} K_{i,j}(y_1, y_2)|^2 d_1 dy_2 \right)^{1/2} \\
\leq C \gamma^{-\sigma} (M_h^2(0))^{1/2} \sum_{|\alpha_1| = p_1, |\alpha_2| = p_2} \left( \int ||\xi_1|^{-n_1 + \epsilon_1} |\xi_2|^{-n_2 + \epsilon_2} \\
\cdot \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} (\xi_1 \xi_2 m_{i,j}(\xi_1, \xi_2))|^2 d\xi_1 d\xi_2 \right)^{1/2}).
\]
Next, we use fractional integration twice on the variables $\xi_1$ and $\xi_2$ respectively. Let $1/q_1 = 1/2 + \epsilon_1/n_1$ and $1/q_2 = 1/2 + \epsilon_2/n_2$. Then

$$|T_{i,j}h(x_1,x_2) - T_{i,j}h(0,x_2) - T_{i,j}h(x_1,0) - T_{i,j}h(0,0)| \leq C\gamma^{-\sigma}(M_s h^2(0))^{1/2} \sum_{|\alpha_1|=p_1} \sum_{|\alpha_2|=p_2} \left(\int \left(\int \|\xi_2\|^{-n_2+\epsilon_2} \right) \right)^{q_1/2} \frac{C_{\gamma}}{q_1} \left(\int \frac{\partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2}(\xi_1 \xi_2 m_{i,j}(\xi_1,\xi_2))^2}{\xi_2^2} d\xi_2 \right)^{1/2}$$

where $s^2$ denotes the convolution operator on the second variable. By Minkowski’s inequality, the last inequality is less than

$$C\gamma^{-\sigma}(M_s h^2(0))^{1/2} \sum_{|\alpha_1|=p_1} \sum_{|\alpha_2|=p_2} \left(\int \left(\int \|\xi_2\|^{-n_2+\epsilon_2} \right) \right)^{q_1/2} \frac{C_{\gamma}}{q_1} \left(\int \frac{\partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2}(\xi_1 \xi_2 m_{i,j}(\xi_1,\xi_2))^2}{\xi_2^2} d\xi_2 \right)^{1/2}$$

Hence

$$VI \leq \sum_{i<0} \sum_{j<0} \text{osc}_R T_{i,j}h \leq C\gamma^{-\sigma}(M_s h^2(0))^{1/2}.$$  

Combining the above estimates, we conclude that

$$T^# f(x_1,x_2) = (M_s f^2(x))^1/2.$$  

(i) is proved.

For the proof of (ii), we use duality. Let $U = W^{-1/(p-1)}$. Then the dual space of $L_{W}^p(R^{n_1} \times R^{n_2})$ is $L_U^p(R^{n_1} \times R^{n_2})$. There exists a function $g \in L_U^p(R^{n_1} \times R^{n_2})$ such that

$$\|T_m f\|_{L_W^p} = \int T_m f \tilde{g} = \int f \tilde{T_m g} \leq \|f\|_{L_W^p} \|T_m g\|_{L_U^p}.$$
It is easy to see that
\[ W^{2/(2-p)} \in A_{p/(2-p)} \iff U \in A_{p'/2}. \]
Applying (i),
\[ \|T_m g\|_{L^p_U} \leq C\|g\|_{L^p_U} \]
where \( U \in A_{p'/2} \). The Theorem is proved.

References

1. L. Carleson, A counterexample for measures bounded for \( H^p \) for the \( B \)-disc, Mittag Leffler report, No. 7, 1974.