ON THE PLURICANONICAL MAP OF THREEFOLDS OF GENERAL TYPE

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Abstract. Let $X$ be a smooth minimal threefold of general type and let $n$ be an integer $> 1$. Assume that the image of the pluricanonical map $\Phi_n$ of $X$ is a curve. Then a simple computation shows that $n$ is necessarily 2 or 3. When $n = 2$ with a numerical condition or when $n = 3$, we obtain two inequalities $\chi(O_X) \leq \min\{-1, 2 - 2q_1\}$ and $q_1 \leq \frac{3}{14} K_X^3 + 1$, where $q_1$ is the irregularity of $X$ and $\chi(O_X)$ is the Euler characteristic of $X$.

Throughout this paper, we are working over the complex number field $\mathbb{C}$.

In this paper, we have studied the case that the image $C$ of the pluricanonical map $\Phi_n$ is a curve for an integer $n > 1$. In this case, a simple calculation shows $n = 2$ or 3. Resolve the base locus of $\Phi_n$. Then we have two terms ‘$b$’ and ‘$c$’ explained just below Proposition 1. When $n = 2$ with additional numerical conditions or when $n = 3$, we may have some information about $\Phi_n$ from these two terms, which are explained in Corollary of Proposition 2 and Proposition 3. Using this information, we obtain two inequalities $\chi(O_X) \leq \min\{-1, 2 - 2q_1\}$ and $q_1 \leq \frac{3}{14} K_X^3 + 1$, which are given in Theorems 2 and 3.

Now, let’s set up our notations. Let $X$ be a smooth projective variety and let $D$ be a divisor on $X$. Denote by $K_X$ the canonical divisor of $X$. Denote by $\Phi_n$ the rational map associated to the complete linear system $|nK_X|$. Denote by $h^i(X, O_X(D))$ the dimension of $H^i(X, O_X(D))$. Let $Bs|D|$ mean the base locus of $|D|$. Let’s denote the genus of $X$ by $p_g(X)$ and $h^i(X, O_X)$ by $q_i(X)$ (or simply $p_g$ and $q_i$ unless there is any confusion.) Denote by $\sim$ the linear equivalence and by $\equiv$ the numerical equivalence. For a real number $r$, $[r]$ means the greatest among the integers less than or equal to $r$.

Theorem 1 (Kawamata-Viehweg vanishing theorem). Let $X$ be a nonsingular projective variety. If $D$ is a nef and big divisor on $X$, then $H^i(X, O_X(K_X + D)) = 0$ for all $i > 0$.

For a reference, see KMM [5].

Lemma. Let $X$ be a smooth projective threefold, and let $D$ be a divisor on $X$. Then we have the following:

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(a) \( \chi(\mathcal{O}_X(D)) = D^3/6 - K_X \cdot D^2/4 + D \cdot (K_X^2 + c_2)/12 + \chi(\mathcal{O}_X) \), where \( c_2 \) is the second Chern class of \( X \). Moreover \( \chi(\mathcal{O}_X) = -c_2 \cdot K_X/24 \).

(b) \( K_X \cdot D^2 \) is even.

(c) \( p_n := h^0(X, \mathcal{O}_X(nK_X)) = \frac{n(n-1)(2n-1)}{12}K_X^3 + (1-2n)\chi(\mathcal{O}_X) \) for \( n \geq 2 \) and \( \chi(\mathcal{O}_X) < 0 \) when \( K_X \) is nef and big.

**Proof.** (a) comes from the Riemann-Roch theorem.

(b) comes from the following:

\[ \chi(\mathcal{O}_X(D)) + \chi(\mathcal{O}_X(-D)) = -K_X \cdot D^2/2 + 2\chi(\mathcal{O}_X) \in \mathbb{Z}. \]

(c) \( \chi(\mathcal{O}_X) < 0 \) since \( -c_2 \cdot K_X \leq -K_X^3/3 \). (See Miyaoka [8].)

**Proposition 1.** Let \( C \) be a nondegenerate curve of degree \( a \) in \( \mathbb{P}^n \). Then we have

(a) If \( n \leq a < 2n \), then \( p_0(C) \leq a - n \).

(b) If \( 2n \leq a \), then \( p_0(C) \leq \frac{m(m-1)}{2}(n-1) + mr \), where \( m = \left[ \frac{a-1}{n-1} \right] \) and \( a-1 = m(n-1) + r \).

**Proof.** See Griffiths & Harris [2], p. 253.

We are going to use the following notations in the rest of this paper.

Let \( X \) be a smooth projective threefold of general type with \( K_X \) nef. Let \( n \) be a positive integer \( \geq 2 \).

Suppose that the dimension of the image \( C \) of \( \Phi_n \) is 1.

Let \( |nK_X| = |M| + Z \), where \( |M| \) and \( Z \) are the moving part and fixed part of \( |nK_X| \) respectively. Let \( f : X' \to X \) be the resolution of the base locus of \( \Phi_n \) if \( Bs|nK_X| \neq \emptyset \), i.e., \( f \) is a succession of blow-ups along nonsingular centers of codim \( \geq 2 \) in the base locus such that \( g = \Phi_n \circ f \) is a morphism. Let \( g = k \circ h \) be the Stein factorization. Observe that \( C'' \) is normal and hence smooth.

\[
\begin{array}{ccc}
X' & \xrightarrow{h} & C' \\
\downarrow f & & \downarrow k \\
X & \xrightarrow{\Phi_n} & C
\end{array}
\]

Let \( a = \deg C \) in \( \mathbb{P}^{p_n-1} \), and let \( b = \deg k \). Recall that \( C \) is a nondegenerate curve in \( \mathbb{P}^{p_n-1} \). Let \( S \) be the general fiber of \( h \).

We have \( K_{X'} = f^*(K_X) + E' \) and \( |f^*nK_X| = |M'| + Z' \), where \( E' \) is the ramification divisor of \( f \) supported on the exceptional locus of \( f \) and \( Z' \) is the fixed part of \( |f^*nK_X| \). We have \( M' = abS \). Since \( f^*K_X \) is nef, \( f^*K_X^2 \cdot Z' \geq 0 \). Hence \( nK_X^3 = n(f^*K_X)^3 = (abS + Z')f^*K_X^2 \geq abf^*K_X^2 \cdot S \).

Let \( c = f^*K_X \cdot S \). Since \( f^*K_X \) is nef and big, and \( S \) is nef and not numerically equivalent to 0, we have \( f^*K_X^2 \cdot S \geq 1 \). Hence

\[
\frac{nK_X^3}{bc} \geq a = p_n - 1.
\]

From the inequality (1) and Lemma, the image of \( \Phi_n \) can be a curve only when \( n \) is 2 or 3. Moreover, we have \( 1 \leq bc \leq 3 \) and in particular, \( bc = 1 \) when \( n = 3 \).
Proposition 2 (cf. Matsuki [7]). If dim Im $\Phi_n = 1$, then $S$ is a surface of general type and $S'$ has $K_{S'}^2 = f^*K_X^2 \cdot S$, where $\pi : S \to S'$ is a morphism of $S$ to its minimal surface $S'$.

Proof. The easy addition formula $\kappa(X) \leq \kappa(S) + \dim C'$ implies that $S$ is the surface of general type, where $\kappa(X)$ means the Kodaira dimension of $X$. (For a reference about easy addition formula, see Ueno [9].) We have that $nK_X \equiv abD + Z$ and $f^*abD \equiv abS + E$, where $D = f_*S$ and $E$ is an effective divisor supported on the exceptional locus of $f$. So,

$$nf^*K_X^2 \cdot S = nK_X^2 \cdot D = (abD + Z) \cdot K_X \cdot D \geq abK_X \cdot D^2.$$ 

Since $D^2$ is an effective 1-cycle and $K_X$ is nef, $K_X \cdot D^2 \geq 0$. If $K_X \cdot D^2 \neq 0$, then $nf^*K_X^2 \cdot S \geq 2a \geq 2(p_a - 1)$ since $K_X \cdot D^2$ is even. This inequality holds true only when $n = 2$, $K_X^3 = 2$, $\chi(O_X) = -1$ and $f^*K_X^2 \cdot S = 3$. (Recall that $2 \leq n \leq 3$ and $1 \leq c \leq 3$.) But it does not satisfy the inequality (1). Hence $K_X \cdot D^2 = 0$. So

$$0 = (abD)^2 \cdot K_X = f^*abD \cdot f^*abD \cdot f^*K_X$$

$$= (abS + E) \cdot f^*abD \cdot f^*K_X$$

$$= abS \cdot f^*abD \cdot f^*K_X$$

$$= abS \cdot (abS + E) \cdot f^*K_X$$

$$= abS \cdot E \cdot f^*K_X.$$

Hence we have $S \cdot E \cdot f^*K_X = 0$. Let $\{E_i\}$ be the irreducible components of $E$. Clearly $S \cdot E_i \cdot f^*K_X = 0$ for each $i$. By the way of taking $f$, supp$(E) = $supp$(E')$. Hence we have $S \cdot E' \cdot f^*K_X = 0$, since $S \cdot E_i \cdot f^*K_X = 0$ for each $i$.

Applying the Hodge index theorem to $S$, we have that $(E'|_{S})^2 = S \cdot E'|^2 \leq 0$ since $(f^*K_X|_{S})^2 = f^*K_X^2 \cdot S \geq 1$. We have $f^*K_X|_{S} + E'|_{S} \sim K_S \sim \pi^*K_{S'} + L$, where $L$ is an effective divisor supported on the exceptional locus of $\pi$. Hence the uniqueness of the Zariski decomposition implies that $f^*K_X|_{S} \sim \pi^*K_{S'}$.

Corollary. If $f^*K_X^2 \cdot S = 1$, then the minimal surface $S'$ of $S$ has $K_{S'}^2 = 1$, $q(S) = 0$ and $1 \leq p_g(S) \leq 2$.

Proof. $K_{S'}^2 = f^*K_X|_{S}^2 = 1$ by Proposition 2. Since $K_{S'}^2 = 1$, we have that $q(S) = 0$ and $p_g(S) \leq 2$. (For a reference, see Bombieri [1], p. 212.) Consider the exact sequence

$$0 \to O_{X'}(K_{X'}) \to O_{X'}(K_{X'} + M') \to \oplus O_{S}(K_{S}) \to 0.$$ 

The above exact sequence shows that $p_g(S) \neq 0$ since $M'$ is not fixed.

We have two facts from the condition $bc = 1$. The first one is that $C'$ is birational to the image $C$ of the pluricanonical map $\Phi_n$. So we may assume that $C$ is smooth since the terms we are interested in are birational invariants. The second one is that a general fiber of $g$ is a surface of general type with its irregularity 0 from Corollary of Proposition 2.

Proposition 3. Suppose that $bc = 1$. Then we have that $1 \leq p_g(S) \leq 2$ and $\chi(O_X) \leq (p_g(S) + 1)(1 - q_1)$. 

Proof. We have the fiber space \( g : X' \to C \) with connected fiber since \( bc = 1 \). By Corollary of Proposition 2, \( q(S) = 0 \) and \( 1 \leq p_g(S) \leq 2 \). We have \( R^1g_*K_{X'} = 0 \) since \( q(S) = h^1(S, \mathcal{O}_S(K_S)) = 0 \). By spectral sequence, we have that

\[
\begin{align*}
p_g &= h^0(X, \mathcal{O}_X(K_X)) = h^0(C, g_*K_{X'}) , \\
q_2 &= h^1(X, \mathcal{O}_X(K_X)) = h^1(C, g_*K_{X'}), \\
q_1 &= h^2(X, \mathcal{O}_X(K_X)) = h^0(C, R^2g_*K_{X'}).
\end{align*}
\]

Since \( R^2g_*K_{X'} = K_C, q_1 = p_g(C) \). (For a reference, see Kollár [6].) It is known that \( g_*K_{X'/C} \overset{\text{def}}{=} g_*(K_X \oplus g^*K_C^{-1}) \) is semipositive and locally free of rank \( p_g(S) \). So \( \deg g_*K_{X'/C} \geq 0 \). (See Kawamata [4].)

\[
\deg g_*K_{X'/C} = \deg g_*K_{X'} + p_g(S)(1 - p_g(C))
\]

\[
\geq p_g(S)(p_g(C) - 1).
\]

So \( p_g - q_2 \geq p_g(S)(q_1 - 1) \). Hence \( -\chi(O_X) = p_g - q_2 + q_1 - 1 \geq (p_g(S) + 1)(q_1 - 1) \).

Now we have \( \chi(O_X) \leq (p_g(S) + 1)(1 - q_1) \).

**Theorem 2.** If \( \dim \text{Im} \Phi_3 = 1 \), then the following hold:

(a) \(-10 \leq \chi(O_X) \leq \min\{-1, 2 - 2q_1\} \). Moreover, \( K_X^3 \geq 8 \).

(b) \( q_1 \leq \frac{1}{12} K_X^3 + 1 \).

Proof. The fact \( bc = 1 \) is given just above Proposition 2. If \( a \geq 2(p_3 - 1) \), then the inequality (1) implies \( 3K_X^3/2 \geq p_3 - 1 \). But it is impossible since \( p_3 = \frac{5}{2} K_X - 5 \chi(O_X) \). So \( bc = 1 \) and \( a < 2(p_3 - 1) \). Hence Lemma and Proposition 3 imply \( \chi(O_X) \leq \min\{-1, 2 - 2q_1\} \). From the inequality (1), we have \( -\frac{1}{10} K_X^3 - \frac{1}{5} \leq \chi(O_X) \). Moreover, since \( \chi(O_X) \leq -1, K_X^3 \geq 8 \).

For (b), by Proposition 1, \( p_g(C) \leq a - (p_3 - 1) \) since the degree \( a \) of \( C \) is less than \( 2(p_3 - 1) \). Recall that we have \( q_1 = p_g(C) \) in the proof of Proposition 3. We already know \( a \leq 3K_X^3 \) from (1). Hence

\[
q_1 = p_g(C) \leq a - (p_3 - 1) \leq 3K_X^3 - p_3 + 1 \leq 1/2K_X^3 + 5\chi(O_X) + 1 .
\]

From (a), \( q_1 \leq 1/2K_X^3 + 5(2 - 2q_1) + 1 \). Hence we have \( q_1 \leq \frac{1}{12} K_X^3 + 1 \).

**Theorem 3.** Suppose that \( \dim \text{Im} \Phi_2 = 1 \). If \( bc = 1 \), then we have the following:

(a) \( \chi(O_X) \leq \min\{-1, 2 - 2q_1\} \).

(b) \( q_1 \leq \frac{3}{14} K_X^3 + 1 \).

Proof. Lemma and Proposition 3 imply (a) clearly.

For (b), we already know \( p_2 - 1 \leq a \leq 2K_X^3 \) from (1). Hence it is enough to consider the following two cases:

**Case 1.** \( p_2 - 1 \leq a < 2(p_2 - 1) \).
Since $p_2 - 1 \leq a < 2(p_2 - 1)$, Proposition 1 implies that $p_g(C) \leq a - (p_2 - 1)$. Since $bc = 1$, the proof of Proposition 3 implies that $q_1 = p_g(C)$ and $\chi(O_X) \leq 2 - 2q_1$.

$$q_1 = p_g(C) \leq a - (p_2 - 1) \leq 2K_X^3 - p_2 + 1 = \frac{3}{2}K_X^3 + 3\chi(O_X) + 1 \leq \frac{3}{2}K_X^3 + 3(2 - 2q_1) + 1.$$ 

Hence we have $q_1 \leq \frac{3}{14}K_X^3 + 1$.

Case 2. $2(p_2 - 1) \leq a \leq 2K_X^3$.

Since $2(p_2 - 1) \leq a \leq 2K_X^3$, Proposition 1 implies that

$$p_g(C) \leq \frac{m(m - 1)}{2} (p_2 - 2) + mr,$$

where $m = \left\lfloor \frac{a - 1}{p_2 - 2} \right\rfloor$ and $a - 1 = m(p_2 - 2) + r$.

So, let’s compute $m$. Since $2(p_2 - 1) \leq a \leq 2K_X^3$, we have that

$$\frac{2(p_2 - 1) - 1}{p_2 - 2} \leq \frac{a - 1}{p_2 - 2} \leq \frac{2K_X^3 - 1}{p_2 - 2}.$$ 

If we modify the above inequalities, we have that

$$2 + \frac{1}{K_X^3/2 - 3\chi(O_X) - 2} \leq \frac{a - 1}{p_2 - 2} \leq \frac{12\chi(O_X) + 7}{K_X^3/2 - 3\chi(O_X) - 2}.$$ 

Since $\chi(O_X) < 0$, we have that

$$\frac{1}{K_X^3/2 - 3\chi(O_X) - 2} > 0 \text{ and } \frac{12\chi(O_X) + 7}{K_X^3/2 - 3\chi(O_X) - 2} < 0.$$ 

Hence we have $m = \left\lfloor \frac{a - 1}{p_2 - 2} \right\rfloor = 2$ or $3$.

When $m = 2$, we have that

$$q_1 = p_g(C) \leq \frac{2 \cdot 1}{2} (p_2 - 2) + 2r \leq (p_2 - 2) + 2(a - 1 - 2(p_2 - 2)) \leq 2a - 3p_2 + 4 \leq 4K_X^3 - 3(\frac{K_X^3}{2} - 3\chi(O_X)) + 4 \leq \frac{5}{2}K_X^3 + 9\chi(O_X) + 4 \leq \frac{5}{2}K_X^3 + 9(2 - 2q_1) + 4.$$ 

Hence $q_1 \leq \frac{5}{38}K_X^3 + \frac{22}{19}$.

When $m = 3$, similarly, we have that $q_1 \leq \frac{3}{37}K_X^3 + \frac{45}{37}$.

Therefore, combining all inequalities about $q_1$, we have $q_1 \leq \frac{3}{14}K_X^3 + 1$.  \[ \square \]
Remark. When \( \dim \text{Im} \Phi_2 = 1 \), we assume the condition \( bc = 1 \). At this moment, we don’t have a necessary and sufficient condition to guarantee \( bc = 1 \). But we have some cases which show \( bc = 1 \).

**Proposition 4.** Suppose that \( \dim \text{Im} \Phi_2 = 1 \). If \( K_X^3 < p_2 - 1 \), then we have that

(a) \( bc = 1 \),

(b) \( -K_X^3/2 - 1/3 \leq \chi(O_X) < -K_X^3/6 - 1/3 \).

**Proof.** Since \( K_X^3 < p_2 - 1 \leq a \), we have \( K_X^3 < p_2 - 1 \leq a \leq \frac{2K_X^3}{bc} \) from (1).

Hence \( bc \) must be 1.

From (1), we have \( -K_X^3/2 - 1/3 \leq \chi(O_X) \). Since \( K_X^3 < p_2 - 1 \), we have \( \chi(O_X) < -K_X^3/6 - 1/3 \). Combining these two inequalities, we have

\[ -K_X^3/2 - 1/3 \leq \chi(O_X) < -K_X^3/6 - 1/3. \]

\( \square \)

**Proposition 5.** Suppose that \( \dim \text{Im} \Phi_2 = 1 \). If \( 2(p_2 - 1) \leq a \), then we have

(a) \( bc = 1 \),

(b) \( -K_X^3/6 - 1/3 \leq \chi(O_X) \leq -1 \).

**Proof.** Since \( 2(p_2 - 1) \leq a \), we have \( 2(p_2 - 1) \leq a \leq 2K_X^3/bc \) from (1). If \( bc \geq 2 \), then we have \( 2(p_2 - 1) \leq K_X^3 \). But, since \( 2(p_2 - 1) = K_X^3 - 6\chi(O_X) - 2 > K_X^3 \), it is impossible. Hence \( bc = 1 \).

From \( 2(p_2 - 1) \leq a \leq 2K_X^3 \), we have \( -K_X^3/6 - 1/3 \leq \chi(O_X) \).

\( \square \)

Remark. When \( \dim \text{Im} \Phi_2 = 1 \), from (1), we may have the following three cases:

\( a \leq 2K_X^3 < 2(p_2 - 1) \), \( a \leq 2(p_2 - 1) \leq 2K_X^3 \), and \( 2(p_2 - 1) \leq a \leq 2K_X^3 \). The case we didn’t cover here is the second one \( p_2 - 1 \leq a \leq 2(p_2 - 1) \leq 2K_X^3 \).

**References**


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