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THE LUSIN-PRIVALOV THEOREM FOR SUBHARMONIC FUNCTIONS

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Abstract. This paper establishes a generalization of the Lusin-Privalov radial uniqueness theorem which applies to subharmonic functions in all dimensions. In particular, it answers a question of Rippon by showing that no subharmonic function on the upper half-space can have normal limit $-\infty$ at every boundary point.

1. Introduction

Let u be a subharmonic function on the upper half-plane D and let

$$A = \{x \in \mathbb{R}\colon u(x,y) \to -\infty \text{ as } y \to 0+\}.$$

Then $A \neq \mathbb{R}$. Indeed, the Lusin-Privalov radial uniqueness theorem for analytic functions [12] has a generalization for subharmonic functions on D (see [1], [3], [13]) which asserts that, if A is metrically dense in an open interval I (i.e. A has positive linear measure in each subinterval of I), then $A \cap I$ is of first category.

Rippon [13, Theorem 6] showed that this result breaks down in higher dimensions by constructing a subharmonic function u on $\mathbb{R}^2 \times (0,+\infty)$ such that

$$u(x_1,x_2,x_3) \to -\infty \qquad (x_3 \to 0+; (x_1,x_2) \in \mathbb{R}^2 \backslash E'),$$

where E' is a first category subset of $\mathbb{R}^2$ with zero area measure. A key observation here is that a line segment is polar in higher dimensions but not in the plane. One way around this problem is to replace normal limits by limits along translates of a somewhat "thicker" set, as in [13]. However, this leaves open the question, posed in [13, p. 479], of whether a subharmonic function on the upper half-space can have normal limit $-\infty$ at every boundary point. In this paper we give a negative answer to this question by establishing a suitable higher dimensional generalization of the Lusin-Privalov theorem.

The *fine topology* on $\mathbb{R}^n$ is the coarsest topology which makes every subharmonic function continuous. We refer to Doob [8, 1.XI] for its basic properties. Let U be a non-empty fine open set. A set A is said to be *metrically fine dense in* U if, for every non-empty fine open subset V of U, the set $A \cap V$ has positive outer λ_n-measure, where λ_n denotes Lebesgue measure on $\mathbb{R}^n$. Also, A is said to be of *first fine category* if it can be expressed as a countable union of sets E_k such that the fine closure of each E_k has empty fine interior. These definitions are given substance by

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the observations that every non-empty fine open set has positive λ_n-measure, and that the fine topology has the Baire property: see §3.2.

Points of $\mathbb{R}^n$ $(n \geq 2)$ will be denoted by X, or by (X', x) where $X' \in \mathbb{R}^{n-1}$, and the upper half-space $\mathbb{R}^{n-1} \times (0, +\infty)$ will be denoted by D. Our generalization of the Lusin-Privalov theorem is as follows.

Theorem 1. *Let u be a subharmonic function on D and let U' be a non-empty fine open subset of $\mathbb{R}^{n-1}$. If the set*

$$\{X' \in \mathbb{R}^{n-1} \colon \liminf_{x \to 0+} u(X', x) = -\infty\}$$

is metrically fine dense in U', then the set

$$\{X' \in U' \colon \limsup_{x \to 0+} u(X', x) < +\infty\}$$

is of first fine category (in $\mathbb{R}^{n-1}$).

Corollary 1. *If U' is a non-empty fine open subset of $\mathbb{R}^{n-1}$, then there is no subharmonic function u on D such that*

$$u(X', x) \to -\infty \qquad (x \to 0+; X' \in U').$$

The fine topology on $\mathbb{R}^n$ is strictly finer than the Euclidean one when $n \geq 2$. This is not true when $n = 1$, since the subharmonic functions on $\mathbb{R}$ are precisely the convex functions and hence are already continuous. Thus, when $n = 2$, Theorem 1 is only a slight refinement of the result cited at the beginning of the paper. However, when $n \geq 3$, Theorem 1 is new and, in view of Corollary 1, gives a negative answer to the question of Rippon noted above.

A variant of the Lusin-Privalov theorem, due to Barth and Schneider [2] has the following generalization to higher dimensions.

Theorem 2. *Let $f \colon (0, 1] \to \mathbb{R}$ be such that $f(x) \to -\infty$ as $x \to 0+$, and let u be a subharmonic function on D. Then the set*

$$E' = \{X' \in \mathbb{R}^{n-1} \colon \limsup_{x \to 0+} \{u(X', x) - f(x)\} < +\infty\}$$

is of first fine category.

Let ρ denote the metric on $[-\infty, +\infty]$ given by $\rho(x, y) = |\tan^{-1} x - \tan^{-1} y|$. The closure of a subset A of $[-\infty, +\infty]$ with respect to ρ will be denoted by $\overline{A}^{\rho}$. Given a point X' in $\mathbb{R}^{n-1}$ and a function $g \colon D \to [-\infty, +\infty]$ we define the *normal* and *fine cluster sets of g at* $(X', 0)$ by

$$C_N(g, X') = \bigcap_{t > 0} \overline{\{g(X', x) \colon 0 < x < t\}}^{\rho}$$

and

$$C_F(g, X') = \bigcap_{V} \overline{\{g(Y) \colon Y \in V \cap D\}}^{\rho}$$

respectively, where the latter intersection is over all fine neighbourhoods V of $(X', 0)$ in $\mathbb{R}^n$. The *minimal fine cluster set g at* $(X', 0)$, denoted by $C_M(g, X')$, is defined analogously with respect to the minimal fine topology for D (see [8, 1.XII] for an account of this topology).

Theorem 3. *If $g\colon D \to [-\infty, +\infty]$ is fine continuous, then there is a first fine category subset E' of $\mathbb{R}^{n-1}$ such that*

$$C_F(g, X') \subseteq C_N(g, X') \qquad (X' \in \mathbb{R}^{n-1} \backslash E').$$

Theorem 3 is similar in spirit to a classical result of Collingwood [4, p. 76] concerning boundary cluster sets of continuous functions on the unit disc. When $n = 2$ a stronger result than Theorem 3 is true; namely, $C_N(g, X')$ is equal to the full cluster set of g at $(X', 0)$ for all but a first category set of points X' in $\mathbb{R}$; see [13, Theorem 1].

Doob [7, Theorem 4.1] has shown that, for any function $g\colon D \to [-\infty, +\infty]$, the inclusion $C_N(g, X') \subseteq C_M(g, X')$ holds for λ_{n-1}-almost every X' in $\mathbb{R}^{n-1}$. In the opposite direction we can now give the following.

Corollary 2. *If $g\colon D \to [-\infty, +\infty]$ is fine continuous, then there is a first fine category subset E' of $\mathbb{R}^{n-1}$ such that*

$$C_M(g, X') \subseteq C_N(g, X') \qquad (X' \in \mathbb{R}^{n-1} \backslash E').$$

The arguments used to prove Theorem 3 and Corollary 2 form part of the proofs of Theorems 1 and 2, so we begin by proving Theorem 3 in §2. Theorems 1 and 2 are then proved in §3 and §4 respectively. Finally, we given an example relating to these results in §5. Since our results are new only when $n \geq 3$, we will restrict our attention to this case in what follows.

2. Proof of Theorem 3 and Corollary 2

2.1. A set A in $\mathbb{R}^n$ is said to be *thin* at a point X if X is not a fine limit point of A. We begin with two preparatory lemmas, the first of which is known (see [5, Lemme 2]).

Lemma 1. *Let $A' \subseteq \mathbb{R}^{n-1}$. Then $A' \times \mathbb{R}$ is polar in $\mathbb{R}^n$ if and only if A' is polar in $\mathbb{R}^{n-1}$.*

Lemma 2. *Let $A' \subseteq \mathbb{R}^{n-1}$ and $(Y', y) \in \mathbb{R}^n$. Then $A' \times \mathbb{R}$ is thin at (Y', y) if and only if A' is thin at Y'.*

To prove Lemma 2, suppose that A' is thin at Y'. If Y' is not a limit point of A', then there is a (Euclidean) neighbourhood U' of Y' such that $(U' \cap A') \times \mathbb{R}$ is contained in the polar set $\{Y'\} \times \mathbb{R}$, and so $A' \times \mathbb{R}$ is thin at (Y', y). If Y' is a limit point of A', then there is a superharmonic function u' on $\mathbb{R}^{n-1}$ such that

$$u'(Y') < \liminf_{\substack{X' \to Y' \\ X' \in A'}} u'(X').$$

If we define $u(X', x) = u'(X')$ on $\mathbb{R}^n$, then u is superharmonic and

$$u(Y', y) < \liminf_{\substack{X \to (Y', y) \\ X \in A' \times \mathbb{R}}} u(X),$$

whence again $A' \times \mathbb{R}$ is thin at (Y', y).

Conversely, suppose that $A' \times \mathbb{R}$ is thin at (Y', y), let ω' denote the open cube in $\mathbb{R}^{n-1}$ of side length $1/2$ centred at Y', and let v denote the Green function for $\omega' \times \mathbb{R}$ with pole at (Y', y). We extend v to all of $\mathbb{R}^n$ by assigning it the value 0 elsewhere. Thus v is subharmonic on $\mathbb{R}^n \backslash \{(Y', y)\}$. Noting that the function

$$\cos(\pi x_1)\cos(\pi x_2)\cdots\cos(\pi x_{n-1})\exp(-\sqrt{n-1}\pi|x|)$$

is positive and superharmonic on $(-1/2,1/2)^{n-1}\times\mathbb{R}$, we see easily that

$$v(X',x)\le c\exp(-\sqrt{n-1}\pi|x-y|)\qquad (X'\in\omega';|x-y|>1)$$

for some positive constant c. In particular, the function v' defined by

$$v'(X')=\int_{-\infty}^{+\infty}v(X',x)\,dx\qquad (X'\in\mathbb{R}^{n-1})$$

is finite on $\mathbb{R}^{n-1}\backslash\{Y'\}$. Further, it follows from [9, Theorems 1, 4] that v' is positive and superharmonic on ω' and subharmonic on $\mathbb{R}^{n-1}\backslash\{Y'\}$. Since $v'=0$ on $\mathbb{R}^{n-1}\backslash\omega'$, we conclude from Bôcher's theorem that $v'=as'$ for some $a>0$, where s' is the Green function for ω' with pole at Y'. Let w be the regularized reduced function (balayage) of v relative to $(A'\cap\omega')\times\mathbb{R}$ in $\omega'\times\mathbb{R}$, and let $w'(X')$ denote the integral of w over $\{X'\}\times\mathbb{R}$, for each X' in ω'. Then $w=v$ on $(A'\cap\omega')\times\mathbb{R}$, except perhaps for the polar subset of $(A'\cap\omega')\times\mathbb{R}$ where that set is thin. By translation invariance, this polar set is of the form $F'\times\mathbb{R}$, and it follows from Lemma 1 that F' is polar in $\mathbb{R}^{n-1}$. Hence w' is a non-negative superharmonic function on ω' which satisfies $w'\le v'$ on ω' and $w'=v'$ on $(A'\cap\omega')\backslash F'$. It follows that $w'\ge at'$, where t' denotes the regularized reduced function of s' relative to $A'\cap\omega'$ in ω'. However, $w\ne v$ since $A'\times\mathbb{R}$ is thin at (Y',y). Thus the superharmonic functions w and v must differ on a set of positive λ_n-measure in $\omega'\times\mathbb{R}$, and so $w'\ne v'$. Hence $t'\ne s'$, and it follows that A' is thin at Y'. This completes the proof of Lemma 2.

We note that one implication in Lemma 2 can be reformulated as follows.

Proposition 1. *The canonical projection from $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$ is a fine open mapping.*

To see this, let W be a fine open subset of $\mathbb{R}^n$, let W' denote its projection onto $\mathbb{R}^{n-1}$ and let $X'\in W'$. Then there exists x in $\mathbb{R}$ such that $(X',x)\in W$. The set $(\mathbb{R}^{n-1}\backslash W')\times\mathbb{R}$, being a subset of $\mathbb{R}^n\backslash W$, is thin at (X',x). Hence, by Lemma 2, $\mathbb{R}^{n-1}\backslash W'$ is thin at X'. It follows that W' is a fine open subset of $\mathbb{R}^{n-1}$.

2.2. We will now prove Theorem 3 using Lemma 2 and an argument of Hayman [13, pp. 472, 473]. Let $g\colon D\to[-\infty,+\infty]$ be fine continuous, let

$$E'=\{X'\in\mathbb{R}^{n-1}\colon C_F(g,X')\backslash C_N(g,X')\ne\varnothing\},$$

and let $\mathcal{I}$ denote the family of closed intervals of $[-\infty,+\infty]$ with endpoints in $\mathbb{Q}\cup\{-\infty,+\infty\}$. Now let $Y'\in E'$. Noting that $C_N(g,Y')$ is a compact subset of $[-\infty,+\infty]$, we can find I in $\mathcal{I}$, a finite union J of intervals from $\mathcal{I}$, and a positive rational number q such that $I\cap C_F(g,Y')\ne\varnothing$, such that

$$\{g(Y',y)\colon 0<y<q\}\subseteq J,$$

and such that $I\cap J=\varnothing$. If I, J and q are as above, then we will say that $Y'\in E'(I,J,q)$. Thus

$$E'\subseteq\bigcup_{I,J,q}E'(I,J,q),$$

where the union is over all possible choices of I, J and q as described above.

Now suppose that one of these sets, $E'(I_0,J_0,q_0)=A'$ say, has the property that its fine closure F' has non-empty fine interior V', and let $X'\in F'$. Then the fine closure of $A'\times(0,q_0)$ contains $F'\times(0,q_0)$, by Lemma 2. Hence, by fine continuity, $g(X',x)\in J_0$ whenever $(X',x)\in F'\times(0,q_0)$. The set $V'\cap A'$ is non-empty, so we can choose a point Z' in it. Then $V' \times (-q_0, q_0)$ is a fine neighbourhood of $(Z', 0)$, by Lemma 2, and so

$$C_F(g, Z') \subseteq J_0 \subseteq [-\infty, +\infty] \backslash I_0.$$

This contradicts the fact that $I_0 \cap C_F(g, Z') \neq \varnothing$. Thus each set $E'(I, J, q)$ must have the property that its fine closure has empty fine interior. It follows that E' is of first fine category, and so Theorem 3 is proved.

2.3. Corollary 2 follows from Theorem 3 and the fact (see [11, §6]) that, if a subset A of D is thin at a boundary point $(X', 0)$, then A is minimally thin at $(X', 0)$ with respect to D.

3. Proof of Theorem 1 and Corollary 1

3.1. Let u and U' be as in the statement of Theorem 1, let

$$E' = \{X' \in U' \colon \limsup_{x \to 0+} u(X', x) < +\infty\},$$

and let

$$E'_{j,k} = \{X' \in U' \colon u(X', x) \leq j \text{ when } 0 < x < k^{-1}\} \qquad (j, k \in \mathbb{N}).$$

Then

$$E' = \bigcup_{j,k} E'_{j,k}.$$

Suppose that E' is not of first fine category. Then there exist j_0 and k_0 such that the fine closure of E'_{j_0,k_0} has non-empty fine interior V'. Since u is fine continuous, we can use Lemma 2 (as we did in §2.2) to see that $u \leq j_0$ on $V' \times (0, k_0^{-1})$. Hence (cf. §2.3) the open set $\{X \in D \colon u(X) < j_0 + 1\}$ is a deleted minimal fine neighbourhood of each point of $V' \times \{0\}$. A result of Doob [6, Theorem 5.1] now shows that u has finite minimal fine limits at λ_{n-1}-almost every point of $V' \times \{0\}$. Since $C_N(u, X') \subseteq C_M(u, X')$ for λ_{n-1}-almost every X' in $\mathbb{R}^{n-1}$ (see §1), we deduce that

$$\lambda_{n-1}(\{X' \in V' \colon \liminf_{x \to 0+} u(X', x) = -\infty\}) = 0.$$

Hence, since $V' \cap U'$ is non-empty and fine open, the set

$$\{X' \in \mathbb{R}^{n-1} \colon \liminf_{x \to 0+} u(X', x) = -\infty\}$$

is not metrically fine dense in U'. This completes the proof of Theorem 1.

3.2. Corollary 1 follows from Theorem 1 in view of the following two facts.

(I) Any fine open set U in $\mathbb{R}^n$ has positive λ_n-measure. To see this, let B be an open ball centred at a point X of U, let v be the Green function for B with pole at X, and let w be the regularized reduced function of v relative to $B \backslash U$ in B. Then $w \not\equiv v$, so $w \neq v$ on a set of positive λ_n-measure, but $w = v$ λ_n-almost everywhere on $B \backslash U$.

(II) The fine topology on $\mathbb{R}^n$ has the Baire property; that is, the intersection of a countable collection of fine open fine dense sets is fine dense (see [8, 1.XI.1]).

4. Proof of Theorem 2

Let f, u and E' be as in the statement of Theorem 2. There is no loss of generality in assuming that f is continuous (see [13, §4]). The function g defined by $g(X', x) = u(X', x) - f(x)$ is then fine continuous on D. Let

$$E'_{j,k} = \{X' \in \mathbb{R}^{n-1}\colon u(X', x) \leq j + f(x) \text{ when } 0 < x < k^{-1}\} \qquad (j, k \in \mathbb{N}).$$

Thus

$$E' = \bigcup_{j,k} E'_{j,k}.$$

As in §3.1 we observe that, if E' is not of first fine category, then there exist j_0, k_0 and a non-empty fine open subset V' of $\mathbb{R}^{n-1}$ such that

$$u(X', x) \leq j_0 + f(x) \qquad (X' \in V'; 0 < x < k_0^{-1}).$$

This contradicts Corollary 1, so Theorem 2 is established.

5. An example

The following example (cf. [13, Theorem 6]) illustrates Theorems 1 and 2.

Example. There is a subset A' of $\mathbb{R}^{n-1}$ such that:
(i) A' is of first fine category and $\lambda_{n-1}(\mathbb{R}^{n-1} \backslash A') = 0$;
(ii) for any continuous function $f : (0, 1] \to \mathbb{R}$ there is a harmonic function u on D such that

$$\limsup_{x \to 0+} \{u(X', x) - f(x)\} \leq 0 \qquad (X' \in A');$$

(iii) there is a negative subharmonic function v on D which has limit $-\infty$ at each point of $(\mathbb{R}^{n-1} \backslash A') \times \{0\}$.

To see this, let (G_k) be a decreasing sequence of dense open subsets of $\mathbb{R}$ such that $\lambda_1(G_k) < k^{-1}$, let $A'_k = (\mathbb{R} \backslash G_k) \times \mathbb{R}^{n-2}$ and $A' = \bigcup_k A'_k$. Clearly $\lambda_{n-1}(\mathbb{R}^{n-1} \backslash A') = 0$. Also, $G_k \times \mathbb{R}^{n-2}$ is open and non-thin at each point of $\mathbb{R}^{n-1}$, so A' is of first fine category. Thus (i) holds. The set

$$E = \bigcup_{k=1}^{\infty} (A'_k \times (0, k^{-1}])$$

is relatively closed in D, and it is easy to see that $D \backslash E$ is non-thin at each point of E. Further, if D^* denotes the Alexandroff (one-point) compactification of D, then $D^* \backslash E$ is connected and locally connected. It now follows from a result of Shaginyan [14] (or see [10, Theorem 3.19]) that, given any continuous function $f\colon (0, 1] \to \mathbb{R}$ there is a harmonic function u on D such that

$$|u(X', x) - (f(x) - 1)| < 1 \qquad ((X', x) \in E).$$

Hence (ii) holds. Finally, (iii) follows from the fact that $(\mathbb{R}^{n-1} \backslash A') \times \{0\}$ has zero harmonic measure for D (see [8, 1.VIII.5(b)]).

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