ON SUMS AND PRODUCTS OF INTEGERS

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Abstract. Erdős and Szemerédi conjectured that if \( A \) is a set of \( k \) positive integers, then there must be at least \( k^2 - \varepsilon \) integers that can be written as the sum or product of two elements of \( A \). Erdős and Szemerédi proved that this number must be at least \( c_k^{1+\delta} \) for some \( \delta > 0 \) and \( k \geq k_0 \). In this paper it is proved that the result holds for \( \delta = 1/31 \).

1. A conjecture of Erdős and Szemerédi

Let \( h \geq 2 \), and let \( A_1, \ldots, A_h \) be finite sets of positive integers. We consider the sumset
\[
A_1 + \cdots + A_h = \{ a_1 + \cdots + a_h \mid a_i \in A_i \text{ for } i = 1, \ldots, h \}
\]
and the product set
\[
A_1 \cdots A_h = \{ a_1 \cdots a_h \mid a_i \in A_i \text{ for } i = 1, \ldots, h \}.
\]
If \( A_i = A \) for all \( i \), we let
\[
hA = \{ a_1 + \cdots + a_h \mid a_i \in A \text{ for } i = 1, \ldots, h \}
\]
derote the \( h \)-fold sumset of \( A \), and we let
\[
A^h = \{ a_1 \cdots a_h \mid a_i \in A \text{ for } i = 1, \ldots, h \}
\]
derote the \( h \)-fold product set of \( A \). We let
\[
E_h(A) = hA \cup A^h
\]
derote the set of all integers that can be written as the sum or product of \( h \) elements of \( A \).

Clearly, if \( |A| = k \), then
\[
|hA| \leq \binom{k+h-1}{h}
\]
and
\[
|A^h| \leq \binom{k+h-1}{h}.
\]
and so the number of sums and products of \( h \) elements of \( A \) is
\[
|E_h(A)| \leq 2 \binom{k + h - 1}{h} = \frac{2k^h}{h!} + O(k^{h-1}).
\]

Erdős and Szemerédi [1, 3] have made the beautiful conjecture that a finite set of positive integers cannot have simultaneously few sums and few products. More precisely, they conjectured that for every \( \varepsilon > 0 \) there exists an integer \( k_0(\varepsilon) \) such that, if \( A \) is a finite set of positive integers and
\[
|A| = k \geq k_0(\varepsilon),
\]
then
\[
|E_h(A)| \gg k^{h-\varepsilon}.
\]

Nothing is known about this conjecture for \( h \geq 3 \).

For \( h = 2 \), Nathanson and Tenenbaum [4] have proved that if \( |A| = k \) and \( |2A| \leq 3k - 4 \), then
\[
|A^2| \gg k^{2-\varepsilon}.
\]

This is the only case in which the full conjecture has been proven.

For an arbitrary set of \( k \) positive integers, Erdős and Szemerédi [3] have shown that there exists a real number \( \delta > 0 \) such that
\[
|E_2(A)| \gg k^{1+\delta}.
\]

Erdős [2] recently observed that “our paper with Szemerédi has nearly been forgotten.” The purpose of this paper is to give a careful version of the Erdős-Szemerédi proof that allows the explicit calculation of an exponent \( \delta \).

**Notation.** For any set \( A \) of integers, let \( |A| \) denote the cardinality of the set \( A \), let \( \max(A) \) denote the largest element of \( A \), and let \( \min(A) \) denote the smallest element of \( A \). For \( x \in \mathbb{R} \), let \( \lfloor x \rfloor \) denote the largest integer not exceeding \( x \). Note that \( \lfloor x \rfloor > x/2 \) if \( x \geq 2 \). Let \( [x_1, x_2) = \{ n \in \mathbb{Z} \mid x_1 \leq n < x_2 \} \).

2. Sets of small diameter

In this section we obtain a result in the special case of sets of small diameter, and in the next section we show that the main theorem reduces to this special case.

**Lemma 1.** Let \( B \) be a nonempty, finite set of positive integers such that
\[
\max(B) \leq 2 \min(B).
\]

Then
\[
|E_2(B)| \geq \left( \frac{|B|}{384} \right)^{16/15}.
\]

**Proof.** Let \( |B| = k \). If \( k < 384 \), the inequality is trivial, so we can assume that
\[
k \geq 384 = 2^5 12.
\]

Then
\[
(k/12)^{1/5} \geq 2.
\]

Let
\[
l = \left\lfloor \left( \frac{k}{12} \right)^{1/5} \right\rfloor.
\]
Then
\[
I \geq \frac{1}{2} \left( \frac{k}{12} \right)^{1/5} = \left( \frac{k}{384} \right)^{1/5}.
\]
Since
\[
k \geq 12l^5,
\]
it follows that
\[(1) \quad \left[ \frac{k}{l} \right] \geq 12l^4.
\]
Let \(B = \{b_1, \ldots, b_k\}\), where
\[
1 \leq b_1 < b_2 < \cdots < b_k \leq 2b_1.
\]
For \(i = 1, 2, \ldots, \lfloor k/l \rfloor\), let
\[
B_i = \{b_{(i-1)l+1}, b_{(i-1)l+2}, \ldots, b_{il}\} \subseteq B
\]
and
\[
d_i = b_{il} - b_{(i-1)l+1}.
\]
Choose \(i_0\) so that
\[
d_{i_0} = \min\{d_i \mid i = 1, \ldots, \lfloor k/l \rfloor\},
\]
and let
\[
B^* = B_{i_0}
\]
and
\[
d^* = d_{i_0}.
\]
Suppose that
\[
1 \leq i < j \leq \left[ \frac{k}{l} \right] \quad \text{and} \quad j - i \geq 3.
\]
If
\[
b_1^*, b_2^* \in B^* \quad \text{and} \quad b_i^* \in B_i, b_j^* \in B_j,
\]
then
\[
x^* = b_2^* - b_1^* \leq d^*
\]
and
\[
x = b_j^* - b_i^* > d_{i+1} + d_{i+2} + \cdots + d_{j-1} \geq 2d^* > 0.
\]
It follows that
\[
b_1^* + b_j^* = b_2^* + b_i^*.
\]
Suppose that
\[
b_1^* b_j^* = b_2^* b_i^*.
\]
Since \(b_i^* < b_j^*\), it follows that \(b_2^* > b_1^*\) and so \(x^* > 0\). Since
\[
b_j^* \leq b_k \leq 2b_1 \leq 2b_1^*,
\]
it follows that
\[
b_1^* b_j^* = b_2^* b_i^* = (b_1^* + x^*)(b_j^* - x) = b_1^* b_j^* + x^* b_j^* - x b_1^* - x^* x,
\]
and so
\[
0 < x^* x = x^* b_j^* - x b_1^* \leq b_1^*(2x^* - x) \leq b_1^*(2d^* - x) < 0,
\]
which is absurd. Therefore,
\[
b_1^* b_j^* \neq b_2^* b_i^*.
\]
It follows that

\((B^* + B_j) \cap (B^* + B_i) = \emptyset\)

and

\((B^* \cdot B_j) \cap (B^* \cdot B_i) = \emptyset\)

for every pair \(i, j\) of integers such that \(j - i \geq 3\).

We shall consider only the sets \(B_1, B_4, B_7, \ldots\), that is, the sets \(B_i\) such that \(i \equiv 1 \pmod{3}\). There are at least

\(\left\lceil \frac{k}{3} \right\rceil\)

such sets. Let

\(0 < \theta < 1\)

and

\(\beta = \theta/3 < 1/3\).

Let

\(E(B^*, B_i) = (B^* + B_i) \cup (B^* \cdot B_i)\),

and let

\[I_1 = \{ i \equiv 1 \pmod{3} | |E(B^*, B_i)| < l^{1+\beta} \}\]

and

\[I_2 = \{ i \equiv 1 \pmod{3} | |E(B^*, B_i)| \geq l^{1+\beta} \}\]

Then

\(|I_1| + |I_2| \geq \frac{1}{3} \left\lceil \frac{k}{7} \right\rceil\).

Suppose that

\(|I_1| \geq \frac{1}{6} \left\lceil \frac{k}{7} \right\rceil\).

Let \(i \in I_1\). For \(m \in B^* \cdot B_i\), let \(\rho(m)\) denote the number of representations of \(m\) in the form \(b^*b'_i\), where \(b^* \in B^*\) and \(b'_i \in B_i\). Choose \(m'\) such that

\(\rho(m') = \max\{\rho(m) | m \in B^* \cdot B_i\}\).

Since \(|B^*| = |B_i| = l\), it follows that

\(l^2 = \sum_{m \in B^* \cdot B_i} \rho(m) \leq \rho(m')|B^* \cdot B_i| < \rho(m')l^{1+\beta}\),

and so

\(\rho(m') > l^{1-\beta}\).

For \(j = 1, \ldots, \rho(m')\), choose \(b_j^* \in B^*\) and \(b_j' \in B_i\) such that

\(b_j^*b_j' = m'\)

and \(b_j^* \neq b_{j'}^*\) for \(j_1 \neq j_2\). There are \(\rho(m')^2\) expressions of the form

\(b_j^* + b_{j'}' \in B^* + B_i\),

where \(j, j' = 1, \ldots, \rho(m')\). Since \(i \in I_1\) and \(\beta < 1/3\), it follows that

\(\rho(m')^2 > l^{2-2\beta} > l^{1+\beta} > |B^* + B'_i|\),

and so there exist \(b_{j_1}^*, b_{j_2}^* \in B^*\) and \(b_{j_3}', b_{j_4}' \in B_i\) such that \(b_{j_1}^* \neq b_{j_2}^*\) and

\(b_{j_1}^* + b_{j_3}' = b_{j_2}^* + b_{j_4}'\).
It follows from (5) that also

\[(7) \quad b'_{j_3} = b'_{j_4}.\]

What we have just shown is that for every \(i \in I_1\) there exist four positive integers \(b^*_j, b^*_j, b_j, b_j \in B^*\) and two positive integers \(b'_{j_3}, b'_{j_4} \in B_i\) that satisfy equations (6) and (7). However, given any positive integers \(b^*_j, b^*_j, b_j, b_j \in B^*\) and two positive integers \(b'_{j_3}, b'_{j_4} \in B_i\) equations (6) and (7) have at most one solution in integers \(b'_{j_3}, b'_{j_4}\). Since the number of quadruples of elements of \(B^*\) is exactly \(l^4\), if \(|I_1| \geq \frac{1}{6} \left\lfloor \frac{k}{l} \right\rfloor\), then \(l^4 \geq |I_1| \geq \frac{1}{6} \left\lfloor \frac{k}{l} \right\rfloor \geq 2l^4\), which is absurd. Therefore,

\[ |I_1| < \frac{1}{6} \left\lfloor \frac{k}{l} \right\rfloor, \]

and so, by (4), we have

\[ |I_2| \geq \frac{1}{6} \left\lfloor \frac{k}{l} \right\rfloor. \]

Let

\[ n \in \bigcup_{i \in I_2} E(B^*, B_i). \]

It follows from (2) and (3) that \(n\) belongs to at most two of the sets \(E(B^*, B_i)\). Therefore,

\[
|E_2(B)| \geq \left| \bigcup_{i \in I_2} E(B^*, B_i) \right| \geq (1/2) \sum_{i \in I_2} |E(B^*, B_i)| \geq (1/2)|I_2|^{1+\beta} \geq (1/12) \left\lfloor \frac{k}{l} \right\rfloor^{1+\beta} \geq (1/12)(12l^4)^{l^{1+\beta}} = l^{5+\beta} \geq \left( \frac{k}{384} \right)^{1+\beta/5} = \left( \frac{k}{384} \right)^{1+\theta/15}.
\]

Since this holds for all \(\theta < 1\), we obtain

\[ |E_2(B)| \geq \left( \frac{k}{384} \right)^{16/15}. \]

This completes the proof of the lemma.

3. The main result

**Theorem 1.** Let \(A\) be a nonempty, finite set of positive integers. Then

\[ |E_2(A)| \geq c|A|^{32/31}, \]

where \(c = 0.00028\ldots\).

**Proof.** For \(j = 1, 2, \ldots\), let

\[ U_j = [2^{j-1}, 2^j) \]

and

\[ V_j = [4^{j-1}, 4^j) = U_{2j-1} \cup U_{2j}. \]

Let

\[ A_j = A \cap U_j = \{a \in A \mid 2^{j-1} \leq a < 2^j\} \]

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for \( j = 1, 2, \ldots \). Then \( A = \bigcup_{j=1}^{\infty} A_j \), the sets \( A_j \) are pairwise disjoint, and
\[
\sum_{j=1}^{\infty} |A_j| = k.
\]

Let \( \alpha > 0 \), and let
\[
c_1 = (384)^{-16/15}
\]
and
\[
c_2(\alpha) = \frac{c_1}{3} \cdot 2^{1+\alpha/15} < c_1 < \frac{1}{32}.
\]

There are two cases. In the first case, we assume that if \( A_j \neq \emptyset \), then
\[
|A_j| \geq k^\alpha.
\]

Since \( \max(A_j) \leq 2 \min(A_j) \), the set \( A_j \) satisfies the conditions of the lemma, and so
\[
|E_2(A_j)| \geq c_1 |A_j|^{16/15}.
\]

Let
\[
n \in \bigcup_{j=1}^{\infty} E_2(A_j).
\]

There exists a unique integer \( t \) such that
\[
n \in V_t = U_{2t-1} \cup U_{2t}.
\]

Observe that if \( a, a' \in A_j \), then \( a + a' \in U_{j+1} \) and \( aa' \in V_j \). Suppose that \( n \in E_2(A_j) \). If \( n \) is a product of two elements of \( A_j \), then \( n \in V_j \) and so \( j = t \). If \( n \) is a sum of two elements of \( A_j \), then \( n \in U_{j+1} \), and so \( j = 2t - 2 \) or \( 2t - 1 \). Therefore, \( n \) belongs to at most three of the sets \( E_2(A_j) \). It follows that
\[
|E_2(A)| \geq \left| \bigcup_{j=1}^{\infty} E_2(A_j) \right|
\]
\[
\geq (1/3) \sum_{j=1}^{\infty} |E_2(A_j)|
\]
\[
\geq (1/3) \sum_{j=1}^{\infty} c_1 |A_j|^{16/15}
\]
\[
= (c_1/3) \sum_{j=1}^{\infty} |A_j| \cdot |A_j|^{1/15}
\]
\[
\geq (c_1/3) k^{\alpha/15} \sum_{j=1}^{\infty} |A_j|
\]
\[
= (c_1/3) k^{\alpha/15} k
\]
\[
> c_2(\alpha) k^{1+\alpha/15}.
\]

In the second case, there exist sets \( A_j \) such that
\[
0 < |A_j| < k^\alpha.
\]

Let
\[
J = \{ j \mid 0 < |A_j| < k^\alpha \}.
\]
If

\[ \left| \bigcup_{j \in J} A_j \right| < k/2, \]

let

\[ A' = A \setminus \left( \bigcup_{j \in J} A_j \right), \]

and let

\[ A'_j = A' \cap U_j = \{ a \in A' \mid 2^{j-1} \leq a < 2^j \}. \]

Then

\[ k/2 < |A'| = k' \leq k. \]

If \( A'_j \neq \emptyset \), then \( A'_j = A_j \) and

\[ |A'_j| = |A_j| \geq k^\alpha \geq (k')^\alpha. \]

Therefore, we can apply the previous case to the set \( A' \), and obtain

\[ |E_2(A)| \geq |E_2(A')| \geq (c_1/3)(k')^{1+\alpha/15} > c_2(\alpha)k^{1+\alpha/15}. \]

On the other hand, if

\[ k/2 \leq \left| \bigcup_{j \in J} A_j \right| < |J|k^\alpha, \]

then

\[ |J| > k^{1-\alpha}/2. \]

Let \( j_1 < j_2 < j_3 < \cdots \) be the elements of \( J \) arranged in increasing order, and choose

\[ a^*_1 \in A_{j_1}, a^*_2 \in A_{j_3}, a^*_3 \in A_{j_3}, \ldots. \]

Let

\[ A^* = \{ a^*_i \mid i = 1, 3, 5, \ldots \} \subseteq A. \]

Then

\[ |A^*| \geq |J|/2 > k^{1-\alpha}/4. \]

Since \( a^*_i \in A_{j_i} \), it follows that

\[ 2a^*_i < 2^{j_i+1} \leq 2^{j_{i+2}} \leq 2^{j_{i+1}+2} \leq a^*_{i+2}, \]

and so the sums of distinct pairs of elements of \( A^* \) are distinct. Therefore,

\[ |E_2(A)| \geq |E_2(A^*)| \geq |2A^*|^2/2 > k^{2-2\alpha}/32 > c_2(\alpha)k^{2-2\alpha}. \]

Choose

\[ \alpha = 15/31. \]

Then

\[ 2 - 2\alpha = 1 + \frac{\alpha}{15}, \]

and we obtain

\[ |E_2(A)| \geq c k^{32/31}, \]

where

\[ c = c_2(15/31) = \frac{1}{6 \cdot (384)^{16/15} \cdot 2^{1/31}} = 0.00028 \ldots. \]

This completes the proof of the theorem.
REFERENCES


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