STABILITY AND ALMOST PERIODICITY OF SOLUTIONS OF ILL-POSED ABSTRACT CAUCHY PROBLEMS

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Abstract. We give simple spectral sufficient conditions for a solution of the linear abstract Cauchy problem, on a Banach space, to be strongly stable or asymptotically almost periodic, without assuming that the associated operator generates a $C_0$-semigroup.

1. Introduction

Let $A$ be a closed linear operator on a Banach space $X$. We shall consider the many physical problems that may be modelled as an abstract Cauchy problem

\[
\begin{align*}
  u'(t) &= Au(t), \quad t \geq 0, \\
  u(0) &= x.
\end{align*}
\]  

(1)

Many results on strong stability and almost periodicity of $t \mapsto u(t)$, when $A$ generates a $C_0$-semigroup, may be found in [1, 2, 7, 13, 14, 16, 17] and [18], to name only a few references. The hypothesis that $A$ generate a $C_0$-semigroup is saying that (1) is well-posed.

We assume in this paper only that all exponentially bounded solutions of Eq. (1) are unique. We do not assume that $A$ generates a $C_0$-semigroup; this is what we mean by saying that (1) is ill-posed. Thus we are considering, as in [2], individual solutions of (1), but without any global well-posedness; our results are truly local in character.

When the spectrum of $A$ on the imaginary axis is countable, we characterize bounded uniformly continuous solutions of (1) that are asymptotically almost periodic, or strongly stable, in terms of the means of $t \mapsto e^{-\lambda t}u(t)$, for purely imaginary $\lambda$ in the spectrum of $A$ (Theorem 4). Theorem 5 is a local version of the theorem of Katznelson-Tzafriri for $C_0$-semigroups. Finally, when the spectrum of $A$ does not intersect the imaginary axis, we show that any uniformly continuous mild solution of (1) is strongly stable (Proposition 7).

Our technique is to use the Hille-Yosida space (see Section 2), introduced, independently, in [10] and [12], as constructed in [3, Chapter V], to deduce a local result from a global result. More specifically, if $t \mapsto u(t)$ is a bounded, uniformly continuous mild solution of (1), then $x$ is in the Hille-Yosida space, on which $A$
generates a $C_0$-semigroup of contractions. We may then apply results about the asymptotic behaviour of orbits of a $C_0$-semigroup.

Throughout this paper, $A$ is a closed linear operator in a Banach space $X$, $T(t)$ is a $C_0$-semigroup in $X$. The spectrum, point spectrum and resolvent set of $A$ are denoted by $\sigma(A)$, $P\sigma(A)$ and $\rho(A)$, respectively.

2. Preliminaries

We consider the abstract Cauchy problem (1), where $A$ is a closed linear operator on a Banach space $X$. We assume throughout the paper that $A$ is such that all exponentially bounded solutions (i.e. all solutions $u(t)$ such that $\|u(t)\| \leq Me^{\alpha t}$ for some $M$ and $\alpha$) of (1) are unique. For instance, it is enough to require that there exists $\lambda_0 \in \mathbb{R}$ such that

$$(\lambda - A)$$ is injective for $\lambda \geq \lambda_0$ (see [12]).

A function $u(t), t \geq 0$, is called a strong solution of Eq. (1), if $u \in C^1(\mathbb{R}, X)$, $u(t) \in \mathcal{D}(A)$ for all $t \geq 0$ and satisfies (1). To introduce the notion of Hille-Yosida space, we need the following definition of mild solutions of Eq. (1) (see [3]).

**Definition.** A continuous function $u : \mathbb{R}^+ \rightarrow X$ is called a mild solution of Eq. (1), if $v(t) \equiv \int_0^t u(s)ds \in \mathcal{D}(A)$ and satisfies $\frac{d}{dt}v(t) = Av(t) + x$, for all $t \geq 0$.

In the sequel, it will sometime be convenient to write this solution as $u(t, x)$.

Let $Z_0$ be the set of all $x$ for which Eq. (1) has a bounded uniformly continuous mild solution $u(t, x), t \geq 0$.

We will write $X \hookrightarrow Y$ to mean that there is a continuous embedding of $X$ into $Y$. If $A$ is an operator on $Y$, we will write $A|_X$ to mean the restriction of $A$ to $X$. Finally, we introduce a norm in $Z_0$ by

$$\|x\|_{Z_0} = \sup_{t \geq 0} \|u(t, x)\|.$$

**Lemma 1.** The space $Z_0$ has the following properties:

1. $Z_0$ is a Banach space;
2. $Z_0 \hookrightarrow X$;
3. $A|_{Z_0}$ generates a $C_0$-semigroup of contractions $T(t)$, given by

$$T(t)x = u(t, x) \quad (t \geq 0);$$
4. $Z_0$ is maximal-unique, i.e., if $W$ is a Banach space such that $W \hookrightarrow X$ and $A|_W$ generates a bounded $C_0$-semigroup, then $W \hookrightarrow Z_0$; and
5. if $B \in B(X)$ and $BA \subseteq AB$, then $B \in B(Z_0)$, with $\|B\|_{Z_0} \leq \|B\|$.

**Proof.** The proof of (1)–(4) can be found in [10], or [3], so we need only to show (5). Since $B$ commutes with $A$, for any $x \in Z_0$,

$$v(t) \equiv Bu(t, x) \quad (t \geq 0)$$

is a mild solution of (1) with the initial value $v(0) = Bx$; that is, $BZ_0 \subseteq Z_0$, with

$$\|Bx\|_{Z_0} = \sup_{t \geq 0} \|u(t, Bx)\| = \sup_{t \geq 0} \|Bu(t, x)\| \leq \|B\|\|x\|_{Z_0} \quad (x \in Z_0).$$

The space $Z_0$, which was introduced in [12] and [10] (independently) is called the Hille-Yosida space for $A$ (cf. [3, 4]).
From Lemma 1-(5) the following corollary immediately follows.

**Corollary 2.** \( \sigma(A|z_0) \subseteq \sigma(A) \).

## 3. Main results

Let \( T(t) \) be a bounded \( C_0 \)-semigroup, with generator \( A \). It is known that the spectrum \( \sigma(A) \) of \( A \) is contained in the closed left half-plane \( \{ \lambda \in \mathbb{C} : \text{Re} \lambda \leq 0 \} \). For \( \lambda \in i\mathbb{R} \), let \( X_\lambda(A) = \{ x \in X : Ax = \lambda x \} \), and \( X_\lambda(A^*) = \{ \phi \in X^* : A^*\phi = \lambda \phi \} \). If \( x \in X_\lambda(A) \) and \( x \neq 0 \), then, by a simple argument using the Markov-Kakutani Fixed Point Theorem, there exists \( \phi \in X_\lambda(A^*) \) such that \( (x, \phi) \neq 0 \). The semigroup \( T(t) \) is called *mean ergodic*, if \( \frac{1}{R} \int_0^R T(t) dt \) converges strongly as \( R \to \infty \) (in this case the limit operator \( P \) is the projection operator from \( X \) onto the subspace \( \{ x \in X : T(t)x = x, \forall t \geq 0 \} \)). By the well known Mean Ergodic Theorem (see e.g. [8]), the semigroup \( \{ T(t) \} \) is mean ergodic if and only if \( \ker(A) + \text{ran}(A) \) is dense in \( X \). It is easy to see that the latter condition is equivalent to the following: \( X_0(A^*) \) separates points of \( X_0(A) \), i.e. for every \( \phi \in X_0(A^*) \) there exists \( x \in X_0(A) \) such that \( (x, \phi) \neq 0 \).

The semigroup \( T(t) \) is called *almost periodic*, if the trajectory \( \{ T(t)x : t \geq 0 \} \) is relatively compact in \( X \) for every \( x \in X \). It is well known that \( T(t) \) is an almost periodic semigroup if and only if, for every \( x \in X \), the function \( u(t) = T(t)x, t \geq 0 \), is asymptotically almost periodic, i.e. there exists an almost periodic function \( v : \mathbb{R} \to X \) such that \( \| u(t) - v(t) \| \to 0 \) as \( t \to \infty \). Moreover, if \( T(t) \) is an almost periodic semigroup, then \( e^{-\lambda t}T(t) \) is ergodic for every \( \lambda \in i\mathbb{R} \) (consequently, \( X_\lambda(A^*) \) separates points of \( X_\lambda(A) \), for every \( \lambda \in P\sigma(A^*) \cap i\mathbb{R} \). Let \( P_\lambda \) denote the corresponding projection operator which is the strong limit, as \( R \to \infty \), of \( \frac{1}{R} \int_0^R e^{-\lambda t}T(t) dt \). Then \( T(t) \) is strongly stable, i.e. \( \| T(t)x \| \to 0 \) as \( t \to \infty \), \( \forall x \in X \), if and only if it is almost periodic and \( P_\lambda = 0 \) for every \( \lambda \in P\sigma(A) \cap i\mathbb{R} \) (see [5, 16]).

In [16] the following almost periodicity theorem has been established: if \( \{ T(t) \} \) is a bounded \( C_0 \)-semigroup, such that \( \sigma(A) \cap i\mathbb{R} \) is countable, and if \( X_\lambda(A^*) \) separates points of \( X_\lambda(A) \) for each \( \lambda \in P\sigma(A^*) \cap i\mathbb{R} \), then \( T(t) \) is almost periodic.

From this theorem we obtain the following result on almost periodicity of individual trajectories.

**Proposition 3.** Let \( \{ T(t) \}_{t \geq 0} \) be a bounded \( C_0 \) semigroup with generator \( A \) such that \( \sigma(A) \cap i\mathbb{R} \) is countable, and let \( x \) be a vector in \( X \). Then

(i) the function \( u(t) = T(t)x \) is asymptotically almost periodic if (and only if) for each \( \lambda \in \sigma(A) \cap i\mathbb{R} \), the function \( e^{-\lambda t}u(t) \) has convergent means; and

(ii) the function \( u(t) \) converges to zero strongly as \( t \to \infty \) if (and only if) for each \( \lambda \in \sigma(A) \cap i\mathbb{R} \), the function \( e^{-\lambda t}u(t) \) has convergent means with the limit equal to 0.

**Proof.** (i) Let \( Y = \overline{\text{span}} \{ T(t)x : t \geq 0 \} \). Then \( Y \) is a closed invariant subspace of \( T(t) \). It is easy to see that \( \sigma(A|Y) \cap i\mathbb{R} \subseteq \sigma(A) \cap i\mathbb{R} \), so it is countable. From the condition it follows that

\[
\frac{1}{R} \int_0^R e^{-\lambda t}T(t) y dt
\]

converges strongly, as \( R \to \infty \), for each \( y \in \text{span} \{ T(t)x : t \geq 0 \} \), and hence it also converges strongly for each \( y \in \overline{\text{span}} \{ T(t)x : t \geq 0 \} \). Therefore, by the quoted
almost periodicity theorem of Lyubich and Vũ Quốc Phong [16], \( T(t)|_Y \) is an almost periodic semigroup, hence \( T(t)x \) is an asymptotically almost periodic function.

(ii) If, in addition,
\[
\lim_{R \to \infty} \frac{1}{R} \int_0^R e^{-\lambda t} u(t) dt = 0
\]
for all \( \lambda \in P\sigma(A) \cap i\mathbb{R} \), then
\[
\lim_{R \to \infty} \frac{1}{R} \int_0^R e^{-\lambda t} T(t) y dt = 0
\]
for all \( y \in Y \), which implies that \( P \lambda (A|_Y) = 0 \) for all \( \lambda \in P\sigma(A|_Y) \cap i\mathbb{R} \), so that \( T(t) y \to 0 \) for all \( y \in Y \). In particular, \( u(t) \to 0 \) as \( t \to \infty \).

Using Proposition 3 and the Hille-Yosida space, we obtain the following result, where we say that a function \( \omega \) has uniformly convergent means if
\[
\lim_{T \to \infty} \frac{1}{R} \int_a^{R+a} \omega(s) ds
\]
exists, uniformly in \( a \geq 0 \).

**Theorem 4.** Suppose \( \sigma(A) \cap i\mathbb{R} \) is countable and \( u(t) \), \( t \geq 0 \), is a bounded uniformly continuous mild solution of Eq. (1). Then

1. \( u(t) \) is asymptotically almost periodic if (and only if) for every \( \lambda \in \sigma(A) \cap i\mathbb{R} \), the function \( e^{-\lambda t} u(t) \) has uniformly convergent means;
2. \( u(t) \) converges strongly to 0 as \( t \to \infty \) if (and only if) for each \( \lambda \in \sigma(A) \cap i\mathbb{R} \), the function \( e^{-\lambda t} u(t) \) has uniformly convergent means with the limit equal to 0.

**Proof.** Let \( Z_0 \) be the Hille-Yosida space and \( T(t) \) be the semigroup generated by \( A|_{Z_0} \). By Corollary 2, \( \sigma(A|_{Z_0}) \cap i\mathbb{R} \) is countable. We will show that, for \( \lambda \in \sigma(A) \cap i\mathbb{R} \), \( e^{-\lambda t} T(t) x = e^{-\lambda t} u(t) \) has uniformly convergent means as a function from \( \mathbb{R}^+ \) to \( Z_0 \).

Fix \( \epsilon > 0 \). There exists \( T_\epsilon \) such that
\[
\| \frac{1}{T} \int_h^{T+h} e^{-\lambda t} u(t) dt - \frac{1}{S} \int_h^{S+h} e^{-\lambda t} u(t) dt \| < \epsilon,
\]
for all \( S, T > T_\epsilon, h > 0 \). From (2) it follows that.
\[
\| \frac{1}{T} \int_h^{T+h} e^{-\lambda t} u(t) dt - \frac{1}{S} \int_h^{S+h} e^{-\lambda t} u(t) dt \|_{Z_0} \leq \epsilon,
\]
\[
\sup_{s \geq 0} \| u(s, \frac{1}{T} \int_h^{T+h} e^{-\lambda t} u(t) dt - \frac{1}{S} \int_h^{S+h} e^{-\lambda t} u(t) dt) \|
\]
\[
= \sup_{s \geq 0} \| \frac{1}{T} \int_h^{T+h} e^{-\lambda t} u(t + s) dt - \frac{1}{S} \int_h^{S+h} e^{-\lambda t} u(t + s) dt \|
\]
\[
= \sup_{s \geq 0} \| \frac{1}{T} \int_h^{T+h+s} e^{-\lambda t} u(t) dt - \frac{1}{S} \int_h^{S+h+s} e^{-\lambda t} u(t) dt \|
\]
\[
\leq \epsilon,
\]
since the convergence is uniform in \( h \).
Thus \( e^{-\lambda t} u(t) = e^{-\lambda t} T(t) x \) has uniformly convergent means, for any \( \lambda \in \sigma(A|Z_0) \cap i\mathbb{R} \). Now the statements (1)–(2) follow from Proposition 3 and the continuous embedding \( Z_0 \hookrightarrow X \).

Theorem 4 is a generalization of the above mentioned result of Lyubich and Vũ Quốc Phong [16], but the proof is based on this result and the Hille-Yosida space.

Part (2) of Theorem 4 is analogous to, but independent of, [2, Theorem 1].

An important corollary of this result, which was obtained independently by Arendt and Batty [1] (see also [14]) (and is sometimes known as the ABLP Theorem), states that \( \|T(t)x\| \to 0 \) as \( t \to \infty \) for all \( x \in X \), if \( \sigma(A) \cap i\mathbb{R} \) is countable and \( P\sigma(A^*) \cap i\mathbb{R} \) is empty.

We note that the results presented in Theorem 4 are new even for the case when \( A \) is a generator of a \( C_0 \)-semigroup (and even for bounded \( A \)). In this case, Theorem 4-(1) (resp., (2)) gives a condition for asymptotic almost periodicity (resp., stability) of individual trajectories of \( \sigma^{\text{ind}} \) semigroups.

Our next result is an individual version of the theorem of Katznelson-Tzafriri type obtained in [7, 17] (independently) for \( C_0 \)-semigroups. A function \( f \in L^1(\mathbb{R}) \) is said to be a \textit{function of spectral synthesis with respect to a closed subset} \( \triangle \) of \( \mathbb{R} \) if there is a sequence \( g_n \in L^1(\mathbb{R}) \), such that, for each \( n \), \( g_n \) vanishes in a neighborhood of \( \triangle \) and \( \|g_n - f\|_{L^1} \to 0 \) as \( n \to \infty \).

**Theorem 5.** Suppose that \( u(t), \, t \geq 0, \) is a bounded uniformly continuous mild solution of Eq. (1) and \( f \in L^1(\mathbb{R}^+) \) is a function of spectral synthesis with respect to \(-i\sigma(A) \cap \mathbb{R}\). Then

\[
\lim_{t \to \infty} \left\| \int_0^\infty f(s)u(t+s)ds \right\| = 0.
\]

**Proof.** Again consider the Hille-Yosida space \( Z_0 \) and the semigroup \( T(t) \) generated by \( A|Z_0 \). By [7, 17],

\[
\lim_{t \to \infty} \left\| \int_0^\infty f(s)T(t+s)ds \right\|_{Z_0} = 0,
\]

from which (3) immediately follows, since \( Z_0 \hookrightarrow X \). \( \square \)

Theorem 5 is a generalization of a result obtained independently by Vũ Quốc Phong [17] and Esterle, Strouse and Zouakia [7], which states that if \( T(t) \) is a bounded \( C_0 \)-semigroup with generator \( A \) and if \( f \in L^1(\mathbb{R}^+) \) is a function of spectral synthesis with respect to \((-i\sigma(A) \cap \mathbb{R})\), then

\[
\lim_{t \to \infty} \left\| \int_0^\infty f(s)T(t+s)ds \right\| = 0.
\]

This result is an extension of an analogous result obtained by Katznelson and Tzafriri [11] for power-bounded operators.

From Theorem 5 we have the following corollary (here \( \hat{u} \) denotes the Laplace transform of \( u \), i.e. \( \hat{u}(\lambda) = \int_0^\infty e^{-\lambda t}u(t)dt \), \( \text{Re} \lambda > 0 \)).

**Corollary 6.** If \( \sigma(A) \cap i\mathbb{R} \subseteq \{0\} \) and \( u(t) \) is a bounded uniformly continuous mild solution of Eq. (1), then

\[
\lim_{t \to \infty} \|\hat{u}_{t+s}(\lambda) - \hat{u}_t(\lambda)\| = 0, \ \forall s \geq 0, \text{Re} \lambda > 0.
\]
In conclusion, we give the following proposition, which gives another simple condition for stability of individual solutions. Assertion (3) is a special case of [9, Theorem 2.5]. Assertion (1) may be deduced from [2, Theorem 1] by the techniques of this paper, by going down to the Hille-Yosida space for $A - \omega$, for any $\omega > 0$. However, the referee has pointed out that an inspection of the proof of [2, Theorem 1] shows that the proof applies, without change, when $A$ does not generate a strongly continuous semigroup.

**Proposition 7.** Suppose $\sigma(A) \cap i\mathbb{R}$ is empty.

1. If $u$ is a uniformly continuous mild solution of Eq. (1), then $\lim_{t \to \infty} u(t) = 0$.
2. If $u$ is a bounded mild solution on $\mathbb{R}^+$ of Eq. (1), then $\lim_{t \to \infty} A^{-1}u(t) = 0$.
3. There does not exist a nontrivial bounded mild solution on $\mathbb{R}$ of Eq. (1).

**Proof.** (1) As we commented above, the proof of [2, Theorem 1] is valid under the hypotheses of this theorem.

(2) From the conditions it follows that $A^{-1}u(t)$ is a bounded uniformly continuous mild solution of Eq. (2), thus (2) follows from assertion (1).

(3) Suppose $u \neq 0$ is a bounded mild solution on $\mathbb{R}$ of Eq. (1). Then $A^{-1}u(t)$ is a nontrivial bounded uniformly continuous mild solution on $\mathbb{R}$ of this equation. By [18, Proposition 3.7], and Corollary 2, $\text{Sp}(A^{-1}(u)) \subset \sigma(A|\mathbb{Z}) \cap i\mathbb{R} \subseteq \sigma(A) \cap i\mathbb{R}$, which is a contradiction to $\sigma(A) \cap i\mathbb{R} = \emptyset$. \qed

Results similar to Proposition 7(1), when $A$ generates a bounded once integrated semigroup, may be found in [6, Theorem 5.6 and Corollary 5.9]. When $\sigma(A) \cap i\mathbb{R}$ is empty, [6, Theorem 5.6 and Corollary 5.9] follow immediately from Proposition 7(1), since, as mentioned in [6], $u(t) \equiv S(t)Ax + x$ is a solution of (1) when $S(t)$ is a once integrated semigroup generated by $A$. However the results in [6, Theorem 5.6 and Corollary 5.9] have weaker hypotheses, analogous to [1] and [14]: $\sigma(A) \cap i\mathbb{R}$ is countable, $\rho(A^*) \cap i\mathbb{R}$ is empty and $0 \in \rho(A)$.

We should also remark that stability results for regularized semigroups (see [3]) also follow immediately from Proposition 7(1): if $\sigma(A) \cap i\mathbb{R}$ is empty, and $A$ generates a bounded regularized semigroup $\{W(t)\}_{t \geq 0}$, then $W(t)x \to 0$, as $t \to \infty$, for all $x \in X$.

**References**


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