PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 1, January 1997, Pages 235–241
S 0002-9939(97)03575-2

STABILITY AND ALMOST PERIODICITY OF SOLUTIONS OF ILL-POSED ABSTRACT CAUCHY PROBLEMS

R. DELAUBENFELS AND VŨ QUÔC PHÓNG

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. We give simple spectral sufficient conditions for a solution of the linear abstract Cauchy problem, on a Banach space, to be strongly stable or asymptotically almost periodic, without assuming that the associated operator generates a C_0-semigroup.

1. INTRODUCTION

Let A be a closed linear operator on a Banach space X. We shall consider the many physical problems that may be modelled as an *abstract Cauchy problem*

$$(1) \qquad \begin{cases} u'(t) = Au(t), & t \geq 0, \\ u(0) = x. \end{cases}$$

Many results on strong stability and almost periodicity of $t \mapsto u(t)$, when A generates a C_0-semigroup, may be found in [1, 2, 7, 13, 14, 16, 17] and [18], to name only a few references. The hypothesis that A generate a C_0-semigroup is saying that (1) is well-posed.

We assume in this paper only that all exponentially bounded solutions of Eq. (1) are unique. We do not assume that A generates a C_0-semigroup; this is what we mean by saying that (1) is ill-posed. Thus we are considering, as in [2], individual solutions of (1), but without any global well-posedness; our results are truly local in character.

When the spectrum of A on the imaginary axis is countable, we characterize bounded uniformly continuous solutions of (1) that are asymptotically almost periodic, or strongly stable, in terms of the means of $t \mapsto e^{-\lambda t}u(t)$, for purely imaginary λ in the spectrum of A (Theorem 4). Theorem 5 is a local version of the theorem of Katznelson-Tzafriri for C_0-semigroups. Finally, when the spectrum of A does not intersect the imaginary axis, we show that any uniformly continuous mild solution of (1) is strongly stable (Proposition 7).

Our technique is to use the Hille-Yosida space (see Section 2), introduced, independently, in [10] and [12], as constructed in [3, Chapter V], to deduce a local result from a global result. More specifically, if $t \mapsto u(t)$ is a bounded, uniformly continuous mild solution of (1), then x is in the Hille-Yosida space, on which A

Received by the editors April 21, 1995 and, in revised form, August 4, 1995.
1991 *Mathematics Subject Classification.* Primary 47D06.

generates a C_0-semigroup of contractions. We may then apply results about the asymptotic behaviour of orbits of a C_0-semigroup.

Throughout this paper, A is a closed linear operator in a Banach space X, $T(t)$ is a C_0-semigroup in X. The spectrum, point spectrum and resolvent set of A are denoted by $\sigma(A)$, $P\sigma(A)$ and $\rho(A)$, respectively.

2. Preliminaries

We consider the abstract Cauchy problem (1), where A is a closed linear operator on a Banach space X. We assume throughout the paper that A is such that all exponentially bounded solutions (i.e. all solutions $u(t)$ such that $\|u(t)\| \leq Me^{\alpha t}$ for some M and α) of (1) are unique. For instance, it is enough to require that there exists $\lambda_0 \in \mathbf{R}$ such that

$$(\lambda - A) \text{ is injective for } \lambda \geq \lambda_0 \quad \text{(see [12])}.$$

A function $u(t), t \geq 0$, is called a *strong solution* of Eq. (1), if $u \in C^1(\mathbf{R}, X)$, $u(t) \in \mathcal{D}(A)$ for all $t \geq 0$ and satisfies (1). To introduce the notion of Hille-Yosida space, we need the following definition of mild solutions of Eq. (1) (see [3]).

Definition. A continuous function $u : \mathbf{R}^+ \to X$ is called a *mild solution* of Eq. (1), if $v(t) \equiv \int_0^t u(s)ds \in \mathcal{D}(A)$ and satisfies $\frac{d}{dt}v(t) = Av(t) + x$, for all $t \geq 0$.

In the sequel, it will sometime be convenient to write this solution as $u(t, x)$.

Let Z_0 be the set of all x for which Eq. (1) has a bounded uniformly continuous mild solution $u(t, x), t \geq 0$.

We will write $X \hookrightarrow Y$ to mean that there is a continuous embedding of X into Y. If A is an operator on Y, we will write $A|_X$ to mean the restriction of A to X. Finally, we introduce a norm in Z_0 by

$$\|x\|_{Z_0} \equiv \sup_{t\geq 0} \|u(t, x)\|.$$

Lemma 1. *The space Z_0 has the following properties:*

(1) *Z_0 is a Banach space;*
(2) *$Z_0 \hookrightarrow X$;*
(3) *$A|_{Z_0}$ generates a C_0-semigroup of contractions $T(t)$, given by*

$$T(t)x \equiv u(t, x) \quad (t \geq 0);$$

(4) *Z_0 is maximal-unique, i.e., if W is a Banach space such that $W \hookrightarrow X$ and $A|_W$ generates a bounded C_0-semigroup, then $W \hookrightarrow Z_0$; and*
(5) *if $B \in B(X)$ and $BA \subseteq AB$, then $B \in B(Z_0)$, with $\|B\|_{Z_0} \leq \|B\|$.*

Proof. The proof of (1)–(4) can be found in [10], or [3], so we need only to show (5). Since B commutes with A, for any $x \in Z_0$,

$$v(t) \equiv Bu(t, x) \ (t \geq 0)$$

is a mild solution of (1) with the initial value $v(0) = Bx$; that is, $BZ_0 \subseteq Z_0$, with

$$\|Bx\|_{Z_0} \equiv \sup_{t\geq 0} \|u(t, Bx)\| = \sup_{t\geq 0} \|Bu(t, x)\| \leq \|B\|\|x\|_{Z_0} \ (x \in Z_0). \qquad \square$$

The space Z_0, which was introduced in [12] and [10] (independently) is called the *Hille-Yosida space* for A (cf. [3, 4]).

From Lemma 1-(5) the following corollary immediately follows.

Corollary 2. $\sigma(A|_{Z_0}) \subseteq \sigma(A)$.

3. Main results

Let $T(t)$ be a bounded C_0-semigroup, with generator A. It is known that the spectrum $\sigma(A)$ of A is contained in the closed left half-plane $\{\lambda \in \mathbf{C} : \operatorname{Re}\lambda \leq 0\}$. For $\lambda \in i\mathbf{R}$, let $X_\lambda(A) = \{x \in X : Ax = \lambda x\}$, and $X_\lambda(A^*) = \{\phi \in X^* : A^*\phi = \lambda\phi\}$. If $x \in X_\lambda(A)$ and $x \neq 0$, then, by a simple argument using the Markov-Kakutani Fixed Point Theorem, there exists $\phi \in X_\lambda(A^*)$ such that $\langle x, \phi\rangle \neq 0$. The semigroup $T(t)$ is called *mean ergodic*, if $\frac{1}{R}\int_0^R T(t)dt$ converges strongly as $R \to \infty$ (in this case the limit operator P is the projection operator from X onto the subspace $\{x \in X : T(t)x = x, \forall t \geq 0\}$). By the well known Mean Ergodic Theorem (see e.g. [8]), the semigroup $\{T(t)\}$ is mean ergodic if and only if $\ker(A) + \operatorname{ran}(A)$ is dense in X. It is easy to see that the latter condition is equivalent to the following: $X_0(A^*)$ separates points of $X_0(A)$, i.e. for every $\phi \in X_0(A^*)$ there exists $x \in X_0(A)$ such that $\langle x, \phi\rangle \neq 0$.

The semigroup $T(t)$ is called *almost periodic*, if the trajectory $\{T(t)x : t \geq 0\}$ is relatively compact in X for every $x \in X$. It is well known that $T(t)$ is an almost periodic semigroup if and only if, for every $x \in X$, the function $u(t) = T(t)x, t \geq 0$, is asymptotically almost periodic, i.e. there exists an almost periodic function $v : \mathbf{R} \to X$ such that $\|u(t) - v(t)\| \to 0$ as $t \to \infty$. Moreover, if $T(t)$ is an almost periodic semigroup, then $e^{-\lambda t}T(t)$ is ergodic for every $\lambda \in i\mathbf{R}$ (consequently, $X_\lambda(A^*)$ separates points of $X_\lambda(A)$, for every $\lambda \in P\sigma(A^*) \cap i\mathbf{R}$). Let P_λ denote the corresponding projection operator which is the strong limit, as $R \to \infty$, of $\frac{1}{R}\int_0^R e^{-\lambda t}T(t)dt$. Then $T(t)$ is strongly stable, i.e. $\|T(t)x\| \to 0$ as $t \to \infty$, $\forall x \in X$, if and only if it is almost periodic and $P_\lambda = 0$ for every $\lambda \in P\sigma(A) \cap i\mathbf{R}$ (see [5, 16]).

In [16] the following almost periodicity theorem has been established: if $\{T(t)\}$ is a bounded C_0-semigroup, such that $\sigma(A) \cap i\mathbf{R}$ is countable, and if $X_\lambda(A^*)$ separates points of $X_\lambda(A)$ for each $\lambda \in P\sigma(A^*) \cap i\mathbf{R}$, then $T(t)$ is almost periodic.

From this theorem we obtain the following result on almost periodicity of individual trajectories.

Proposition 3. *Let $\{T(t)\}_{t\geq 0}$ be a bounded C_0 semigroup with generator A such that $\sigma(A) \cap i\mathbf{R}$ is countable, and let x be a vector in X. Then*

(i) *the function $u(t) = T(t)x$ is asymptotically almost periodic if (and only if) for each $\lambda \in \sigma(A) \cap i\mathbf{R}$ the function $e^{-\lambda t}u(t)$ has convergent means; and*

(ii) *the function $u(t)$ converges to zero strongly as $t \to \infty$ if (and only if) for each $\lambda \in \sigma(A) \cap i\mathbf{R}$, the function $e^{-\lambda t}u(t)$ has convergent means with the limit equal to* 0.

Proof. (i) Let $Y = \overline{span}\{T(t)x : t \geq 0\}$. Then Y is a closed invariant subspace of $T(t)$. It is easy to see that $\sigma(A|_Y) \cap i\mathbf{R} \subseteq \sigma(A) \cap i\mathbf{R}$, so it is countable. From the condition it follows that

$$\frac{1}{R}\int_0^R e^{-\lambda}T(t)ydt$$

converges strongly, as $R \to \infty$, for each $y \in span\{T(t)x : t \geq 0\}$, and hence it also converges strongly for each $y \in \overline{span}\{T(t)x : t \geq 0\}$. Therefore, by the quoted almost periodicity theorem of Lyubich and Vũ Quôc Phóng [16], $T(t)|_Y$ is an almost periodic semigroup, hence $T(t)x$ is an asymptotically almost periodic function.

(ii) If, in addition,

$$\lim_{R\to\infty} \frac{1}{R}\int_0^R e^{-\lambda t}u(t)dt = 0$$

for all $\lambda \in P\sigma(A) \cap i\mathbf{R}$, then

$$\lim_{R\to\infty} \frac{1}{R}\int_0^R e^{-\lambda t}T(t)ydt = 0$$

for all $y \in Y$, which implies that $P_\lambda(A|_Y) = 0$ for all $\lambda \in P\sigma(A|_Y) \cap i\mathbf{R}$, so that $T(t)y \to 0$ for all $y \in Y$. In particular, $u(t) \to 0$ as $t \to \infty$. □

Using Proposition 3 and the Hille-Yosida space, we obtain the following result, where we say that a function ω has *uniformly convergent means* if

$$\lim_{T\to\infty} \frac{1}{R}\int_a^{R+a} \omega(s)ds$$

exists, uniformly in $a \geq 0$.

Theorem 4. *Suppose $\sigma(A)\cap i\mathbf{R}$ is countable and $u(t), t \geq 0$, is a bounded uniformly continuous mild solution of Eq. (1). Then*

(1) *$u(t)$ is asymptotically almost periodic if (and only if) for every $\lambda \in \sigma(A)\cap i\mathbf{R}$, the function $e^{-\lambda t}u(t)$ has uniformly convergent means;*

(2) *$u(t)$ converges strongly to 0 as $t \to \infty$ if (and only if) for each $\lambda \in \sigma(A)\cap i\mathbf{R}$, the function $e^{-\lambda t}u(t)$ has uniformly convergent means with the limit equal to 0.*

Proof. Let Z_0 be the Hille-Yosida space and $T(t)$ be the semigroup generated by $A|_{Z_0}$. By Corollary 2, $\sigma(A|_{Z_0}) \cap i\mathbf{R}$ is countable. We will show that, for $\lambda \in \sigma(A) \cap i\mathbf{R}$, $e^{-\lambda t}T(t)x = e^{-\lambda t}u(t)$ has uniformly convergent means as a function from $\mathbf{R}^+$ to Z_0.

Fix $\epsilon > 0$. There exists T_ϵ such that

$$\|\frac{1}{T}\int_h^{T+h} e^{-\lambda t}u(t)\,dt - \frac{1}{S}\int_h^{S+h} e^{-\lambda t}u(t)\,dt\| < \epsilon, \tag{2}$$

for all $S, T > T_\epsilon, h > 0$. From (2) it follows that

$$\begin{aligned}
&\|\frac{1}{T}\int_h^{T+h} e^{-\lambda t}u(t)\,dt - \frac{1}{S}\int_h^{S+h} e^{-\lambda t}u(t)\,dt\|_{Z_0}\\
&\equiv \sup_{s\geq 0}\|u(s, \frac{1}{T}\int_h^{T+h} e^{-\lambda t}u(t)\,dt - \frac{1}{S}\int_h^{S+h} e^{-\lambda t}u(t)\,dt)\|\\
&= \sup_{s\geq 0}\|\frac{1}{T}\int_h^{T+h} e^{-\lambda t}u(t+s)\,dt - \frac{1}{S}\int_h^{S+h} e^{-\lambda t}u(t+s)\,dt\|\\
&= \sup_{s\geq 0}\|\frac{1}{T}\int_{h+s}^{T+h+s} e^{-\lambda t}u(t)\,dt - \frac{1}{S}\int_{h+s}^{S+h+s} e^{-\lambda t}u(t)\,dt\|\\
&\leq \epsilon,
\end{aligned}$$

since the convergence is uniform in h.

Thus $e^{-\lambda t}u(t) = e^{-\lambda t}T(t)x$ has uniformly convergent means, for any $\lambda \in \sigma(A|_{Z_0}) \cap i\mathbf{R}$. Now the statements (1)–(2) follow from Proposition 3 and the continuous embedding $Z_0 \hookrightarrow X$. □

Theorem 4 is a generalization of the above mentioned result of Lyubich and Vũ Quôc Phóng [16], but the proof is based on this result and the Hille-Yosida space.

Part (2) of Theorem 4 is analogous to, but independent of, [2, Theorem 1].

An important corollary of this result, which was obtained independently by Arendt and Batty [1] (see also [14]) (and is sometimes known as the ABLP Theorem), states that $\|T(t)x\| \to 0$ as $t \to \infty$ for all x in X, if $\sigma(A) \cap i\mathbf{R}$ is countable and $P\sigma(A^*) \cap i\mathbf{R}$ is empty.

We note that the results presented in Theorem 4 are new even for the case when A is a generator of a C_0-semigroup (and even for bounded A). In this case, Theorem 4-(1) (resp., (2)) gives a condition for asymptotic almost periodicity (resp., stability) of individual trajectories of C_0-semigroups.

Our next result is an individual version of the theorem of Katznelson-Tzafriri type obtained in [7, 17] (independently) for C_0-semigroups. A function $f \in L^1(\mathbf{R})$ is said to be a *function of spectral synthesis with respect to a closed subset* $\triangle$ *of* $\mathbf{R}$ if there is a sequence $g_n \in L^1(\mathbf{R})$, such that, for each n, $\hat{g}_n$ vanishes in a neighborhood of $\triangle$ and $\|g_n - f\|_{L^1} \to 0$ as $n \to \infty$.

Theorem 5. *Suppose that* $u(t)$, $t \geq 0$, *is a bounded uniformly continuous mild solution of Eq. (1) and* $f \in L^1(\mathbf{R}_+)$ *is a function of spectral synthesis with respect to* $-i\sigma(A) \cap \mathbf{R}$. *Then*

$$\lim_{t\to\infty} \left\| \int_0^\infty f(s)u(t+s)ds \right\| = 0. \tag{3}$$

Proof. Again consider the Hille-Yosida space Z_0 and the semigroup $T(t)$ generated by $A|_{Z_0}$. By [7, 17],

$$\lim_{t\to\infty} \left\| \int_0^\infty f(s)T(t+s)xds \right\|_{Z_0} = 0,$$

from which (3) immediately follows, since $Z_0 \hookrightarrow X$. □

Theorem 5 is a generalization of a result obtained independently by Vũ Quôc Phóng [17] and Esterle, Strouse and Zouakia [7], which states that if $T(t)$ is a bounded C_0-semigroup with generator A and if $f \in L^1(\mathbf{R}_+)$ is a function of spectral synthesis with respect to $(-i\sigma(A) \cap \mathbf{R})$, then

$$\lim_{t\to\infty} \left\| \int_0^\infty f(s)T(t+s)ds \right\| = 0.$$

This result is an extension of an analogous result obtained by Katznelson and Tzafriri [11] for power-bounded operators.

From Theorem 5 we have the following corollary (here $\hat{u}$ denotes the Laplace transform of u, i.e. $\hat{u}(\lambda) = \int_0^\infty e^{-\lambda t}u(t)dt, \operatorname{Re}\lambda > 0$).

Corollary 6. *If* $\sigma(A) \cap i\mathbf{R} \subseteq \{0\}$ *and* $u(t)$ *is a bounded uniformly continuous mild solution of Eq. (1), then*

$$\lim_{t\to\infty} \|\hat{u}_{t+s}(\lambda) - \hat{u}_t(\lambda)\| = 0, \ \forall s \geq 0, \operatorname{Re}\lambda > 0.$$

In conclusion, we give the following proposition, which gives another simple condition for stability of individual solutions. Assertion (3) is a special case of [9, Theorem 2.5]. Assertion (1) may be deduced from [2, Theorem 1] by the techniques of this paper, by going down to the Hille-Yosida space for $A-\omega$, for any $\omega>0$. However, the referee has pointed out that an inspection of the proof of [2, Theorem 1] shows that the proof applies, without change, when A does not generate a strongly continuous semigroup.

Proposition 7. *Suppose $\sigma(A)\cap i\mathbf{R}$ is empty.*

(1) *If u is a uniformly continuous mild solution of Eq. (1), then* $\lim_{t\to\infty} u(t)=0$.
(2) *If u is a bounded mild solution on $\mathbf{R}^+$ of Eq. (1), then* $\lim_{t\to\infty} A^{-1}u(t)=0$.
(3) *There does not exist a nontrivial bounded mild solution on $\mathbf{R}$ of Eq. (1).*

Proof. (1) As we commented above, the proof of [2, Theorem 1] is valid under the hypotheses of this theorem.

(2) From the conditions it follows that $A^{-1}u(t)$ is a bounded uniformly continuous mild solution of Eq. (2), thus (2) follows from assertion (1).

(3) Suppose $u\neq 0$ is a bounded mild solution on $\mathbf{R}$ of Eq. (1). Then $A^{-1}u(t)$ is a nontrivial bounded uniformly continuous mild solution on $\mathbf{R}$ of this equation. By [18, Proposition 3.7], and Corollary 2, $\mathrm{Sp}(A^{-1}(u))\subset\sigma(A|_Z)\cap i\mathbf{R}\subseteq\sigma(A)\cap i\mathbf{R}$, which is a contradiction to $\sigma(A)\cap i\mathbf{R}=\emptyset$. □

Results similar to Proposition 7(1), when A generates a bounded once integrated semigroup, may be found in [6, Theorem 5.6 and Corollary 5.9]. When $\sigma(A)\cap i\mathbf{R}$ is empty, [6, Theorem 5.6 and Corollary 5.9] follow immediately from Proposition 7(1), since, as mentioned in [6], $u(t)\equiv S(t)Ax+x$ is a solution of (1) when $S(t)$ is a once integrated semigroup generated by A. However the results in [6, Theorem 5.6 and Corollary 5.9] have weaker hypotheses, analogous to [1] and [14]: $\sigma(A)\cap i\mathbf{R}$ is countable, $P\sigma(A^*)\cap i\mathbf{R}$ is empty and $0\in\rho(A)$.

We should also remark that stability results for regularized semigroups (see [3]) also follow immediately from Proposition 7(1): if $\sigma(A)\cap i\mathbf{R}$ is empty, and A generates a bounded regularized semigroup $\{W(t)\}_{t\geq 0}$, then $W(t)x\to 0$, as $t\to\infty$, for all $x\in X$.

References

1. W. Arendt and C.J.K. Batty, *Tauberian theorems and stability of one-parameter semigroups*, Trans. Amer. Math. Soc. **306** (1988), 837–852. MR **89g:**47053
2. C.J.K. Batty and Vũ Quôc Phóng, *Stability of individual elements under one-parameter semigroups*, Trans. Amer. Math. Soc. **322** (1990), 805–818. MR **91c:**47072
3. R. deLaubenfels, *Existence families, functional calculi and evolution equations. Lecture Notes in Math.*, vol. 1570, Springer, Berlin, 1994. MR **96b:**47047
4. R. deLaubenfels and S. Kantorovitz, *Laplace and Laplace-Stieltjes spaces*, J. Functional. Anal. **116** (1993), 1–61. MR **94g:**47015
5. K. DeLeeuw and I. Glicksberg, *Applications of almost periodic compactifications*, Acta Mathematica **105** (1961), 63–97. MR **24:**A1632
6. O. ElMennaoui, *Asymptotic behaviour of integrated semigroups*, J. Comp. Appl. Math. **54** (1994), 351–369. MR **96a:**47067
7. J. Esterle, E. Strouse and F. Zouakia, *Stabilité asymptotique de certains semigroupes dópérateurs*, J. Operator Theory **28** (1992), 203–227. MR **95f:**43001
8. J.A. Goldstein, *Semigroups of linear operators and applications*, Oxford Univ. Press, Oxford, 1985. MR **87c:**47056
9. S. Huang, *Characterizing spectra of closed operators through existence of slowly growing solutions of their Cauchy problems*, Studia Math. **116** (1995), 23–41. CMP 96:02

10. S. Kantorovitz, *The Hille-Yosida space of an arbitrary operator*, J. Math. Anal. and Appl. **136** (1988), 107–111. MR **90a:**47097
11. Y. Katznelson and L. Tzafriri, *On power-bounded operators*, J. Functional Analysis **68** (1986), 313–328. MR **88e:**47006
12. S.G. Krein, G.I. Laptev and G.A. Cvetkova, *On Hadamard correctness of the Cauchy problem for the equation of evolution*, Soviet Math. Dokl. **11** (1970), 763–766. MR **42:**637
13. B.M. Levitan and V.V. Zhikov, *Almost periodic functions and differential equations*, Cambridge Univ. Press, Cambridge, 1982. MR **84g:**34004
14. Yu.I. Lyubich and Vũ Quôc Phóng, *Asymptotic stability of linear differential equations on Banach spaces*, Studia Math. **88** (1988), 37–42. MR **89e:**47062
15. R. Nagel, *One-parameter semigroups of positive operators. Lecture Notes in Math. 1184*, Springer, Berlin, 1986. MR **88i:**47022
16. Vũ Quôc Phóng and Yu.I. Lyubich, *A spectral criterion for almost periodicity of one-parameter semigroups*, J. Soviet Math. **48** (1990), 644–647. Originally published in: "Teor. Funktsii, Funktsional. Anal. i Prilozhenia", **47**, 36–41 (1987). MR **89a:**47067
17. Vũ Quôc Phóng, *Theorems of Katznelson-Tzafriri type for semigroups of operators*, J. Functional Analysis **103** (1992), 74–84. MR **93e:**47050
18. Vũ Quôc Phóng, *On the spectrum, complete trajectories and asymptotic stability of linear semidynamical systems*, J. Differential Equations **105** (1993), 30–45. MR **94f:**47049

Department of Mathematics, Ohio University, Athens, Ohio 45701
E-mail address, R. deLaubenfels: 72260.2403@compuserve.com
E-mail address, Vũ Quôc Phóng: qvu@oucsace.cs.ohiou.edu