CONVEX SOLUTIONS OF THE SCHRÖDER EQUATION IN BANACH SPACES

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Abstract. The problem of the existence and uniqueness of increasing and convex solutions of the Schröder equation, defined on cones in Banach spaces, is examined on a base of the Krein-Rutman theorem.

The aim of this paper is to obtain a theorem on the existence and uniqueness of increasing and convex solutions \( \varphi \) of the Schröder equation

\[
\varphi(f(x)) = \rho \varphi(x),
\]

one of the most important equations of linearization, having many applications in various fields of mathematics (see [4] and [5]). Our result generalizes the theorem of F.M. Hoppe [1], in particular for functions defined on infinite-dimensional Banach spaces. The main point is to obtain an infinite-dimensional analogue of [2, Theorem 1] by A. Joffe and F. Spitzer exploiting the famous Krein-Rutman theorem [3, pp. 267-270], cf. also [6, Theorem 2.1].

1. Preliminaries

Fix a non-degenerate Banach space \((X, \lVert \cdot \rVert)\) and a closed cone \(K \subset X\) with non-empty interior, i.e. (cf. [3, p. 217, Definition 2.1]), \(K\) is a closed subset of \(X\) such that \(K + K \subset K\), \(tK \subset K\) for every \(t \geq 0\), \(K \cap (-K) = \{\theta\}\) and \(\text{Int } K \neq \emptyset\).

We define a (partial) order \(\leq\) on \(X\) by \(x \leq y\) iff \(y - x \in K\), and we assume that the norm \(\lVert \cdot \rVert\) is an increasing function on \(K\), i.e. \(\theta \leq x \leq y\) implies \(\lVert x \rVert \leq \lVert y \rVert\). (According to [7, p. 216], if \(X\) is a real space and there exists a real constant \(\gamma \geq 1\) such that \(\theta \leq x \leq y\) implies \(\lVert x \rVert \leq \gamma \lVert y \rVert\), then in the space \(X\) there exists an equivalent norm which is increasing on \(K\).)

Let \(A : X \to X\) be a completely continuous linear operator such that \(AK \subset K\) and for every \(x \in K \setminus \{\theta\}\) there exists a positive integer \(n\) such that \(A^n x \in \text{Int } K\).

By the Krein-Rutman theorem [3, p. 267] the spectral radius \(\rho\) of \(A\) is positive and there exists exactly one vector \(u \in \text{Int } K\) and exactly one continuous linear functional \(g : X \to \mathbb{R}\) such that \(Au = \rho u, g(Ax) = \rho g(x)\) for every \(x \in X\), \(g(x) > 0\) for every \(x \in K \setminus \{\theta\}\), \(\lVert u \rVert = 1\) and \(g(u) = 1\). Moreover [3, p. 269-270], the spectral radius of the operator \(B : X \to X\) defined by

\[
Bx = Ax - \rho g(x)u
\]

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is less than $\rho$ and
\[ \| \rho^{-n} A^n x - g(x) u \| \leq \rho^{-n} \| B^n \| \| x \| \] for $n \in \mathbb{N}$.

We assume also that a function $f : K \to K$ is given and such that
\[ f(x) \neq \theta \quad \text{for } x \in K \setminus \{\theta\}, \]
\[ \lim_{x \to \theta} (f(x) - Ax)/\| x \| = \theta \]
and there exists a positive $c$ such that
\[ g(x) \geq c \| g \| \| x \| \] for $x \in f(K)$.

Let us note that in the case where $X$ is finite-dimensional the last condition is always satisfied.

2. The Joffe-Spitzer sequence

The main result of this section reads:

**Theorem 1.** Assume that either
\[ \rho < 1, \]
or
\[ \rho = 1 \] and $f(x) \leq Ax$ for $x \in K$.

If $x_0 \in K \setminus \{\theta\}$ and $\lim_{n \to \infty} f^n(x_0) = \theta$, then
\[ \lim_{n \to \infty} f^n(x_0)/g(f^n(x_0)) = u. \]

**Proof.** Fix $r_0 > 0$ such that the closed ball centered at $u$ with the radius $r_0$ is contained in $K$. Then
\[ x \leq r_0^{-1} \| x \| \quad \text{for } x \in X. \]

Put
\[ \alpha_n := \rho^{-n} \| B^n \| (c r_0 \| g \|)^{-1} \]
for $n \in \mathbb{N}$.

According to the last part of the Krein-Rutman theorem
\[ \lim_{n \to \infty} \alpha_n = 0 \]
and
\[ \rho^n (1 - \alpha_n) g(x) u \leq A^n x \leq \rho^n (1 + \alpha_n) g(x) u \]
for every positive integer $n$ and $x \in f(K)$. Define $F : K \to X$ by
\[ F(x) := f(x) - Ax \]
and put
\[ \beta_n := (c r_0 \| g \|)^{-1} \| F(f^n(x_0)) \| / \| f^n(x_0) \| \]
for $n \in \mathbb{N}$.

It follows from (2) and (3) that
\[ \lim_{n \to \infty} \beta_n = 0. \]

We shall show
\[ \pm A^{n-k-1} F(f^{k+m}(x_0)) \leq \rho^{n-k-1} \beta_{k+m} g(f^{k+m}(x_0)) u \leq \rho^{n-k-1} \beta_{k+m} g(f^m(x_0)) u \]

for \( n \in \mathbb{N}, k \in \{0,\ldots,n-1\} \) and \( m \) large enough, say \( m > M \).

Applying (7) and (4) we obtain

\[
\pm F(f^{k+m}(x_0)) \leq \beta_{k+m} g(f^{k+m}(x_0))u \quad \text{for } k, m \in \mathbb{N}.
\]

Hence, as \( A \) increases,

\[
\pm A^{n-k-1} F(f^{k+m}(x_0)) \leq \beta_{k+m} g(f^{k+m}(x_0)) A^{n-k-1} u = \rho^{n-k-1} \beta_{k+m} g(f^{k+m}(x_0)) u
\]

(14)

for \( m, n \in \mathbb{N} \) and \( k \in \{0,\ldots,n-1\} \). To get the right-hand-side of (13) assume first (5), fix a \( \lambda \in (\rho, 1) \) and, making use of (3), \( \delta > 0 \) such that

\[
\|F(x)\|/\|x\| \leq (\lambda - \rho)c_0\|g\| \quad \text{for } x \in K \text{ with } 0 < \|x\| \leq \delta.
\]

Then, applying also (7),

\[
F(x) \leq (\lambda - \rho)g(x)u \quad \text{for } x \in f(K) \text{ with } 0 < \|x\| \leq \delta,
\]

whence

\[
g(f(x)) = g(Ax) + g(F(x)) \leq \lambda g(x) \quad \text{for } x \in f(K) \text{ with } 0 < \|x\| \leq \delta.
\]

Now, if \( M \) is a positive integer such that \( \|f^m(x_0)\| \leq \delta \) for \( m > M \), then

\[
g(f^{k+m}(x_0)) \leq \lambda^k g(f^m(x_0)) \leq g(f^m(x_0))
\]

for \( m > M \). This jointly with (14) ends the proof of (13) in case (5). In case (6) we have \( g(f(x)) \leq g(Ax) = g(x) \) for \( x \in K \) which jointly with (14) gives (13).

Since

\[
f^n(x) = A^n x + \sum_{k=0}^{n-1} A^{n-k-1} F(f^k(x)) \quad \text{for } n \in \mathbb{N}, x \in K,
\]

it follows from (10) that

\[
\rho^n (1 - \alpha_n) g(x)u + \sum_{k=0}^{n-1} A^{n-k-1} F(f^k(x)) \leq f^n(x)
\]

\[
\leq \rho^n (1 + \alpha_n) g(x)u + \sum_{k=0}^{n-1} A^{n-k-1} F(f^k(x))
\]

for \( n \in \mathbb{N}, x \in f(K) \). Using these inequalities for \( x = f^m(x_0) \) and applying (13) we get

\[
[\rho^n (1 - \alpha_n) - \sum_{k=0}^{n-1} \rho^{n-k-1} \beta_{k+m}] g(f^m(x_0))u \leq f^{m+n}(x_0)
\]

(15)

\[
\leq [\rho^n (1 + \alpha_n) + \sum_{k=0}^{n-1} \rho^{n-k-1} \beta_{k+m}] g(f^m(x_0))u
\]

for \( n \in \mathbb{N} \) and \( m > M \). Let \( N \) be a positive integer such that \( \alpha_n < 1 \) for \( n > N \), and for each \( n > N \) let \( M_n > M \) be a positive integer such that

\[
\rho^n (1 - \alpha_n) - \sum_{k=0}^{n-1} \rho^{n-k-1} \beta_{k+m} > 0 \quad \text{for } m > M_n.
\]
Making use of (15) and the facts that $g$ increases and $g(u) = 1$ we obtain

$$-2 \alpha_n + \sum_{k=0}^{n-1} \rho^{-(k+1)} \beta_k + u \leq -2 \alpha_n + \sum_{k=0}^{n-1} \rho^{-(k+1)} \beta_k + u \leq \frac{\sum_{k=0}^{n-1} \rho^{-(k+1)} \beta_k + u}{1 + \beta_k}$$

for $n > N$, $m > M_n$. Moreover,

$$-y \leq x \leq y \text{ implies } \|x\| \leq 3\|y\| \text{ for } x, y \in X.$$

Consequently,

$$\left\| \frac{f^{m+n}(x_0)}{g(f^{m+n}(x_0))} - u \right\| \leq 6 \frac{\alpha_n + \sum_{k=0}^{n-1} \rho^{-(k+1)} \beta_k + u}{1 + \beta_k}$$

for $n > N$ and $m > M_n$. Hence and from (12) we get

$$\limsup_{m \to \infty} \left\| \frac{f^m(x_0)}{g(f^m(x_0))} - u \right\| \leq 6 \frac{\alpha_n}{1 - \alpha_n} \text{ for } n > N$$

which jointly with (9) ends the proof.

**Corollary 1.** Under the assumptions of Theorem 1 we have

$$\lim_{n \to \infty} f^n(x_0)/\|f^n(x_0)\| = u$$

and

$$\lim_{n \to \infty} \|f^{n+1}(x_0)\|/\|f^n(x_0)\| = \rho.$$

2. The Szekeres sequence

Passing to solutions of (S) we assume additionally that the function $f$ is increasing, convex and

$$\lim_{n \to \infty} f^n(x) = \theta \text{ for } x \in K \setminus \{\theta\}.$$

Observe that then in such a case zero is the only fixed point of $f$ and

$$\lim_{n \to \infty} g(f^{n+1}(x))/g(f^n(x)) = \rho \text{ for } x \in K \setminus \{\theta\}.$$

Fix arbitrarily an $a \in \text{Int} K$. We shall show that for every $x \in K$ the sequence $(g(f^n(x))/g(f^n(a)))_{n \in \mathbb{N}}$ is bounded in order to define the function $\varphi_0 : K \to [0, \infty)$ by the formula

$$\varphi_0(x) := \limsup_{n \to \infty} g(f^n(x))/g(f^n(a)).$$

In fact, if $x \in K \setminus \{\theta\}$, then $f^N(x) \leq a$ for a positive integer $N$. Consequently, $g(f^n[f^N(x)]) \leq g(f^n(a))$ and, on the other hand,

$$\frac{g(f^{n+N}(x))}{g(f^{n+N}(a))} = \frac{g(f^n[f^N(x)])}{g(f^n(a))} \prod_{k=1}^{N} \frac{g(f^{n+k-1}(x))}{g(f^{n+k}(a))} \text{ for } n \in \mathbb{N}.$$

Hence and from (16) we obtain $\limsup_{n \to \infty} g(f^n(x))/g(f^n(a)) \leq \rho^{-N}$.

Arguing as F.M. Hoppe did in [1], but using our Theorem 1 instead of [2, Theorem 1] by A. Joffe and F. Spitzer, we can prove what follows.
Theorem 2. If $\rho < 1$, then $\varphi_0$ is an increasing and convex solution of (S) and if $\varphi : K \to \mathbb{R}$ is an increasing and convex solution of (S), then $$\varphi(x) = \varphi(a)\varphi_0(x) \quad \text{for } x \in K.$$  

Corollary 2. If $\rho < 1$, then $$\varphi_0(x) = \lim_{n \to \infty} g(f^n(x))/g(f^n(a)) \quad \text{for } x \in K.$$  

Applying Theorem 1 and Corollary 2 we obtain also a representation of the solution $\varphi_0$ in which the functional $g$ does not occur.

Corollary 3. If $\rho < 1$, then $$\varphi_0(x) = \lim_{n \to \infty} \|f^n(x)/\|f^n(a)\| \quad \text{for } x \in K.$$  

Example. Let $I$ denote the interval $[0, 1]$, $X$ denote the Banach space of all continuous real functions on $I$ with the supremum norm and $K$ denote the cone of all non-negative functions on $X$. Let $a : I^2 \to (0, 1)$ be a continuous function. It is easy to check that the function $f : K \to K$ given by the formula $$f(x)(t) := \int_0^1 \left[a(s,t) + \frac{x(s)}{1 + x(s)} \right] x(s) \, ds$$ satisfies all the assumptions of our theorems, with $$Ax(t) := \int_0^1 a(s,t) x(s) \, ds \quad \text{for } t \in I \text{ and } x \in X,$$ except, maybe, condition (4). To get (4) let us observe that putting $\gamma = \inf_{T \times T} a$ we have $$f(x)(t)/\|f(x)\| \geq \frac{\gamma}{2} =: c > 0 \quad \text{for } t \in I$$ and for every $x \in K \setminus \{\theta\}$. In other words, the ball centered at $f(x)/\|f(x)\|$ and with the radius $c$ is contained in $K$ for every $x \in K \setminus \{\theta\}$. This jointly with [3, p. 210, Lemma 1.2] proves (4).

Remarks. 1. For the sake of simplicity we considered functions defined on the whole cone $K$ but similar results hold if we replace $K$ by $\{x \in K : x \leq a\}$, or by $\{x \in K : x < a\}$, with $a \in \text{Int } K$.

2. Assuming that the function $f$ is concave we can consider increasing and concave solutions of (S) replacing in the definition of $\varphi_0$ the upper limit by the lower limit.

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References  

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