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NORMAL OPERATORS IN C^*-ALGEBRAS WITHOUT NICE APPROXIMANTS

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Abstract. The second author constructed a separable direct limit C^*-algebra with real rank zero containing a normal element whose spectrum is the closed unit disk that is not the limit of normal elements in the limiting algebras, and is not a limit of normals in the algebra having finite spectrum. We use Fredholm index theory to modify and simplify this construction to obtain such examples that are not limits of any "nice" types of elements.

In [5] the second author constructed an increasing sequence $\{\mathcal{A}_n\}$ of C^*-subalgebras of $B(l^2(\mathbb{Z}))$ with direct limit $\mathcal{A}$ (i.e., $\mathcal{A} = [\bigcup \mathcal{A}_n]^-$), and he constructed a normal element N in $\mathcal{A}$ that is not a limit of normals in the $\mathcal{A}_n$'s. (See also a related construction [6].) We modify the construction, putting it in the context of BDF [1], leading to a richer class of examples. By replacing the general K-theory arguments in [5] with Fredholm index theory arguments we show that N cannot be approximated in the union of the $\mathcal{A}_n$'s by operators from any other nice classes of operators, such as hyponormals.

Lemma 1. *If $\mathcal{S}$ is a norm separable subset of a C^*-algebra $\mathcal{A}$, and $\mathcal{A}$ has real rank zero, then there is a separable C^*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ such that $\mathcal{S} \subset \mathcal{B}$ and $\mathcal{B}$ has real rank zero.*

Proof. Let $\mathcal{A}_1$ be the C^*-algebra generated by $\mathcal{S}$, and, given $\mathcal{A}_n$ for $n \geq 1$, define $\mathcal{A}_{n+1}$ to be a C^*-algebra generated by $\mathcal{A}_n$ and a countable set of finite-spectrum Hermitian elements of $\mathcal{A}$ whose closure contains the set of Hermitian elements in $\mathcal{A}_n$. Let $\mathcal{B}$ be the closure of the union of the $\mathcal{A}_n$'s. □

It is easy to find a unitary element in a separable real rank zero C^*-algebra that cannot be approximated by normal elements with finite spectrum, e.g., the image u of the unilateral shift operator in the Calkin algebra is, by the above lemma, contained in a separable real rank zero algebra $\mathcal{A}$, and the nonzero Fredholm index of u implies that u is not a limit of any (possibly non-normal) elements with finite spectrum. To construct such an example with a normal element whose spectrum is the unit disk, choose a real rank zero algebra $\mathcal{B}$ containing a normal element b whose spectrum is the closed unit disk and consider the element $u \oplus b$ in the algebra $\mathcal{A} \oplus \mathcal{B}$.

We now turn to normal elements in direct limits. Suppose $\mathcal{A}_i \subset B(H_i)$, $i = 1, 2$, and define $\mathcal{A}_1 \# \mathcal{A}_2 = \mathcal{A}_1 \oplus \mathcal{A}_2 + \mathcal{K}(H_1 \oplus H_2)$. It is easy to see that if $\mathcal{A}_1$ and $\mathcal{A}_2$

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both have real rank zero, then so does $\mathcal{A}_1\#\mathcal{A}_2$. If $T \in B(H_1 \oplus H_2)$, we denote the 2×2 operator matrix for T by (T_{ij}). Let $\sigma_e(T)$ denote the essential spectrum of an operator T.

Lemma 2. *If $\mathcal{A}_1$ is finite-dimensional, $T \in \mathcal{A}_1\#\mathcal{A}_2$, T_{22} is Fredholm and $\varepsilon > 0$, then there is a t, $0 < t < \varepsilon$, such that $T - t$ is Fredholm and* $\operatorname{ind}(T - t) = \operatorname{ind} T_{22}$.

Proof. Since $\sigma_e(T_{11})$ is finite, we can choose t in $(0, \varepsilon)$ such that $T_{11} - t$ is Fredholm with index 0 and $T_{22} - t$ is still Fredholm with the same index as that of T_{22}. Since T is a compact perturbation of $T_{11} \oplus T_{22}$, it follows that $T - t$ is Fredholm and $\operatorname{ind}(T - t) = \operatorname{ind} T_{22}$. □

Theorem 3. *Suppose H is a separable infinite-dimensional Hilbert space, $N, S \in B(H)$, N is normal, S is essentially normal, and $\sigma(S) \subset \sigma_e(N)$. Then there are subspaces H_1, H_2 of H, a separable commutative AF C^*-subalgebra $\mathcal{A}_1$ of $B(H_1)$, and a separable real rank zero C^*-subalgebra $\mathcal{A}_2$ of $B(H_2)$ such that*

(1) $H = H_1 \oplus H_2$,
(2) $N \in \mathcal{A}_1\#\mathcal{A}_2$,
(3) *N_{22} is unitarily equivalent to a compact perturbation of S.*

Proof. It follows from [1] that N is unitarily equivalent to a compact perturbation of $N \oplus S$. Thus we can write $H = H_1 \oplus H_2$, with respect to which N has an operator matrix such that $N_{11} = N_1 + K_1$, with N_1 unitarily equivalent to N and K_1 compact, N_{22} unitarily equivalent to a compact perturbation of S, and both N_{21} and N_{12} compact. Choose a separable commutative AF C^*-subalgebra $\mathcal{A}_1$ of $B(H_1)$ containing N_1 and a separable real rank zero C^*-subalgebra $\mathcal{A}_2$ of $B(H_2)$ containing N_{22}. □

To apply the preceding proposition to obtain examples like those of [5], we choose an increasing sequence $\{\mathcal{F}_n\}$ of finite-dimensional C^*-subalgebras of the algebra $\mathcal{A}_1$ whose union is dense in $\mathcal{A}_1$. It is clear that $\mathcal{A}_1\#\mathcal{A}_2$ is the direct limit of $\{\mathcal{F}_n\#\mathcal{A}_2\}$. Since $\mathcal{A}_1$ and $\mathcal{A}_1$ have real rank zero, so does $\mathcal{A}_1\#\mathcal{A}_2$.

Example 4. Let N be a normal operator whose spectrum is the closed unit disk, and let S be the unilateral shift with multiplicity 1. The distance from S to the set of operators with 0 Fredholm index is 1. Suppose $T \in \mathcal{F}_n\#\mathcal{A}_2$ and $\|T - N\| < 1$. Then there is an arbitrarily small $r > 0$ such that $\|(T - r) - N\| < 1$, $T - r$ is Fredholm, and $\operatorname{ind}(T - r) = \operatorname{ind} N_{22} = \operatorname{ind} S = -1$. The set of Fredholm operators with index -1 is open; hence $\sigma(T)$ has nonempty interior. Thus the distance from N to the set of elements in the union of the $\mathcal{F}_n\#\mathcal{A}_2$'s that are either normal, invertible or have nowhere dense spectrum is 1. The set of invertible elements in $\mathcal{A}_1\#\mathcal{A}_2$ is open; thus the distance from N to the set of invertible elements in $\mathcal{A}_1\#\mathcal{A}_2$ is 1. Note that the last two sentences remain true when $\mathcal{A}_2$ is replaced with the much larger algebra $B(H_2)$. □

Example 5. Let A be the unilateral shift, let $S = (A^* - 2) \oplus (A + 2)$, and let N be a normal operator whose spectrum is the closed disk centered at the origin with radius 3. We then have $\operatorname{ind}(N_{22} \pm 2) = \operatorname{ind}(S \pm 2) = \pm 1$. Furthermore, in the Calkin algebra, the inverses of $N_{22} \pm 2$ have norm 1. Hence, if W is any operator and $\|W - N_{22}\| < 1$, then $\operatorname{ind}(W \pm 2) = \operatorname{ind}(N_{22} \pm 2)$. Thus if T is in some $\mathcal{F}_n\#\mathcal{A}_2$, and $\|T - N\| < 1$, then, for some arbitrarily small $r > 0$, $\operatorname{ind}((T - r) \pm 2) = \operatorname{ind}(N_{22} \pm 2) = \pm 1$. This rules out approximation of N by a vast array of elements

in $\bigcup_n \mathcal{F}_n \# \mathcal{A}_2$. For example an operator T is *hyponormal* [2] if $T^*T - TT^* \geq 0$. Equivalently, T is hyponormal if and only if $\|T^*f\| \leq \|Tf\|$ for every vector f. In particular, if T is hyponormal, then $\ker T^* \subset \ker T$, which means $\operatorname{ind} T \leq 0$ if T is Fredholm. Thus N cannot be approximated by elements T in $\bigcup_n \mathcal{F}_n \# \mathcal{A}_2$ such that either T or T^* is hyponormal. Note that the hyponormal elements include the quasinormal ones (T commutes with T^*T) and the subnormals [2] (the $n \times n$ matrix $((T^*)^i T^j) \geq 0$ for $1 \leq n < \infty$). An operator T is n-normal if every irreducible representation of $C^*(T)$ has dimension at most n. An operator T is strongly quasidiagonal if every irreducible representation of $C^*(T)$ is quasidiagonal (i.e., contained in the sum of the compact operators and a countable direct product of finite-dimensional C^*-algebras). A Fredholm quasidiagonal operator has index 0. Thus N cannot be approximated by strongly quasidiagonal (and hence n-normal) elements of $\bigcup_n \mathcal{F}_n \# \mathcal{A}_2$. $\square$

Note that if we choose S so that $\operatorname{ind}(S - \lambda)$ assumes infinitely many values as λ ranges over the Fredholm resolvent of S, then N cannot be approximated by elements T in $\bigcup_n \mathcal{F}_n \# \mathcal{A}_2$ with a bound on the cardinality of the set of values of $\operatorname{ind}(T - \lambda)$.

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