INFINITE LOOP SPACES
AND NEISENDORFER LOCALIZATION

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Abstract. There is a localization functor \( L \) with the property that \( L(X) \) is the \( p \)-completion of \( X \) whenever \( X \) is a finite dimensional complex. This same functor is shown to have the property that \( L(E) \) is contractible whenever \( E \) is a connected infinite loop space with a torsion fundamental group. One consequence of this is that many finite dimensional complexes \( X \) are uniquely determined, up to \( p \)-completion, by the homotopy fiber of any map from \( X \) into the classifying space \( BE \).

1. Introduction

Fix a rational prime \( p \) and let \( \mathcal{L}_p \) denote the homotopy functor defined by localizing with respect to the constant map \( \varphi : B\mathbb{Z}/p \to \ast \), in the sense of Dror Farjoun [4], and then completing at the prime \( p \) in the sense of Bousfield-Kan [2]. In symbols, \( \mathcal{L}_p(X) = (L_\varphi(X))_p \). This is the localization functor to which the title refers. Now if \( X \) is a finite nilpotent complex, it follows from Miller’s solution to the Sullivan conjecture that \( \mathcal{L}_p(X) \cong X_p \). At first glance, this would suggest that the functor \( \mathcal{L}_p \) is unlikely to yield any new information. However, in [10], Neisendorfer showed that this functor has a remarkable property when applied to the \( n \)-connected cover, \( X(n) \), of certain spaces \( X \). His main result was the following.

Theorem 1. Let \( X \) be a 1-connected finite dimensional complex with \( \pi_2(X) \) a torsion group. Then \( \mathcal{L}_p(X(n)) \cong X_p \) for any positive integer \( n \).

Thus, up to \( p \)-completion, no information is lost when one passes to the \( n \)-connected cover of such a complex! Of course, this is false for more general spaces, where the first \( n \) homotopy groups and the corresponding \( k \)-invariants are irretrievably lost in such a process. Thus Theorem 1 reveals a subtle homotopy property of certain finite dimensional complexes. It is worth noting that Theorem 1 also applies to those iterated loop spaces of \( X \) which have the required connectivity.

The main result of this paper is another application of the functor \( \mathcal{L}_p \). It shows that the circle and its \( n \)-fold products are essentially the only infinite loop spaces of finite type which are not destroyed by Neisendorfer localization.

Theorem 2. Let \( E \) be a connected infinite loop space whose fundamental group is a torsion group. Then \( \mathcal{L}_p(E) \cong \ast \) for each prime \( p \).
This result is about as close as one can get to saying that, in general, the classifying space $B\mathbb{Z}/p$ is the prime ingredient or the major building block in the $p$-primary part of infinite loop spaces of the sort just described. However, under certain special conditions (see Theorem 2.1) one can say more; in that case there is a cellular inequality $B\mathbb{Z}/p \ll E$ in the sense of Dror Farjoun, [5]. Roughly this means that $E$ can be constructed from $B\mathbb{Z}/p$ and its suspensions in much the same way that a $CW$-complex is built out of the sphere $S^0$ and its suspensions. More accurately, it means that $E$ is a member of the smallest class of spaces which contains $B\mathbb{Z}/p$ and is closed under weak equivalences and pointed homotopy colimits, [5].

The following theorem, from [8], is similar to Theorem 2 although the precise connection between the two results is not clear to me. This result is another offspring of the Sullivan conjecture.

**Theorem 3.** Let $E$ be a connected infinite loop space whose fundamental group is a torsion group, and assume that $Y$ is a nilpotent finite $CW$-complex. Then the function space of based maps from $E$ to the $p$-completion of $Y$ is weakly contractible.

At any rate, we see that as far as maps into finite complexes are concerned, most infinite loop spaces behave just like $B\mathbb{Z}/p$. It was this fact that led me to wonder if Theorem 2 might be true.

There is a crucial step in the proof of Neisendorfer’s theorem where one shows that $L_p(Y) \simeq \ast$ when $Y$ is a connected loop space with a torsion fundamental group and only finitely many nonzero higher homotopy groups. Theorem 2 plays the same role in the next result, which deals with the infinite suspension. As usual, let $QX = \text{colim}_n \Omega^n\Sigma^n X$ for a pointed space $X$ and let $\iota : X \to QX$ be the usual inclusion. More precisely, $\iota$ is the map whose adjoint is the identity on $\Sigma^\infty X$. Let $\text{Fib}(\iota_X)$ denote the homotopy fiber of the map $\iota$. The final result deals with the homotopy fibration

$$\text{Fib}(\iota_X) \xrightarrow{\ i \ } X \xrightarrow{\ i \ } QX.$$  

**Corollary 4.1.** Let $X$ be a 1-connected finite dimensional complex with $\pi_2(X)$ a torsion group. There is a homotopy equivalence $L_p(\text{Fib}(\iota_X)) \simeq X_p$ induced by the map $\iota$.

In other words, up to $p$-completion, such a space can be recovered from the fiber of its infinite suspension! Notice that the map $X \to QX$ is an equivalence through some finite range of dimensions (the stable range—which, of course, depends on the connectivity of $X$) and one might think that this stable information is lost when one passes to the fiber. However, just as in Neisendorfer’s theorem, the fiber somehow retains this data.

## 2. Proofs

Each space in this paper is assumed to have the homotopy type of a $CW$-complex. All spaces and maps are assumed to be pointed. To simplify notation, let $B = B\mathbb{Z}/p$. We will sometimes assume certain spaces are $p$-local without burdening the notation with this assumption. Let $C(B)$ denote the smallest class of connected spaces which contains $B$ and is closed under both weak equivalences and pointed homotopy colimits. It is not difficult to see that the one-point space $\ast$, is in this class, as are the iterated suspensions $\Sigma^nB$ for $n = 1, 2, \ldots$ Following Dror Farjoun, [7], write $X \gg B$ iff $X$ belongs to $C(B)$. Note also that $L_p(X) \simeq \ast$ whenever
$X \gg B$ \textit{(ibid., Prop. A7)}. In the same paper it is noted that for connected spaces $X$ and $Y$, one has
\[ \Sigma X \ll Y \iff X \ll \Omega Y. \]
Thus the inequality $\Sigma^n B \ll \Sigma^n B$ implies that $B \ll \Omega^n \Sigma^n B$ for each $n$. Taking colimits, it follows that $QB$ is a member of $C(B)$. Let $QS^0(0)$ denote the base point component of $QS^0$. This component is a retract of $QB$ by the Kahn-Priddy Theorem, \cite{1}. It is not difficult to see that if $Y$ is a retract of a connected $H$-space $X$, then $Y \gg X$, and hence
\[ B \ll QS^0(0). \]
Since $QS^n(n) \simeq \Omega QS^{n+1}(n+1)$, it follows that
\[ \Sigma^n B \ll QS^n(n) \]
for each $n$. Thus $L_p(QS^n(n)) \simeq \ast$ for each $n \geq 0$. The next step is to establish the same result for $QS^n$ for each $n \geq 2$. To do this we will use the fibration
\[ QS^n(n) \rightarrow QS^n \rightarrow K(Z, n) \]
and the following important result due to Dror Farjoun, \cite{5}.

\textbf{Theorem 5.} Given a map $f : X \rightarrow Y$ and a fibration $F \rightarrow C \rightarrow D$, if $L_f(F) \simeq \ast$, then $L_f(\pi)$ is a homotopy equivalence.

There is a technical point that should be mentioned before using this result. Recall that $L_p$ was defined as the composite of two functors: $L_\omega$, followed by $p$-completion. Thus it is not obvious that it can be used with Theorem 5. However, Neisendorfer shows in \cite{10}, Lemma 1.4, that when applied to simply connected spaces, $L_p = L_f$ where $f : B \vee M \rightarrow \ast$. Here $B$ is again $BZ/p$ and $M$ is the Moore space with one nontrivial reduced homology group isomorphic to $Z[1/p]$ in dimension 1. Thus Theorem 5 can be used with $L_p$.

Now $L_p(K(\pi, n)) \simeq \ast$ for each $n \geq 2$ and for any abelian group $\pi$, according to Casacuberta (\cite{3}, §7). Therefore, using Theorem 5 on the fibration over $K(Z, n)$ mentioned earlier, it follows that the Neisendorfer localization of $QS^n$ is contractible for each $n \geq 2$.

Let $X$ be a 1-connected finite CW-complex. Consider the cofibrations which inductively construct its skeleta,
\[ \vee S^n \rightarrow X_{n-1} \rightarrow X_n, \]
starting with $X_1 = \ast$. Each of these cofiber sequences involves only a finite number of spheres. The functor $Q(\ )$ converts these cofibrations into fibrations,
\[ \times Q(S^n) \rightarrow Q(X_{n-1}) \rightarrow Q(X_n). \]
The functor $L_p$ commutes with finite products, \cite{6}, and so Theorem 5 together with the result for $QS^n$ implies that $L_p(QX) = L_p(QX_n) = \cdots = L_p(QX_1) = \ast$.

Now let $E$ denote a 1-connected infinite loop space. Choose a CW-decomposition of $E$ in which every finite subcomplex is 1-connected. It is clear that
\[ QE \simeq \operatorname{colim} Q(K). \]
where $K$ runs through the finite subcomplexes of $E$. Apply the functor $L_p$ to both sides of this equation and note that, in general,
\[ L_f(\operatorname{colim}_\alpha X_\alpha) \simeq L_f(\operatorname{colim}_\alpha L_f(X_\alpha)) \]
according to Dror Farjoun, [7]. This implies that $L_p(QE) \simeq \ast$. Since $E$ is an infinite loop space, it is a retract of $QE$. It follows at once that $L_p(E)$ must also be contractible.

Finally consider the case where $E$ is not simply connected. Its fundamental group $\pi$ must be an abelian torsion group. Apply Theorem 5 to the fibration

$$E(1) \longrightarrow E \longrightarrow K(\pi, 1).$$

Since $L_p$ annihilates both the fiber and base space, the conclusion of Theorem 2 follows.

The following result gives some conditions under which $B \ll E$. I am not certain that this result is best possible.

**Theorem 2.1.** Assume that $E$ is an infinite loop space which is $p$-local, rationally trivial, and $n$-connected where $n \geq 1$. Then $\Sigma^{n-1}B\mathbb{Z}/p \ll E$.

**Proof.** Since $E$ is $n$-connected and rationally trivial it can be obtained as the homotopy colimit of a direct system of finite complexes with the same two properties. These finite complexes, in turn, can be constructed using mod-$p^r$ Moore spaces, for various $r$, instead of spheres. Express this particular colimit as

$$E \simeq \operatorname{colim}_\alpha L_{\alpha}.$$ 

Write $A \ll X$ to signify that the localization of $X$ with respect to the map $A \rightarrow \ast$ is contractible. The implication $\Sigma A \ll X \implies A \ll X$ is due to Dror Farjoun [5] and plays a crucial role in what follows.

**Lemma 2.1.1.** If $0 \leq k < n$, then $\Sigma^k B\mathbb{Z}/p < Q(S^n \cup_{p^r} e^{n+1})$.

Assume for the moment that this inequality is true. Let $L$ denote one of the finite complexes in the expression of $E$ as a colimit. Then $L$ can be obtained in a finite sequence of cofibrations

$$M_q \longrightarrow L_{q-1} \longrightarrow L_q$$

where each $M_q$ is a finite wedge of Moore spaces, each of which is at least $n$-connected. Apply $Q(\ )$ to this cofiber sequence to obtain a fibration. Using Lemma 2.1.1 it follows that $\Sigma^n B < Q(M_q)$. Apply Theorem 5 to this fibration, where $f : \Sigma^n B \rightarrow \ast$. It follows from a finite induction that $\Sigma^n B < QL$. Since $QE \simeq \operatorname{colim}_\alpha QL_{\alpha}$, it follows that $\Sigma^n B < E$ (and hence, that $\Sigma^{n-1} B \ll E$) by the same arguments that were used in the proof of Theorem 2.

To verify the lemma, take the cofiber sequence

$$S^n \longrightarrow \frac{p^r}{\partial} S^n \longrightarrow S^n \cup_{p^r} e^{n+1},$$

apply the functor $Q$ to it, and then take $n$-connected covers. It was shown earlier that $\Sigma^n B < Q(S^n)\langle n \rangle$. Thus Theorem 5, applied to the fibration just obtained, implies that $\Sigma^n B < Q(S^n \cup_{p^r} e^{n+1})\langle n \rangle$. Theorem 5 can also be used to show first

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1This result seems best possible, since $L_f$ does not commute with direct limits in general. A good example is the case of $QX = \operatorname{colim} \Omega^n \Sigma^n X$ when $X$ is a mod $p$ Moore space. It has been shown that $L_p(QX) = \ast$, and yet $L_p$ acts like the identity on each term in the colimit.
that $B < K(\mathbb{Z}/p^r, 1)$ and then that $\Sigma^{n-1}B < K(\mathbb{Z}/p^r, n)$). The conclusion of the lemma then follows from a final application of Theorem 5 to the fibration

$$
\Omega E \longrightarrow \text{Fib}(\rho) \longrightarrow X.
$$

Final remarks

The results in this paper give further testament to the power of recent advances made in unstable localization by Bousfield, Dror Farjoun, and others; e.g., see [3]. The preprints of Dror Farjoun, listed below, are available on the Hopf Topology archive. The material they contain will soon appear in a Springer monograph by him.

REFERENCES


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