AFFINE AND HOMEOMORPHIC EMBEDDINGS INTO $\ell^2$

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Abstract. It is shown that
(1) a locally compact convex subset $C$ of a topological vector space that
admits a sequence of continuous affine functionals separating points of $C$
affinely embeds into a Hilbert space;
(2) an infinite-dimensional locally compact convex subset of a metric linear
space has a central point;
(3) every $\sigma$-compact locally convex metric linear space topologically embeds
onto a pre-Hilbert space.

Let $C$ be a convex subset of a separable metric linear space $E$. The question
arises of whether $C$ can be embedded onto a convex subset of $\ell^2$, or onto a linear
subspace of $\ell^2$ provided $C$ is linear. A positive answer to this question would reduce
the topological identification of such arbitrary convex sets $C$ to those contained in
$\ell^2$. Let us mention that this identification problem has a satisfactory solution for
complete sets $C$ which are AR’s. It is shown in [DT1] and [DT2] that an infinite-
dimensional set $C$ which is an AR is either a (contractible) Hilbert cube manifold
or a copy of $\ell^2$. The case of incomplete $C$, in particular, the $\sigma$-compact case, is
far from being settled (even for $C$ which are AR’s). By Dugundji’s theorem (see
[BP2, p. 61]), every convex subset $C$ of a locally convex $E$ is an AR. It is unknown
whether this is true for an arbitrary metric linear space $E$. Therefore the AR-
property of $C$ could possibly be an obstacle for embedding $C$ onto a convex subset
of $\ell^2$.

The first part of the paper is devoted to affine embeddings of locally compact
convex sets $C$ into $\ell^2$. Due to an elementary observation of Klee (see [BP2, p. 98])
every compact convex subset of a locally convex $E$ affinely embeds into $\ell^2$. Following
this, the notion of a Keller set was introduced which proved to be important in
solving some identification problems, see [BP2]. An infinite-dimensional compact
convex set $C$ is a Keller set if it affinely embeds into $\ell^2$. It is known that not
all infinite-dimensional compact convex sets $C$ are Keller sets. The examples of
compact convex subsets $C$ without extreme points given by Roberts (see [R1],
[R2], and [KPR]) obviously cannot be affinely embedded into $\ell^2$. Refining the
construction of Roberts one can find such $C$ with the AR-property ([DM2] and

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negative; see [Mar] and [Ca].

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We show that every locally compact (closed) convex subset \( C \) of \( E \) whose dual \( E^* \) separates points of \( C \) admits an affine (closed) embedding into \( \ell^2 \); this answers [BP2, Problems 1 and 2, p. 99].

The importance of a Keller space \( C \) comes from the fact that \( C \) has a so-called central point (see [BP2, p. 157]). This fact was recovered in [vBDHvM] for an arbitrary infinite-dimensional compact convex set \( C \). We show that this remains true for locally compact convex \( C \), solving a question raised in [vBDHvM]. Similarly as in [vBDHvM], we apply this fact to show that the AR-property of a locally compact convex \( C \) is equivalent to the homeomorphism extension property between so-called \( Z_\infty \)-sets in \( C \).

In the second part of the paper we show that every \( \sigma \)-compact linear subspace \( E \) that is locally convex embeds onto a dense linear subspace of \( \ell^2 \), which answers [DM1, Problem 587]. The proof is obtained by constructing a pre-Hilbert space \( H \) so that the classes of compacta embeddable in \( E \) and \( H \) coincide. Then we apply the uniqueness theorem on absorbing sets to get a homeomorphism of the linear completions \( \hat{E} \) and \( \hat{H} \) of \( E \) and \( H \), respectively, that sends \( E \) onto \( H \). Earlier, using the same approach, it was shown [Do3] that such \( E \) embeds onto a linear subspace of the countable product of lines. We leave as an open problem the question of whether an arbitrary convex \( \sigma \)-compact set \( C \) of such \( E \) embeds onto a convex subset of \( \ell^2 \).

1. Affine embeddings of convex local compacta

Let us start with the following fact which belongs to mathematical folklore.

**1.0. Lemma.** If \( K \) is a locally compact, \( \sigma \)-compact space which admits a sequence \( \{g_n\}_{n=1}^\infty \) of continuous real functions separating points of \( K \), then \( K \) is metrizable.

**Proof.** By our assumption, \( K \) locally embeds in the countable product of lines, and therefore is locally metrizable. Since \( K \) is \( \sigma \)-compact, it is Lindelöf, and consequently it is paracompact. Applying [En, Ex. 5.4.A, p. 415], \( K \) is metrizable. \( \square \)

Assume that a locally compact convex set \( C \) of a topological vector space \( E \) contains the origin \( 0 \in E \). Then there exists a closed symmetric neighborhood \( U \) of \( 0 \) in \( E \) such that

\[
(*) \quad U \cap C \text{ is compact.}
\]

We also have for every \( \lambda \geq 1 \)

\[
(**) \quad \lambda(U) \cap C \subset \lambda(U \cap C).
\]

**1.1. Theorem.** Let \( C \) be a locally compact (closed) convex subset of a topological vector space \( E \). If there exists a sequence of continuous affine functionals \( \{g_n\}_{n=1}^\infty \) on \( C \) separating points of \( C \), then \( C \) admits an affine (closed) embedding into \( \ell^2 \). Here \( \ell^2 \) can be replaced by an arbitrary infinite-dimensional, complete metric linear space.

**Proof.** We can assume that \( 0 \in C \). Pick a closed symmetric neighborhood \( U \) of \( 0 \) in \( E \) such that \( (*) \) is satisfied. Let \( p_n = \max \{|g_n(x)| : x \in U \cap C\} \). Without loss of generality we may assume that \( p_n \leq 2^{-n} \) (if not, replace the functionals \( g_n \) by suitable positive multiples). Define \( T(x) = (g_n(x)) \) for \( x \in C \). The estimate for \( p_n \)'s and \( (** \) together with the assumptions of the theorem imply that \( T \) is an injective affine map into \( \ell^2 \) and that, for every \( \lambda \geq 1 \), the restricted map \( T|\lambda(U)\cap C \) is continuous. Hence \( T : C \to \ell^2 \) is continuous.
Let $Y$ be an infinite-dimensional, complete metric linear space and $|\cdot|$ be a monotone $F$-norm on $Y$, i.e., for every number $|\lambda| \leq 1$ and every $y \in Y$ we have $|\lambda y| \leq |y|$. By S. Mazur’s lemma (see the proof of [BP2, Prop. 2.2, p. 268]), there are (linearly independent) vectors $y_n \in Y$ such that for every sequence of reals $(t_n)_{n=1}^\infty$ with $|t_n| \leq 2^{-n}$, $n \geq 1$, the series $\sum_{n=1}^\infty |t_n y_n|$ converges and the map $(t_n) \mapsto \sum_{n=1}^\infty t_n y_n$ is injective. Let $T(x) = \sum_{n=1}^\infty g_n(x)y_n$ for $x \in C$. It is easy to check that $T$ is a well-defined, injective, affine, continuous transformation of $C$ into $Y$.

Using $(**) \text{ and } (**), C$ is $\sigma$-compact. By 1.0, $C$ is metrizable. Therefore the following lemma completes the proof of the theorem.

1.2. Lemma. If $C$ is a (closed) locally compact metrizable, convex subset of a topological vector space $E$ and $T : C \to Y$ is an injective continuous affine map into a metric linear space, then $T$ is a (closed) embedding.

Proof. Let $|\cdot|$ be a monotone $F$-norm on the space $Y$. First we shall show that

$$S = T^{-1} : T(C) \to E$$

is continuous at each point $y_0 = T(x_0) \in T(C)$. Replacing $T$ by the map $x \mapsto T(x + x_0) - T x_0$, it is no loss of generality to assume that $x_0 = 0_E$, $y_0 = 0_Y$. (In the sequel we shall use one symbol 0 for denoting the points $0_E$, $0_Y$ and the number zero.) Let $U$ be a neighborhood of 0 in $E$ satisfying $(*)$. We claim that there exists an $\epsilon > 0$ such that

$$(**) \text{ whenever } |T(x)| < \epsilon, x \in C, \text{ then } x \in \text{int } U \cap C.$$

If the claim were not true, then there would exist a sequence $(x_n)$ of elements of $C$ such that $x_n \in \partial U \cap C$ and such that $T x_n \to 0$ (use monotonicity of the $F$-norm $|\cdot|$). By $(*)$, there would exist a subsequence $(z_n)$ of $(x_n)$ with $T z_n \to 0$ such that $z_n \to z \in \partial U \cap C$, contradicting the continuity of $T$.

Since every neighborhood of 0 in $E$ contains a neighborhood satisfying $(*)$, the condition $(**)$ implies the continuity of $S$ at 0.

Assume that $C$ is closed in $E$ and that $y_n = T x_n \to y \in Y$, $x_n \in C$. Since the set $\{T x_n\}_{n=1}^\infty \cup \{y\}$ is compact, there exists $0 < a \leq 1$ such that $|a T x_n| < \epsilon$ for all $n$, where $\epsilon$ is that of $(**)$. Using the facts that $0 \in C$ and $a \leq 1$, all the points $ax_n$ are in $C$. By $(**) \text{ and } (**)$ and the fact that $T$ is affine, $ax_n \in \text{int } U \cap C$. By $(*)$, there is a subsequence $(a z_n)$ of the sequence $(ax_n)$ such that $a z_n \to z \in U \cap C$. Therefore $z_n \to a^{-1} z$ which is in $C$ (since $C$ is closed) and

$$y = \lim T x_n = \lim T z_n = T(a^{-1} z) \in T(C).$$

This completes the proof.

In connection with the last part of our proof let us notice the following elementary fact.

1.3. Remark. If a locally compact convex set $C$ is closed in a metric linear space, then $C$ is also closed in the (linear) completion $E$ of $E$. Apply 1.2 to the inclusion map of $C$ into $E$.

Let us note the following generalization of [B, Cor. 1].

1.4. Corollary. Every metrizable locally compact convex (closed) subset $C$ of a topological vector space $E$ whose dual $E^*$ separates points of $E$ admits an affine (closed) embedding into $\ell^2$. 

By the compactness of $C$. Let $U$ be a closed neighborhood of $0$ in $E$ satisfying $(\ast)$. Fix a metric $d$ on $C$. Since the set $A_r = \{k_1 - k_2 : k_1, k_2 \in U \cap C$, and $d(k_1, k_2) \geq \epsilon\}$, $\epsilon > 0$, is compact, there are finitely many functionals that separate points of $A_r$ from 0. It follows that there exists a sequence of functionals $\{x_n^r\}_{n=1}^\infty \subset E^*$ that separate 0 from points of $(U \cap C) - \{U \cap C\} \{0\}$. Consequently, $\{x_n^r\}_{n=1}^\infty$ separates points of $U \cap C$. By $(\ast\ast)$, $\{x_n\}_{n=1}^\infty$ separates points of $C$ and 1.1 is applicable. 

2. Central points and application to the AR-property

A closed subset $A$ of a metric space $X$ is called a $Z_\infty$-set [Tor], if every map of an $n$-dimensional cube $I^n$, $n \geq 1$, into $X$ can be approximated by maps whose images miss $A$. If the space $X$ is an ANR, then the set $A$ is simply called a $Z$-set. The distinction of the terminology comes from the fact that for ANR spaces $X$ the above property guarantees that every map of $X$ into $X$ can be approximated by maps whose images miss $X$ (the property commonly understood to describe the fact that $A$ is a $Z$-set in $X$). If $A$ is not necessarily closed and satisfies the mapping condition for a $Z_\infty$-set, then $A$ is called locally homotopy negligible in $X$. We will use the above notions for the case where $X$ is a convex subset of a metric linear space and hence, in general, it is merely contractible and locally contractible (and perhaps, is not an AR).

Throughout this section $C$ will denote an infinite-dimensional locally compact convex subset of a complete metric linear space $E$ endowed with an $F$-norm $|\cdot|$. We say that $x_0 \in C$ is a central point for $C$ if the set $x_0 + [0,1) \cdot (C - x_0)$ is a countable union of $Z_\infty$-sets in $C$. Our main result is

2.1. Theorem. There exists a central point for $C$.

The proof employs a few auxiliary lemmas.

2.2. Lemma. The set $x_0 + [0,1) \cdot C$ is a countable union of $Z_\infty$-sets iff each $x_0 + [0,1 - \frac{1}{n}) \cdot C$ is locally homotopy negligible in $C$.

Proof. Since each $x_0 + [0,1 - \frac{1}{n}) \cdot C$ is a $\sigma$-compact subset of $C$ and since a compact subset of a locally homotopy negligible set is a $Z_\infty$-set, the fact that each $x_0 + [0,1 - \frac{1}{n}) \cdot C$ is locally homotopy negligible implies that $x_0 + [0,1) \cdot C$ is a countable union of $Z_\infty$-sets in a complete metrizable space $C$. By [Tor, Cor. 2.7] it follows that $x_0 + [0,1) \cdot C$ is locally homotopy negligible. \hfill $\square$

2.3. Lemma. For every compact convex subset $C_0 \subseteq C$ the set $C - C_0$ is convex and locally compact.

Proof. Clearly $C - C_0$ is convex and we need to verify that $C - C_0$ is locally compact. By $(\ast\ast)$ it is enough to check the local compactness at $0 \in C - C_0$. Using the local compactness of $C$, for every $x \in C$ there exists $\delta_x > 0$ such that

$$C_x = \{y \in C : |x - y| \leq 2\delta_x\}$$

is compact.

By the compactness of $C_0$, there are $x_1, x_2, \ldots, x_k \in C_0$ such that

$$C_0 \subseteq \bigcup_{i=1}^k B_{x_i},$$
where \(B_x = \{ y \in C : |x - y| < \delta_x \}, x \in C_0 \). Set \( \delta = \min \{ \delta_{x_1}, \delta_{x_2}, \ldots, \delta_{x_k} \} \). We claim that

\[
\{ y - x : y \in C, x \in C_0 \text{ and } |y - x| \leq \delta \}
\]

is compact. Given an arbitrary \( x \in C_0 \), there exists \( i, 1 \leq i \leq k \), with \( x \in B_{x_i} \), i.e., \(|x - x_i| < \delta_{x_i}\). Now, pick any \( y \in C \) with \(|y - x| \leq \delta\). We have

\[
|y - x_i| \leq |y - x| + |x - x_i| \leq \delta + \delta_{x_i} \leq 2\delta_{x_i};
\]

hence \( y \in C_{x_i} \). Consequently,

\[
\{ y - x : y \in C, x \in C_0 \text{ and } |y - x| < \delta \} \subseteq \bigcup_{i=1}^{k} C_{x_i} - C_0.
\]

By (1), the lemma follows.

Before we provide a proof of 2.1 let us modify [vBDHvM, Lm. 4.1]. Every point \( x_0 \in C \) satisfying the assertion of 2.4 turns out to be a central point of \( C \).

2.4. Lemma. There exists \( x_0 \in C \) such that for every compact convex \( C_0 \subseteq C \) we have

\[
\inf_{0 \neq z \in C} \text{diam}_{|1|} \left( [0, \infty) \cdot (z - x_0) \cap (C - C_0) \right) = 0.
\]

Proof (cf. [vBDHvM, proof of Lm. 4.1]). Assume \( 0 \in C \). Take \( \delta > 0 \) so that \( U = \{ x \in C : |x| \leq \delta \} \) is compact. Fix a dense subset \( \{ x_n \}_{n=1}^{\infty} \) in \( C \) such that \( x_n \neq x_m \) for \( n \neq m \). Pick a sequence of positive reals \( \{ \lambda_n \}_{n=1}^{\infty} \) satisfying

\[
\sum_{n=1}^{\infty} \lambda_n \leq 1 \quad \text{and} \quad \sum_{n=1}^{\infty} |\lambda_n x_n| \leq \delta.
\]

It follows that the series \( \sum_{n=1}^{\infty} \lambda_n x_n \) converges to some \( x_0 \in U \). We claim that \( x_0 \) satisfies the assertion of the lemma. Otherwise, there would exist a compact convex subset \( C_0 \subseteq C \) such that

\[
\text{diam}_{|1|} \left( [0, \infty) \cdot (z - x_0) \cap (C - C_0) \right) > \epsilon
\]

for some \( \epsilon > 0 \) and for all \( z \in C \setminus \{ x_0 \} \). From the definition of \( x_0 \) it follows that if \( k \neq l \) (and therefore \( x_k \neq x_l \)) and if \( 0 < t < \min(\lambda_k, \lambda_l) \), then

\[
x_0 + t(x_k - x_l) = \sum_{n=1}^{\infty} \lambda_n x_n + tx_k - tx_l = \sum_{n=1}^{\infty} \mu_n x_n,
\]

where all \( \mu_k \) are positive and \( \sum_{n=1}^{\infty} \lambda_n = \sum_{n=1}^{\infty} \mu_n \leq 1 \), whence \( x_0 + t(x_k - x_l) \in C \). Therefore

\[
\text{diam}_{|1|} \left( [0, \infty) \cdot (x_k - x_l) \cap (C - C_0) \right) > \epsilon
\]

for \( k \neq l \). Hence

\[
B = \left( \bigcup_{k,l} [0, \infty) \cdot (x_k - x_l) \right) \cap \{ x \in E : |x| < \frac{\epsilon}{2} \} \subset C - C_0.
\]

Since \( \{ x_n \}_{n=1}^{\infty} \) is dense in \( C \), we conclude that

\[
\text{span}(C) = [0, \infty) \cdot (C - C) \subseteq \bigcup_{k,l} [0, \infty) \cdot (x_k - x_l).
\]
This would imply
\[ \{ x \in \text{span}(C) : |x| < \frac{\varepsilon}{2} \} \subset B \subset C - C_0 \]
(the closure taken in \( E \)). Since \( C - C_0 \) is locally compact (Lemma 2.3), so is \( C - C_0 \). By the infinite-dimensionality of \( \text{span}(C) \), no neighborhood of it can be locally compact, a contradiction.

2.5. Remark. If \( C \) is finite-dimensional, it has nonempty interior with respect to its affine hull and therefore the assertion of Lemma 2.4 is then false.

Proof of 2.1 (see [BP1, Lm. 2.7]). We will show that any point \( x_0 \) satisfying the assertion of 2.4 is a central point. We may assume that \( x_0 = 0 \). By 2.2, it is enough to show that \([0,1 - \frac{1}{n}] \cdot C \) is locally homotopy negligible in \( C \). Fix a map \( f : I^n \to C \) and \( \varepsilon > 0 \). Find \( f_1 : I^n \to C \cap L = C' \), where \( L \) is a finite-dimensional linear subspace of \( E \), such that \( d(f_1) < \frac{\varepsilon}{2} \) and \( C_0 = \text{conv}\{f_1(I^n)\} \subset \text{int}_L C' \). By Remark 2.5,

\[ \inf_{0 \neq z \in C'} \text{diam}_1(\{(0, \infty) \cdot z \cap (C - C_0)\}) \geq \inf_{0 \neq z \in C'} \text{diam}_1(\{(0, \infty) \cdot z \cap (C' - C_0)\}) > 0, \]

and by 2.4, there exists \( q \in C \setminus L \) such that
\[ \text{diam}_1(\{(0, \infty) \cdot q \cap (C - C_0)\}) < \frac{\varepsilon}{2}. \]

Write \( X = \text{span}(L, q) \), \( K = (1 - \frac{1}{n+1}) \cdot (C \cap X) \) and \( A = f_1(I^n) \). Define \( \{f_2(x)\} = (x + [0, \infty) \cdot q) \cap \partial X(K) \) for \( x \in A \). Since \( A \subset \text{int}_L C' \), we conclude that \( f_2(x) \) consists precisely of one point. We have \( f_2(x) \in C \) and \( f_2(x) \in C \setminus [0, 1 - \frac{1}{n}] \cdot C \). Moreover, \( f_2 : A \to C \) is continuous, and
\[ d(f_2, \text{id}) \leq \text{diam}_1(\{(0, \infty) \cdot q \cap (C - C_0)\}) < \frac{\varepsilon}{2}. \]

Writing \( g = f_2 \circ f_1 \) we see that \( d(g, f) < \varepsilon \) and \( g(I^n) \subset C \setminus [0, 1 - \frac{1}{n}]C \).

2.6. Remark. The above argument shows that every \( x_0 \in C \) satisfying the assertion of 2.4 is also a central point of \( C \).

2.7. Corollary. If \( C \) has the homeomorphism extension property for \( Z_{\infty} \)-sets, then \( C \) is an AR.

Proof. By a result of [Do2], \( C \) is an AR iff for every compact subset \( A \subset C \) the identity map \( \text{id}_A \) can be approximated by maps with finite-dimensional ranges. Take a central point \( x_0 \) for \( C \) (Theorem 2.1). We can assume that \( x_0 = 0 \). Clearly the sequence of maps \( x \to (1 - \frac{1}{n+1})\cdot x \), \( x \in A \), converges to \( \text{id}_A \) and has ranges which are \( Z_{\infty} \)-sets. Consequently, we may assume that \( A \) itself is a \( Z_{\infty} \)-set. By [CDM, Prop. 3.5], there is a copy of the Hilbert cube \( Q \) contained in \( \frac{1}{2} \cdot C \). Since \( Q \) is a \( Z_{\infty} \)-set in \( C \), by our assumption, there is a homeomorphism \( h \) of \( C \) with \( h(A) \subset Q \).

The sequence \( (h^{-1}\pi_nh) \), where \( \pi_n \) are standard projections \( Q = \prod_{k=1}^{\infty} I_k, I_k = I \),
onto \( \prod_{k=1}^{n} I_k \), approximates \( \text{id}_A \) and \( \dim(h^{-1}\pi_nh(A)) < \infty, n = 1, 2, \ldots \).
Applying [DT1, Th. 2] we obtain

2.8. Corollary. $C$ has the homeomorphism extension property for $Z_{\infty}$-sets iff $C$ is a Hilbert cube manifold.

In [B, Rm. 3] it was shown that if $E$ is locally convex, then $C$ is homeomorphic to $Q \setminus K$, where $K$ is a $Z$-set in $Q$. It suggests the following

2.9. Question. Assume that $C$ is an AR. Is then $C$ homeomorphic to $Q \setminus K$ for some $Z$-set $K$ in the Hilbert cube $Q$?

Let us also ask

2.10. Question. Assume $C$ is an AR. Is then $\overline{C}$ an AR?

Note that if $C$ is homogeneous, then the answer to 2.10 is “yes”.

2.11. Remark. Assume that $C$ is closed in $E$ and that $C$ has the homeomorphism extension property for its $Z_{\infty}$-sets. Write $cc(C) = \{x \in E : \exists (y \in C) (y + [0, \infty) \cdot x \subset C)\}$ for the characteristic cone for $C$. We have:

(i) If $cc(C) = \{0\}$, then $C$ is homeomorphic to $Q$.

(ii) If $cc(C)$ is a linear subspace of $E$, then $C$ is homeomorphic to $Q \times R^n$, where $n = \dim(cc(C))$, $n \geq 1$.

(iii) If $cc(C)$ is not linear, then $C$ is homeomorphic to $Q \times [0, \infty)$.

Apply a result of [Do1].

In connection with 2.3 let us note that if $C$ is a cone over a Keller set, then $C - C$ is not locally compact (it contains a closed infinite-dimensional linear subspace). Therefore the assumption that $C_0$ is compact is essential in 2.3. We have the following corresponding fact for topological groups.

2.12. Remark. Let $G$ be a topological group. Assume $L \subset G$ is locally compact and $K \subset G$ is compact. If $L$ is closed in $G$, then $L + K$ is locally compact. To justify this, pick $g = l + k$, $l \in L$ and $k \in K$. Since $(g - K) \cap L$ is a compact subset of $L$, it is contained in a compact neighborhood $U$ in $L$. We claim that $U + K$ contains a neighborhood $g$ (then, necessarily, relatively compact) in $L + K$. Otherwise, $g$ will be an accumulation point of elements $l' + k'$, $l' \in L \setminus U$ and $k' \in K$. Using the compactness of $K$ we easily get a contradiction.

If $L$ happens not to be closed, then $L + K$ might not be locally compact. (Take $G = R^2$ with the addition, $L = \{(0, t) : |t| < 1\}$, and $K = \{\left(\frac{1}{n}, -1\right) : n = 1, 2, \ldots\} \cup \{(0, -1)\}$. Clearly $(0, 0) \in L + K$ has no compact neighborhood in $K + L$.)

The assertion of 2.12 and the above example are due to J. Grabowski.

3. Embedding of $\sigma$-compact linear spaces onto pre-Hilbert spaces

Let $T$ be a topological class of compacta. A subset $Y$ of a copy $X$ of $\ell^2$ is called $T$-absorbing provided $Y = \bigcup_{n=1}^{\infty} K_n$, $K_n \in T$, and the following condition is satisfied:
(abs) for every \( K \in T \) and a closed set \( A \subset K \), every map \( f : K \to X \) that restricts on \( A \) to an embedding into \( Y \) can be arbitrarily closely approximated by embeddings into \( Y \) that agree with \( f \) on \( A \).

We will make use of the following version of the uniqueness theorem for absorbing sets [BP2, p. 123].

**3.1. Theorem.** Let \( Y_1 \) and \( Y_2 \) be \( \sigma \)-compact subsets of the copies \( X_1 \) and \( X_2 \) of \( \ell^2 \), respectively. Assume \( Y_i \) is a \( K(Y_i) \)-absorbing set, where \( K(Y_i) \) is the class of compacta embeddable in \( Y_i \), \( i = 1, 2 \). If \( Y_i \) can be represented as \( \bigcup_{n=1}^{\infty} K^i_n \), where \( K^i_n \) are compacta such that \( K^i_n \subset K(Y_2) \) and \( K^i_n \subset K(Y_1) \) for all \( n = 1, 2, \ldots \) and \( i = 1, 2 \), then there exists a homeomorphism \( h \) of \( X_1 \) onto \( X_2 \) with \( h(Y_1) = Y_2 \).

The proof of 3.1 can be obtained by using a standard back and forth argument [BP2, p. 123]. Note that if \( Y \) is \( T \)-absorbing, then the condition (abs) remains true for an arbitrary compactum which is a union of two elements of \( T \), see [DM1, p. 412] (consequently, there is no need to require that \( T \) is additive).

We apply 3.1 with \((X,Y)\) a pair of locally convex separable metric linear spaces so that \( Y = E \) is infinite-dimensional and \( \sigma \)-compact and \( X = \hat{E} \) is its linear completion. By the Kadec-Anderson theorem [BP2, p. 189], \( \hat{E} \) is a copy of \( \ell^2 \). The following fact was proved in [Do3].

**3.2. Proposition.** The space \( E \) is a \( K(E) \)-absorbing subset of \( \hat{E} \).

**3.3. Lemma.** Expressing \( E = \bigcup_{n=1}^{\infty} K_n \), where \( K_n \) are compacta, there exists a linear (not necessarily continuous) injective transformation \( T \) of \( E \) into \( \ell^2 \) such that \( T \mid K_n \) is continuous, \( n = 1, 2, \ldots \).

**Proof.** Assume \( K_1 \subseteq K_2 \subseteq \cdots \). Pick a sequence of continuous linear functionals \( \{ x^*_n \}_{n=1}^{\infty} \) which separates points of \( E \). Let

\[
p_n = \max_{x \in K_n} \left( |x^*_1(x)|, |x^*_2(x)|, \ldots, |x^*_n(x)| \right).
\]

Write \( T(x) = \left( \frac{x^*_1(x)}{p_1}, \frac{x^*_2(x)}{p_2}, \ldots, \frac{x^*_n(x)}{p_n}, \frac{x^*_n(x)}{p_n} \right) \) and observe that for every \( x \in K_n \) and every \( i \geq n \) we have

\[
\left| \frac{x^*_i(x)}{p_i} \right| \leq \frac{1}{p_i}.
\]

It follows that \( T(K_n) \) is contained in a compact subset of \( \ell^2 \). Since the topology on compacta in \( \ell^2 \) coincides with the coordinatewise convergence topology, \( T \mid K_n \) is continuous. Clearly, \( T \) is a linear transformation of \( E \) into \( \ell^2 \).

**3.4. Theorem.** Every \( \sigma \)-compact locally convex metric linear space \( E \) is homeomorphic to a pre-Hilbert space \( H \). Moreover, there exists a homeomorphism of the linear completions \( \hat{E} \) of \( \hat{H} \) of \( E \) and \( H \), respectively, which sends \( E \) onto \( H \).

**Proof.** Represent \( E = \bigcup_{n=1}^{\infty} K_n \), where \( K_n \) are compacta. Pick a transformation \( T \) of 3.3. Apply 3.1 with \( Y_1 = E \) and \( Y_2 = T(E) \). By 3.2, each \( Y_i \) is a \( K(Y_i) \)-absorbing set. Since \( T \mid K_n \) is continuous the other assumption of 3.1 is also satisfied. Now, the theorem follows from 3.1.

**3.5. Remark.** The Hilbert space \( \hat{H} \) in Theorem 3.4 can be replaced by an arbitrary locally convex complete separable metric linear space \( F \). Let \( \{(x_n, x^*_n)\}_{n=1}^{\infty} \) be a
biorthogonal sequence (i.e., $x_k^*(x_n) = 1$ and $x_k^*(x_m) = 0$ for $n \neq k$) such that the $x_n$’s are linearly dense in $F$ and the $x_n$’s separate points of $F$ (see, e.g., [Kl, Cor. 2.3]). Replacing, if necessary, the $x_n$’s by suitable scalar multiples the formula $S((t_n)) = \sum_{n=1}^{\infty} t_n x_n$ defines a continuous linear injection of $\ell^2$ onto a dense subspace of $F$. Now, the argument of 3.4 applies, and there exists a homeomorphism of $\hat{H}$ onto $F$ which carries $T(E)$ onto $S(T(E))$.

We do not know whether the assertion of 3.4 can be extended over all convex $\sigma$-compacta $C$ of locally convex metric linear spaces $E$. The case where the closure $\overline{C}$ of $C$ in $\hat{E}$ is locally compact is obviously settled by Theorem 1.1. Assume that $\overline{C}$ is nonlocally compact. By a result of [DT2], $\overline{C}$ is then homeomorphic to $\ell^2$. Note that 3.3 easily extends for the convex case. However, we are not able to show that $C$ is a $\mathcal{K}(C)$-absorbing set in $\overline{C}$ and consequently we cannot apply 3.1 (it is even unclear how to check the condition (abs) for such $C$ contained in $\ell^2$). In general, we do not know whether $C$ must be homogeneous. The following example shows that not all convex sets with nonlocally compact closures are homogeneous.

3.6. Example. Let $C = B \cup A$, where $B$ is the open unit ball in $\ell^2$ and $A$ is a copy of the rationals embedded into its sphere. Clearly, $C$ is convex and each point $p \in A \subset C$ has no complete neighborhood while $0 \in C$ has a basis of complete neighborhoods in $C$.

Let us finally recall that, according to [CDM], $C$ is a $\mathcal{K}(C)$-absorbing set in $\overline{C}$ in the following instances:

- (1) if $C$ is a countable union of finite-dimensional compacta,
- (2) if $C$ contains a Keller set.

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