

**A FORMULA WITH NONNEGATIVE TERMS
FOR THE DEGREE OF THE DUAL VARIETY
OF A HOMOGENEOUS SPACE**

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ABSTRACT. Let G be a reductive group and P a parabolic subgroup. For every P -regular dominant weight λ let $X(\lambda)$ denote the variety G/P embedded in the projective space by the embedding corresponding to the ample line bundle $\mathcal{L}(\lambda)$. Writing $\lambda = \rho_P + \sum_{i=1}^n m'_i \omega_i$, we prove that the degree $d(\lambda)^\vee$ of the dual variety to $X(\lambda)$ is a polynomial with nonnegative coefficients in m'_1, \dots, m'_n . In the case of homogeneous spaces G/B we find an expression for the constant term of this polynomial.

1. INTRODUCTION

Let X be a nonsingular projective variety of dimension l embedded in the projective space P^n . Let H be the hyperplane section divisor on X . We are interested in the degree of the dual variety X^\vee contained in the dual projective space $(P^n)^\vee$. There are various formulas for this degree, notably the formula in terms of H and the Chern classes of a sheaf of differentials of X (due to Kleiman, cf. [K1]), and in terms of Cayley-Koszul type complexes (cf. [G-K-Z], [W]). The drawback of these formulas is that they give a degree as the alternating sum of positive terms, and therefore it is difficult to decide when the degree is equal to 0 (which means that X^\vee has codimension ≥ 2).

Therefore it is desirable to give a formula for the degree which expresses it as a sum of nonnegative terms. The goal of this note is to reformulate Kleiman's formula in such a way.

The contents of the paper are as follows. In section 2 we prove the general result on nonnegativity. The proof is a simple extension of Katz's proof of Kleiman's formula (cf. [K]). In section 3 we apply our result to the duals of homogeneous spaces G/P . We show that in this case it gives the degree as a sum of nonnegative terms. More precisely, if G/P is embedded by a very ample line bundle $\mathcal{L}(\lambda)$ where λ is a weight such that $\lambda - \rho_P = \sum_{i=1}^s m'_i \omega_i$ where ω_i are the fundamental weights corresponding to the generators of $\text{Pic } G/P$ (see section 3 for the notation), then the degree $d(\lambda)^\vee$ of the dual variety equals $Q(m'_1, \dots, m'_s)$ where Q is a polynomial with nonnegative coefficients. We apply the formula to give a short proof of one

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of the results of Knop and Mentzel from [K-M]. In section 4 we investigate the constant term of the polynomial Q when $P = B$ is a Borel subgroup in G . We find the expression for this constant term in terms of permanents of certain matrices associated to a root system of G .

2. THE CONORMAL CONSTRUCTION, KLEIMAN'S FORMULA AND POSITIVITY

We start with some notation. For any smooth projective variety T we will denote by $A(T)$ the Chow ring of T . Given $z \in A(T)$ we will denote by

$$\int_T z$$

the degree of the graded component of z of dimension 0. We will call this number the degree of z . The class in $A(T)$ of an invertible sheaf \mathcal{L} on T will be also denoted by \mathcal{L} .

Let X be a nonsingular variety of dimension 1. Let us consider an independent variable t and the polynomial

$$P(t) = \sum_{i=0}^l c_{l-i}(\Omega_X^1) t^{i+1}.$$

We will denote by $P^{(j)}(t)$ the j th derivative of $P(t)$.

Let \mathcal{L} be a very ample line bundle over X . Throughout this section we assume that \mathcal{L} can be written

$$\mathcal{L} = \mathcal{L}_1 + \cdots + \mathcal{L}_r$$

where \mathcal{L}_i are the bundles such that the corresponding linear systems have no base points and such that for each i there is a representative of the linear system corresponding to \mathcal{L}_i which is a smooth subvariety of X . The bundle \mathcal{L} defines the embedding of X into the projective space P^n . We denote by X^\vee the projectively dual variety to X .

Proposition 1. *let X and \mathcal{L} be as above. Then for each $j = 1, \dots, l$ and for each monomial $\mathcal{L}_1^{u_1}, \dots, \mathcal{L}_r^{u_r}$ of degree j ,*

$$(1) \quad \int_X P^{(j)}(\mathcal{L}) \cdot \mathcal{L}_1^{u_1} \cdots \mathcal{L}_r^{u_r} \geq 0.$$

Proof. The main tool in the proof is the conormal construction. Let X , \mathcal{L} and X^\vee be as above and let I be the sheaf of ideals on P^n defining X . We denote by N the conormal sheaf I/I^2 . This is a locally free sheaf on X of dimension $n-l$. Let $P(N)$ be the projective bundle on X of lines in N^\vee . The bundle $P(N)$ can be identified with a subvariety of $X \times (P^n)^\vee$ consisting of pairs (x, H) such that H is tangent to X at x . Let us denote by

$$\nu: P(N) \rightarrow X \times (P^n)^\vee$$

the embedding of $P(N)$ into $X \times (P^n)^\vee$, and by

$$\phi: P(N) \rightarrow (P^n)^\vee$$

the composition of ν with the second projection. By definition X^\vee is the image of ϕ . Let us notice that ν can be given locally in the following form. If F is a local

section of I/I^2 , then

$$(2) \quad P(I/I^2)_x \ni F \mapsto (x, \text{hyperplane } \sum \frac{\partial F}{\partial X_i}(x)T_i).$$

Let Y be a smooth subvariety of codimension y in X . Let us denote by P_Y the total space of $P(N)|_Y$ and by Y^\vee the image $\phi_*(P_Y)$.

If H^\vee denotes the class of hyperplane section in $A((P^n)^\vee)$, then by definition

$$(3) \quad \deg(Y^\vee) = \int_{(P^n)^\vee} (H^\vee)^{n-y-1} \cdot (Y^\vee).$$

We can rewrite this as

$$(4) \quad \deg(Y^\vee) = \frac{1}{\deg(\phi)} \int_{(P^n)^\vee} (H^\vee)^{n-y-1} \cdot \phi_*(P_Y)$$

where $\deg(\phi)$ denotes the degree of the morphism from P_Y to Y^\vee induced by ϕ . Using the projection formula we get

$$(5) \quad \deg(Y^\vee) = \frac{1}{\deg(\phi)} \int_{P_Y} (\phi^*(H^\vee))^{n-y-1}.$$

It remains to calculate the class $\phi^*(H^\vee)$. Let us denote by ψ the invertible sheaf $\mathcal{O}_{P(N)}(1)$ on $P(N)$, and by π the structural morphism $\pi: P(N) \rightarrow X$. Then one can see easily that in $A(P(N))$ we have

$$(6) \quad \phi^*(H^\vee) = \psi - \pi^*(H).$$

Restricting to P_Y we can finally write

$$(7) \quad \deg(Y^\vee) = \frac{1}{\deg(\phi)} \int_{P_Y} (\psi - \pi^*(H))^{n-y-1}.$$

This means that we can rewrite formula (5) as

$$(8) \quad \deg(Y^\vee) = \frac{1}{\deg(\phi)} \int_{P_Y} (\psi - \pi^*(H))^{n-y-1}.$$

Let us recall that the total Chern class of the vector bundle \mathcal{V} of dimension v over a scheme X is $C(\mathcal{V}) := \sum_{i=0}^v c_i(\mathcal{V})$.

Now we can apply Lemmas 5.3 and 5.4 from [K] to the variety Y and the bundle $N|_Y$.

We get the formula

$$(9) \quad \deg(Y^\vee) = \frac{(-1)^{l+y}}{\deg(\phi)} \int_Y \frac{(1 + \mathcal{L})^{n-y-1}}{C(N|_Y)}.$$

Let us denote by $C(X)$ the total Chern class of the tangent bundle T_X . Then we have

$$C(N)C(X) = C(P^n)|_X = (1 + \mathcal{L})^{n+1}.$$

Putting these two formulas together gives

$$(10) \quad \deg(Y^\vee) = \frac{(-1)^{l+y}}{\deg(\phi)} \int_Y \frac{C(X)|_Y}{(1 + \mathcal{L})^{y+1}}.$$

Using the projection formula for the embedding of Y into X we get

$$(11) \quad \deg(Y^\vee) = \frac{(-1)^{l+y}}{\deg(\phi)} \int_X \frac{[Y]C(X)}{(1+\mathcal{L})^{y+1}}$$

where $[Y]$ denotes the fundamental class of Y .

Let us recall that we assumed that $\mathcal{L} = \mathcal{L}_1 + \dots + \mathcal{L}_r$ where each \mathcal{L}_i is base point free, and such that each class \mathcal{L}_i in $A(X)$ has a smooth representative. Let us consider the morphism of X to the product \mathcal{P} of projective spaces defined by the bundles $\mathcal{L}_1, \dots, \mathcal{L}_r$. By repeatedly applying to \mathcal{P} the Bertini type theorem ([Ha, Ch. III, Theorem 10.8]) we see that for each exponent u_1, \dots, u_r there exists a nonsingular subvariety Y in X such that $[Y] = \mathcal{L}_1^{u_1} \dots \mathcal{L}_r^{u_r}$. Applying (11) to such Y , we get the statement of the theorem. \square

The proof of the proposition also has the following consequence.

Corollary 2. *Let X, \mathcal{L} and $\mathcal{L}_1, \dots, \mathcal{L}_r$ be as in Proposition 1. Let us consider the smooth subvariety Y such that $[Y] = \mathcal{L}_1^{u_1} \dots \mathcal{L}_r^{u_r}$. Let us assume that $\dim \phi_*(P_Y) = \dim P_Y$. Then the number*

$$\int_X P^{(j)}(\mathcal{L}) \cdot \mathcal{L}_1^{u_1} \dots \mathcal{L}_r^{u_r}$$

is positive.

3. APPLICATIONS TO DUAL VARIETIES OF HOMOGENEOUS SPACES

Let G be a reductive group and let P be a parabolic subgroup. Let Φ denote the root system of G , let Δ be a basis of Φ and let Δ' be the subset of Δ corresponding to P . Let $\omega_1, \dots, \omega_s$ be the set of fundamental weights corresponding to the roots from Δ' . For any weight λ we will denote by $\mathcal{L}(\lambda)$ the corresponding line bundle on G/P . It is well known that the bundles $\mathcal{L}(\omega_1), \dots, \mathcal{L}(\omega_s)$ generate the Picard group of G/P . Moreover, very ample line bundles on G/P are the bundles $\mathcal{L}(\lambda)$ where $\lambda = \sum_{i=1}^s m_i \omega_i$, where $m_i > 0$ for $i = 1, \dots, s$. We denote by ρ_P the weight $\sum_{i=1}^s \omega_i$. Let $\lambda = \sum_{i=1}^s m_i \omega_i$ be a very ample line bundle on G/P . Let $X(\lambda)$ denote the variety G/P embedded in the projective space by the embedding corresponding to λ . We are interested in the degree of the variety $X(\lambda)^\vee$. Let us denote this degree by $d(\lambda)^\vee$.

Finally we recall Kleiman's formula ([Kl] (IV, 64) or [K], section 5). It says that if X is a smooth projective variety of dimension l in P^n and if $\mathcal{L} = \mathcal{O}_{P^n}(1)|_X$, then

$$d(X^\vee) = \sum_{i=0}^l (i+1) \int_X c_{l-i}(\Omega_X^1) \mathcal{L}^i.$$

In our situation it follows that $d(\lambda)^\vee$ is a polynomial in m_1, \dots, m_s .

Let us write $\lambda = \lambda' + \rho_P$. Let us denote $\lambda' = \sum_{i=1}^s m'_i \omega_i$, where $m'_i = m_i - 1$ for each i and therefore $m'_i \geq 0$ for each i . We can certainly find a polynomial $Q(m'_1, \dots, m'_s)$ such that $d(\lambda)^\vee = Q(m'_1, \dots, m'_s)$. Our main application is

Theorem 3. *The polynomial $Q(m'_1, \dots, m'_s)$ has nonnegative coefficients.*

Proof. We write $d(\lambda)^\vee = P(m_1, \dots, m_s)$ where $P(m_1, \dots, m_s)$ is a polynomial in m_1, \dots, m_s . Then

$$Q(m'_1, \dots, m'_s) = P(m'_1 + 1, \dots, m'_s + 1).$$

We apply Proposition 1 to $\mathcal{L} = \rho_P, \mathcal{L}_i = \omega_i$. It is well known that the assumptions of Proposition 1 are satisfied for these bundles. Now the Taylor formula for the values of Q and Proposition 1 imply Theorem 3. \square

Examples. (1) Let $G = SL(n)$ and let P be the stabilizer of the line in the n -dimensional space. Then $G/P = P^{n-1}$ and $\lambda = m\omega_1$. The degree of $X(\lambda)^\vee$ is equal to the degree of the discriminant of a homogeneous polynomial of degree m in n variables. Thus $d(m)^\vee = n(m-1)^{n-1}$. Substituting $m' = m-1$ we see that $d(m')^\vee = n(m')^{n-1}$, so it is a polynomial with nonnegative coefficients.

(2) Let $G = SL(3)$ and let $P = B$ be the Borel subgroup. The weight λ can be written as $\lambda = m_1\omega_1 + m_2\omega_2$. It is easy to see using Kleiman's formula that

$$d(\lambda)^\vee = 12(m_1^2 m_2 + m_1 m_2^2) - 6(m_1^2 + 4m_1 m_2 + m_2^2) + 12(m_1 + m_2) - 6.$$

Substituting $m'_1 = m_1 - 1$ and $m'_2 = m_2 - 1$ we see that

$$\begin{aligned} d(\lambda)^\vee &= 12((m'_1)^2 m'_2 + m'_1 (m'_2)^2) + 6((m'_1)^2 + 4m'_1 m'_2 + (m'_2)^2) \\ &\quad + 12(m'_1 + m'_2) + 6. \end{aligned}$$

We will denote the polynomial Q constructed above by $Q_{G,P}$. It is an interesting (but probably very difficult) problem to find the combinatorial description of the coefficients of polynomials $Q_{G,P}$.

We finish this section with two quick applications of Theorem 3.

Corollary 4. *Let us assume that λ satisfies the condition $m_i \geq 2$ for $i = 1, \dots, s$. Then $d(\lambda)^\vee > 0$, i.e., the variety $X(\lambda)^\vee$ is a hypersurface.*

Proof. Our assumption means that $m'_i > 0$ for each i . This means that the value of the highest term of Q evaluated at λ is positive. This, together with Theorem 3, proves the corollary. \square

Corollary 4 was proven in the paper of Knop and Mentzel [K-M] by a different method.

Corollary 5. *Let G be a simple group. Then for any positive number N there are only finitely many dominant integral weights λ such that $d(\lambda)^\vee \leq N$.*

Proof. It is enough to prove this statement for the weights λ corresponding to the embeddings of the fixed G/P . If $X(\rho)^\vee$ is a hypersurface, then by [K], Proposition 3.5, ϕ is a birational isomorphism. Thus for each monomial $\mathcal{L}(\omega_1)^{u_1} \cdots \mathcal{L}(\omega_r)^{u_r}$ we can choose the smooth representative Y of this class, so $\dim \phi_*(P_Y) = \dim P_Y$. By Corollary 2 (applied to $\mathcal{L} = \rho_P, \mathcal{L}_i = \mathcal{L}(\omega_i)$) we see that each monomial in $Q(m'_1, \dots, m'_s)$ occurs with a positive coefficient. The result follows immediately. Now let us assume that $\text{codim } X(\rho_P)^\vee > 1$. Then by the classification of Knop-Mentzel ([K-M]) we see that P is a maximal parabolic subgroup, so $s = 1$. Then $Q(m'_1)$ is a polynomial of positive degree, and the result follows. \square

4. THE DEGREE OF THE DUAL VARIETY TO G/B AND PERMANENTS

In this section we investigate the constant term of the polynomial $Q_{G,P}$ in the case when $P = B$ is a Borel subgroup of G . By definition this term is the degree $d(\rho)^\vee$ of the variety $X(\rho)^\vee$. By Kleiman's formula this degree equals

$$(12) \quad d(\rho)^\vee = \sum_s (N - s + 1) \int_{G/B} c_s(G/B) \rho^{N-s}.$$

Therefore we want to compute the terms $\int_{G/B} c_s(G/B) \rho^{N-s}$. These terms have a nice description in terms of permanents of certain matrices associated to root systems.

Before we start our calculations we change our setup slightly.

Since G/B has a cellular decomposition into locally closed cells (Schubert cells), the Chow ring $A^\bullet(G/B)$ is canonically isomorphic to the cohomology ring.

We can therefore make our computation in the cohomology ring. We can also assume we work with cohomology with rational coefficients, since we are going to compute integrals of forms (or make computations of degree).

Now let G be a topological group; denote by BG its classifying space, and by $E \rightarrow BG$ the classifying fibration. By definition of equivariant cohomology we have

$$H_G^*(X, Q) = H^*(E \times_G X, Q).$$

Now let us take G to be a semisimple simply connected group, T a maximal torus, $B \supset T$ a Borel subgroup and $X = G/B$. We also denote by K a maximal compact subgroup of G such that $T_K = T \cap K$ is a maximal torus in K so that $G/B = K/T_K$. Then $E \times_G X \cong E \times_K X$ (\cong means homotopy equivalent) since G/K is contractible. But $E \times_K X = E/T_K = BT_K$.

Choosing a basis of the character ring of T_K , we get an isomorphism of this group with $(S^1)^r$, where $r = \dim T = \text{rank } G$. Now $BS^1 \cong P^\infty$, so $H^*(P^\infty, Q) = Q[x]$, where x is the hyperplane class.

It follows immediately that

$$(13) \quad H_G^*(X, Q) = H^*(BT_K, Q) = S^*(X(T) \otimes_Z Q),$$

since $B(S^1)^r = (BS^1)^r$.

Now it is well known that $H^*(BG, Q) = H^*(BK, Q) - H^*(BT_K Q)^W$, W being the Weyl group (compare [B]).

Now consider the spectral sequence of the fibration $BT_K \rightarrow BK$ with fiber X . All spaces involved (total, base, fiber) have only cohomology in even degree, so the spectral sequence collapses at the E_2 term. We deduce that

$$(14) \quad H^*(X, Q) = H^*(BT_K, Q)/I$$

where I is the ideal generated by the elements of positive degree in $H^*(BK, Q)$, i.e., by the positive degree W -invariants in $S^*(X(T) \otimes_Z Q)$, as desired.

The lattice $X(T)$ can be identified with the weight lattice Λ . Denoting $\Lambda_Q = \Lambda \otimes Q$ we can rewrite (13) and (14) as

$$(13') \quad H_G^*(X, Q) = \text{Sym}(\Lambda_Q),$$

$$(14') \quad H^*(G/B, Q) = H_G^*(G/B, Q)/I,$$

where I is the ideal generated by W -invariant elements of positive degree.

Let $\lambda \in \Lambda$. Then we denote the equivariant Chern class of the line bundle $\mathcal{L}(\lambda)$ which admits a canonical G -linearisation by λ .

For any coroot $\alpha^\vee \in \Lambda_Q^*$ we define the operator D_{α^\vee} on $H_G^*(G/B, Q)$ by

$$(15) \quad D_{\alpha^\vee}(\lambda) = \langle \alpha^\vee, \lambda \rangle \quad \text{for all } \lambda \in \Lambda_Q.$$

We consider now the differential operator on $H_G^*(G/B, Q)$:

$$(16) \quad D = \frac{\prod_{\alpha > 0} D_{\alpha^\vee}}{\prod_{\alpha > 0} \langle \alpha^\vee, \rho \rangle}.$$

Let N be the number of positive roots in Φ . Then the operator D decreases the degree by N , so in particular we get a linear form

$$D: H_G^N(G/B, Q) \rightarrow H_G^0(G/B, Q) = Q.$$

Proposition 6. *Let $x \in H^N(G/B, Q)$. Then*

$$\int_{G/B} x = D\tilde{x}$$

where $\tilde{x} \in H_G^N(G/B, Q)$ is a lifting of x .

Proof. Every simple reflection $s_i \in W$ permutes the positive coroots other than α_i^\vee and $s_i\alpha_i^\vee = -\alpha_i^\vee$. Therefore we deduce that if $y \in H_G^N(G/B, Q)$, then $D(wy) = (-1)^{l(w)}D(y)$. In particular D kills every irreducible component of $H_G^N(G/B, Q)$ not isomorphic to the sign representation. Since the kernel K of the map

$$H_G^N(G/B, Q) \rightarrow H_G^0(G/B, Q)$$

is W -stable and contains no irreducible representation isomorphic to the sign representation, we have

$$D|_K = 0.$$

This means that $D\tilde{x}$ does not depend on the choice of \tilde{x} and that

$$D\tilde{x} = a \int_{G/B} x.$$

To finish the proof it is enough to show that $a = 1$. For this we consider the bundle $\mathcal{L}(\rho)$. This is a very ample bundle on G/B , and by the Weyl dimension formula we have

$$h^0(G/B, \mathcal{L}(\rho)^{\otimes n}) = (n+1)^N.$$

It follows that (again denoting by ρ the Chern class $c_1(\mathcal{L}(\rho))$)

$$\int_{G/B} \rho^N = N!.$$

On the other hand, calculating directly from the definition we see that

$$D\rho^N = N!,$$

so $a = 1$ and the proposition is proved. \square

We now want to compute

$$(17) \quad \int_{G/B} c_s(G/B) \rho^{N-s}.$$

Let us recall that

$$c_s(G/B) = \sum_{\Gamma \subset \Phi^+, \text{card } \Gamma = s} \prod_{\alpha \in \Gamma} \alpha.$$

By Proposition 6 we need to compute

$$D \left(\sum_{\Gamma \subset \Phi^+, \text{card } \Gamma = s} \left(\prod_{\alpha \in \Gamma} \alpha \right) \rho^{N-s} \right) = (N-s)! \sum_{\Gamma \subset \Phi^+, \Delta \subset \Phi^+} \prod_{\alpha \in \Gamma, \beta^\vee \in \Delta} \frac{(\beta^\vee, \alpha)}{(\beta^\vee, \rho)}.$$

We can summarize this result in the following way. Fix an order $\alpha_1, \dots, \alpha_N$ on Φ^+ and $\beta_1^\vee, \dots, \beta_N^\vee$ on $\Phi^{\vee+}$. Let us define the $N \times N$ matrix

$$(18) \quad B = \left(\begin{array}{c} (\beta_i^\vee, \alpha_j) \\ (\beta_i^\vee, \rho) \end{array} \right)_{1 \leq i, j \leq N}.$$

Then we have

Proposition 7.

$$\int_{G/B} c_s(G/B) \rho^{N-s} = (N-s)! P_s(B)$$

where $P_s(B)$ is the sum of all the permanents of $s \times s$ submatrices of the matrix B .

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