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ON A THEOREM OF PRIVALOV AND NORMAL FUNCTIONS

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ABSTRACT. A well known result of Privalov asserts that if f is a function which is analytic in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, then f has a continuous extension to the closed unit disc and its boundary function $f(e^{i\theta})$ is absolutely continuous if and only if f' belongs to the Hardy space H^1. In this paper we prove that this result is sharp in a very strong sense. Indeed, if, as usual, $M_1(r, f') = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f'(re^{i\theta})\right| d\theta$, we prove that for any positive continuous function ϕ defined in $(0,1)$ with $\phi(r) \to \infty$, as $r \to 1$, there exists a function f analytic in Δ which is not a normal function and with the property that $M_1(r, f') \leq \phi(r)$, for all r sufficiently close to 1.

1. Introduction and statement of results

Let Δ denote the unit disc $\{z \in \mathbb{C} : |z| < 1\}$. For $0 < r < 1$ and g analytic in Δ we set

$$M_p(r, g) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|g(re^{i\theta})\right|^p d\theta\right)^{1/p}, \quad 0 < p < \infty,$$

$$M_\infty(r, g) = \max_{|z|=r} |g(z)|.$$

For $0 < p \leq \infty$ the Hardy space H^p consists of those functions g, analytic in Δ, for which

$$||g||_{H^p} = \sup_{0<r<1} M_p(r, g) < \infty.$$

A classical result of Privalov [8, Th. 3.11] asserts that a function f analytic in Δ has a continuous extension to the closed unit disc $\overline{\Delta}$ whose boundary values are absolutely continuous on $\partial\Delta$ if and only if $f' \in H^1$. In particular, $f' \in H^1 \Rightarrow f \in H^\infty$. The question of studying the possibility of obtaining results of this kind if the condition $f' \in H^1$ is slightly weakened has been considered by several authors. A result of Bennet and Stoll [3] shows that if the function f is analytic in Δ and f' is the Cauchy-Stieltjes integral of a finite complex Borel measure on $\partial\Delta$, then f belongs to $BMOA$, the space of all H^1-functions whose boundary values have bounded mean oscillation on $\partial\Delta$. A stronger result was obtained by Baernstein and Brown in [2]. Indeed, Proposition 3 of [2] shows that if the function f is analytic

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in Δ and $f' \in$ weak-H^1, then f belongs to the mean Lipschitz space $\Lambda(2,1/2)$, and it is well known that $\Lambda(2,1/2) \subset BMOA$ (see [7] and [6]).

On the other hand, Yamashita proved in [18] that there exists a function f analytic in Δ with $f' \in H^p$ for all $p \in (0,1)$ but such that f is not even a normal function in the sense of Lehto and Virtanen [9]. We recall that a function f which is meromorphic in Δ is a normal function if and only if

$$\sup_{z\in\Delta}(1-|z|^2)\frac{|f'(z)|}{1+|f(z)|^2} < \infty.$$

We refer to [1] and [15] for the theory of normal functions.

In view of these results, it seems natural to study what happens if we substitute the condition $f' \in H^1$ by a condition of the type "$M_1(r,f')$ grows to ∞ slowly enough".

Let us start considering some very simple functions. For every $\epsilon > 0$, we set

$$f_\epsilon(z) = \left(\log\frac{1}{1-z} + i\pi\right)^{1+\epsilon}, \quad z \in \Delta. \tag{1}$$

Then, for every $\epsilon > 0$, f_ϵ is holomorphic in Δ and it is easy to see that

$$M_1(r, f'_\epsilon) = O\left(\left(\log\frac{1}{1-r}\right)^{1+\epsilon}\right), \quad \text{as} \quad r \to 1, \tag{2}$$

while,

$$f'_\epsilon(r) \approx (1+\epsilon)\frac{1}{1-r}\left(\log\frac{1}{1-r}\right)^{\epsilon}, \quad \text{as} \quad r \to 1. \tag{3}$$

Notice that (3) shows that f_ϵ is not a Bloch function (see [1] for the theory of Bloch functions) and, hence, $f \notin BMOA$. Consequently, we see that the condition

$$M_1(r, f') = O\left(\left(\log\frac{1}{1-r}\right)^{1+\epsilon}\right), \quad \text{as} \quad r \to 1,$$

for some $\epsilon > 0$ does not even imply that f is a Bloch function. However, we can prove a much stronger result showing that no restriction on the growth of $M_1(r,f')$ other than its boundedness is enough to conclude that f is a normal function. More precisely, we can prove the following result.

Theorem 1. *Let ϕ be any positive continuous function defined in $[0,1)$ with $\phi(r) \to \infty$, as $r \to 1$. Then, there exists a function f analytic in Δ which is not a normal function and having the property that*

$$M_1(r,f') \le \phi(r), \quad \text{for all } r \text{ sufficiently close to } 1.$$

2. Proof of Theorem 1

Clearly, it suffices to prove that there exists a function f which is analytic and non-normal in Δ and a constant $C > 0$ such that

$$M_1(r,f') \le C\phi(r), \quad \text{for all } r \text{ sufficiently close to } 1. \tag{4}$$

Also, we may assume without loss of generality that ϕ satisfies also the following two conditions:

$$\phi \text{ is increasing and } \phi(r) \ge 1 \text{ for all } r \in [0,1), \tag{5}$$

and

$$(1-r)^2\phi(r) \to 0, \quad \text{as } r \to 1. \tag{6}$$

Indeed, let ϕ be as in Theorem 1. There exists a positive constant A such that $A\phi(r) \geq 1$ for all $r \in [0,1)$. Then, we set

$$\phi_1(r) = \min\left(A\phi(r), \frac{2}{1-r}\right), \quad 0 < r < 1,$$

and we let ϕ_2 denote the highest increasing minorant of ϕ_1, that is,

$$\phi_2(r) = \inf_{r \leq s < 1} \phi_1(s), \quad 0 \leq r < 1.$$

Then it is clear that ϕ_2 is a positive and continuous function in $[0,1)$ with $\phi_2(r) \leq A\phi(r)$, for all $r \in [0,1)$ and $\phi_2(r) \to \infty$, as $r \to 1$. Furthermore, (5) and (6) hold with ϕ_2 in the place of ϕ.

Hence we shall assume that ϕ satisfies (5) and (6) in addition to the conditions of Theorem 1.

Let $\omega : [0,1] \to \mathbb{R}$ be defined as follows:

$$\begin{cases} \omega(0) = 0, \\ \omega(\delta) = \delta\phi(1-\delta)^{1/2}, \quad 0 < \delta \leq 1. \end{cases} \tag{7}$$

Hence,

$$\phi(r) = \left[\frac{\omega(1-r)}{1-r}\right]^2, \quad 0 < r < 1. \tag{8}$$

Using (6), it is easy to see that ω is positive and continuous in $[0,1]$. Moreover,

$$\frac{\omega(\delta)}{\delta} \to \infty, \quad \text{as } \delta \to 0, \tag{9}$$

and (5) implies

$$\frac{\omega(\delta)}{\delta} \quad \text{is decreasing in } (0,1] \tag{10}$$

and

$$\omega(\delta) \geq \delta, \quad \text{for all } \delta \in [0,1]. \tag{11}$$

Take a fixed number λ with $0 < \lambda < 1$ and let us consider the sequence of numbers $\{\delta_n\}_{n=0}^{\infty}$, defined inductively as

$$\begin{cases} \delta_0 = 1, \\ \delta_{n+1} = \min\left\{\delta \in [0,1) : \max\left[\frac{\omega(\delta)}{\omega(\delta_n)}, \frac{\omega(\delta_n)\delta}{\delta_n\omega(\delta)}\right] = \lambda\right\}, \quad n \geq 0. \end{cases} \tag{12}$$

This sequence was defined by K. I. Oskolkov in [11, 12, 13] and [14] (see also [10]) under the hypothesis of ω being a modulus of continuity, hence, (see Proposition 2.1 of [4]) in these papers ω is assumed to be increasing and subadditive. However, it is clear that the definition of $\{\delta_n\}$ makes sense in our setting. In the following lemma we shall list the main properties of the sequence $\{\delta_n\}$ which will be used in the sequel.

Lemma 1. *Let ω and λ be as above and let $\{\delta_n\}_{n=0}^{\infty}$ be defined by (12). Then $\{\delta_n\}$ is a decreasing sequence of positive numbers with $\delta_n \to 0$, as $n \to \infty$. Moreover, for all $n \geq 0$, we have*

$$\omega(\delta_{n+1}) \leq \lambda\omega(\delta_n), \tag{13}$$

$$\delta_{n+1} \leq \lambda^2 \delta_n, \tag{14}$$

$$\omega(\delta_{n+1})\delta_{n+1} \leq \lambda^3 \omega(\delta_n)\delta_n. \tag{15}$$

Furthermore, there exists an absolute constant $\beta > 0$ (which depends only on λ) such that

$$\sum_{k=0}^{\infty} \omega(\delta_k) \min\left(1, \frac{\delta_n}{\delta_k}\right) \leq \beta\omega(\delta_n), \quad n \geq 1. \tag{16}$$

We remark that (16) is the substitute in our setting of the inequality (2.12) of [13].

Proof of Lemma 1. First let us notice that (13) and (14) are direct consequences of the definition of the sequence $\{\delta_n\}$, and then (15) and the fact that δ_n tends monotonically to zero follow trivially.

Since $\{\delta_n\}$ is decreasing, we have

$$\sum_{k=0}^{\infty} \omega(\delta_k) \min\left(1, \frac{\delta_n}{\delta_k}\right) = \sum_{k=0}^{n} \omega(\delta_k)\frac{\delta_n}{\delta_k} + \sum_{k=n+1}^{\infty} \omega(\delta_k). \tag{17}$$

Notice that (12) implies that

$$\frac{\omega(\delta_k)}{\delta_k} \leq \lambda \frac{\omega(\delta_{k+1})}{\delta_{k+1}}, \quad k \geq 0,$$

and, hence,

$$\frac{\omega(\delta_k)}{\delta_k} \leq \lambda^{n-k} \frac{\omega(\delta_n)}{\delta_n}, \quad 0 \leq k \leq n. \tag{18}$$

On the other hand, (13) implies that

$$\omega(\delta_k) \leq \lambda^{k-n}\omega(\delta_n), \quad k \geq n. \tag{19}$$

Then, using (17), (18) and (19), we deduce that

$$\begin{aligned}
\sum_{k=0}^{\infty} \omega(\delta_k) \min\left(1, \frac{\delta_n}{\delta_k}\right) &\leq \sum_{k=0}^{n} \lambda^{n-k}\omega(\delta_n) + \sum_{k=n+1}^{\infty} \lambda^{k-n}\omega(\delta_n) \\
&\leq 2\left(\sum_{j=0}^{\infty} \lambda^j\right)\omega(\delta_n) \\
&= \frac{2}{1-\lambda}\omega(\delta_n).
\end{aligned}$$

This proves (16) with $\beta = \frac{2}{1-\lambda}$ finishing the proof of Lemma 1. □

Once Lemma 1 has been proved, we continue the proof of Theorem 1. The function f that we are going to construct to prove Theorem 1 will be of the form $f(z) = B(z)F(z)$, where B will be a Blaschke product while the function F will be given by a series of analytic functions in Δ which converges uniformly on every compact subset of Δ. We start with the construction of the Blaschke product B, but first let us remark that from now on we shall be using the convention that C will denote an absolute positive constant which may be different at each occurrence.

Notice that (13) implies that $\omega(\delta_n) \to 0$, as $n \to \infty$, and, hence, there exists a positive integer N such that $\omega(\delta_n) < 1$, if $n \geq N$. Define

$$a_n = 1 - \delta_n \omega(\delta_n), \quad n \geq N. \tag{20}$$

Notice that (15) implies that the sequence $\{a_n\}_{n=N}^{\infty}$ satisfies the Blaschke condition, that is, $\sum_{n=N}^{\infty}(1-|a_n|) < \infty$. Let B denote the Blaschke product whose zeros are $\{a_n\}_{n=N}^{\infty}$, that is,

$$B(z) = \prod_{n=N}^{\infty} \frac{a_n - z}{1 - a_n z}, \quad z \in \Delta. \tag{21}$$

Now, we set

$$r_n = 1 - \delta_n, \quad n \geq N. \tag{22}$$

Protas proved in [16, p. 394] that

$$\int_{-\pi}^{\pi} |B'(re^{i\theta})|\, d\theta \leq 8\pi \sum_k \frac{1-|a_k|}{1-r+1-|a_k|}, \quad 0 < r < 1.$$

Using this inequality, (22) and (20), we deduce that, for every $n \geq N$, we have

$$\begin{aligned} (1-r_n)M_1(r_{n+1}, B') &\leq C(1-r_n)\sum_{k=N}^{\infty} \frac{1-|a_k|}{1-r_{n+1}+1-|a_k|} \\ &= C\sum_{k=N}^{\infty} \omega(\delta_k)\left[\frac{\delta_n\delta_k}{\delta_{n+1}+\delta_k\omega(\delta_k)}\right]. \end{aligned} \tag{23}$$

Now, (11) implies that $\delta_k^2 \leq \delta_{n+1} + \delta_k\omega(\delta_k)$ and hence it follows that

$$\frac{\delta_k\delta_n}{\delta_{n+1}+\delta_k\omega(\delta_k)} \leq \frac{\delta_n}{\delta_k}, \quad k, n \in \mathbb{N}. \tag{24}$$

On the other hand, since the sequence $\{\delta_n\}$ is decreasing, we easily see that

$$\frac{\delta_k\delta_n}{\delta_{n+1}+\delta_k\omega(\delta_k)} \leq 1, \quad \text{if } k > n,$$

which, using again the fact that $\{\delta_n\}$ is decreasing and (24), implies

$$\frac{\delta_k\delta_n}{\delta_{n+1}+\delta_k\omega(\delta_k)} \leq \min\left(1, \frac{\delta_n}{\delta_k}\right), \quad k, n \in \mathbb{N}. \tag{25}$$

Then (23), (25) and (16) give

$$(1-r_n)M_1(r_{n+1}, B') \leq C\omega(\delta_n), \quad n \geq N,$$

or, equivalently,

$$M_1(r_{n+1}, B') \leq C\frac{\omega(\delta_n)}{\delta_n}, \quad n \geq N. \tag{26}$$

Now we turn to construct the above mentioned function F. We set

$$F(z) = \sum_{j=N}^{\infty} \frac{\omega(\delta_j)\delta_j}{1 - z + \omega(\delta_j)\delta_j}, \quad z \in \Delta. \tag{27}$$

Clearly, this series converges uniformly on each compact subset of Δ, and therefore it defines a function which is analytic in Δ. Using (22), (25) and (16), we deduce that, for every $n \geq N$, we have

$$(1 - r_n) M_\infty(r_{n+1}, F) \leq \delta_n \sum_{j=N}^{\infty} \frac{\omega(\delta_j)\delta_j}{\delta_{n+1} + \omega(\delta_j)\delta_j} \leq C\omega(\delta_n),$$

or, equivalently,

$$M_\infty(r_{n+1}, F) \leq C \frac{\omega(\delta_n)}{\delta_n}. \tag{28}$$

Now, we have

$$F'(z) = \sum_{j=N}^{\infty} \frac{\omega(\delta_j)\delta_j}{(1 - z + \omega(\delta_j)\delta_j)^2}, \quad z \in \Delta,$$

and therefore we conclude that

$$\begin{aligned} M_1(r, F') &\leq C \sum_{j=N}^{\infty} \omega(\delta_j)\delta_j \int_{-\pi}^{\pi} \frac{d\theta}{|1 + \omega(\delta_j)\delta_j - re^{i\theta}|^2} \\ &= C \sum_{j=N}^{\infty} \frac{\omega(\delta_j)\delta_j}{(1 + \omega(\delta_j)\delta_j)^2} \int_{-\pi}^{\pi} \frac{d\theta}{\left|1 - \frac{re^{i\theta}}{1+\omega(\delta_j)\delta_j}\right|^2} \\ &\leq C \sum_{j=N}^{\infty} \frac{\omega(\delta_j)\delta_j}{(1 + \omega(\delta_j)\delta_j)^2} \frac{1}{1 - \frac{r}{1+\omega(\delta_j)\delta_j}} \\ &\leq C \sum_{j=N}^{\infty} \frac{\omega(\delta_j)\delta_j}{1 - r + \omega(\delta_j)\delta_j}, \end{aligned} \tag{29}$$

which, with (22), implies

$$(1 - r_n) M_1(r_{n+1}, F') \leq C \sum_{j=N}^{\infty} \omega(\delta_j) \frac{\delta_n \delta_j}{\delta_{n+1} + \omega(\delta_j)\delta_j}, \quad n \geq N, \tag{30}$$

and then, noticing that the right hand side of (30) and the right hand side of (23) are the same and arguing as in the proof of (26), we obtain

$$M_1(r_{n+1}, F') \leq C \frac{\omega(\delta_n)}{\delta_n}, \quad n \geq N. \tag{31}$$

Finally, notice that for every j

$$\frac{\omega(\delta_j)\delta_j}{1 - r + \omega(\delta_j)\delta_j}$$

is a positive increasing function of r in $(0, 1)$ and hence, using Lebesgue's monotone convergence theorem, we deduce that

$$\lim_{r \to 1^-} F(r) = \sum_{j=N}^{\infty} \lim_{r \to 1^-} \frac{\omega(\delta_j)\delta_j}{1 - r + \omega(\delta_j)\delta_j} = \infty. \tag{32}$$

Once the functions B and F have been constructed, we set

$$f(z) = B(z)F(z), \quad z \in \Delta. \tag{33}$$

Then, since $|B(z)| \le 1$ for all z, using (26), (28), (31), (11), (8) and (22), we deduce that, for every $n \ge N$,

$$\begin{aligned} M_1(r_{n+1}, f') &\le M_1(r_{n+1}, B')M_\infty(r_{n+1}, F) + M_1(r_{n+1}, F') \\ &\le C\left(\frac{\omega(\delta_n)}{\delta_n}\right)^2 + C\frac{\omega(\delta_n)}{\delta_n} \\ &\le C\left(\frac{\omega(\delta_n)}{\delta_n}\right)^2 \\ &= C\phi(r_n). \end{aligned} \tag{34}$$

Now, since $M_1(r, f')$ and $\phi(r)$ are increasing functions of r, using (34), we deduce that

$$M_1(r, f') \le M_1(r_{n+1}, f') \le C\phi(r_n) \le C\phi(r), \quad r_n \le r \le r_{n+1}, \quad n \ge N.$$

Hence

$$M_1(r, f') \le C\phi(r), \quad r_N \le r < 1. \tag{35}$$

Now observe that (15) and (20) imply that the sequence $\{a_n\}$ is uniformly separated (see Chapter 9 of [15]). Hence, there exists $\gamma > 0$ such that

$$(1 - |a_n|^2)|B'(a_n)| = \prod_{\substack{j=N \\ j\ne n}}^{\infty} \left|\frac{a_j - a_n}{1 - a_j a_n}\right| \ge \gamma, \quad n \ge N. \tag{36}$$

Since $B(a_n) = 0$, computing the spherical derivative of f at a_n yields

$$\begin{aligned} (1 - |a_n|^2)\frac{|f'(a_n)|}{1 + |f(a_n)|^2} &= (1 - |a_n|^2)\frac{|F'(a_n)B(a_n) + F(a_n)B'(a_n)|}{1 + |B(a_n)F(a_n)|^2} \\ &= (1 - |a_n|^2)|B'(a_n)||F(a_n)| \end{aligned}$$

which, using (36) and (32), implies

$$(1 - |a_n|^2)\frac{|f'(a_n)|}{1 + |f(a_n)|^2} \to \infty, \quad \text{as } n \to \infty,$$

and, hence, we see that f is not a normal function. Notice that (35) shows that f satisfies (4) and so this finishes the proof.

3. Final remarks and some further results

(i) If f is a function which is analytic in Δ and the non-tangential limit $f(e^{i\theta})$ exists almost everywhere on $\partial\Delta$, then (see [8, p. 72]), for $p > 0$, $\omega_p(f, .)$ denotes the integral modulus of continuity of order p of the boundary function $f(e^{i\theta})$, that is,

$$\omega_p(f, \delta) = \sup_{0<h<\delta} \left(\int_{-\pi}^{\pi} |f(e^{i(t+h)}) - f(e^{it})|^p \, dt\right)^{1/p}, \quad -\pi < \delta < \pi.$$

It is well known that there is a close connection between the behaviour of $\omega_p(f, \delta)$, as $\delta \to 0$, and the growth of integral means of the derivative $M_p(r, f')$ as $r \to 1$ (see Chapter 5 of [8] and [5]). I wish to express my gratitude to Alexei Solianik who, in a private communication, showed to the author that for a certain modulus of continuity $\omega(\delta)$ with $\frac{\omega(\delta)}{\delta} \to \infty$, as $\delta \to 0$, there exists a function $f \in H^1$ with $\omega_1(f,\delta) = O(\omega(\delta))$, as $\delta \to 0$, and $f \notin BMOA$. This result motivated our work and, in fact, using Theorem 2.1 of [5] and Theorem 1, we can state the following improvement of Solianik's result.

Theorem 2. *Let $\rho(t)$ be a positive increasing function in $[0,1)$ satisfying the following two conditions.*

(a) Dini's condition: $\rho(t)/t \in L^1((0,1))$ and there is a constant C such that

$$\int_0^t \frac{\rho(s)}{s}\, ds \le C\rho(t), \quad 0 < t < 1.$$

(b) The condition b_1: There exists a constant C such that

$$\int_t^1 \frac{\rho(s)}{s^2}\, ds \le C\frac{\rho(t)}{t}, \quad 0 < t < 1.$$

If $\frac{\rho(\delta)}{\delta} \to \infty$, as $\delta \to 0$, then there exists a function $f \in H^1$ which is not a normal function and satisfying

$$\omega_1(f,\delta) = O(\rho(\delta)), \quad \text{as } \delta \to 0.$$

(ii) It is well known that if f is a function which is analytic in Δ and has finite Dirichlet integral, that is, if

$$\iint_{|z|<1} |f'(z)|^2\, dx\, dy < \infty,$$

then $f \in \Lambda(2,1/2) \subset BMOA$. On the other hand, Yamashita proved in [17] that given $0 < p < 2$ there exists a function f analytic in Δ with $\iint_\Delta |f'(z)|^p\, dx\, dy < \infty$ but such that f is not a normal function.

These results lead us to consider the question of whether or not some restriction on the growth of

$$\iint_{|z|<r} |f'(z)|^2\, dx\, dy, \quad \text{as } r \to 1,$$

other than its boundedness, is enough to conclude that f is a normal function. Theorem 3 asserts that the answer to this question is negative.

Theorem 3. *Let ϕ be any positive continuous function defined in $[0,1)$ with $\phi(r) \to \infty$, as $r \to 1$. Then, there exists a function f analytic in Δ which is not a normal function and having the property that*

$$\left(\iint_{|z|<r} |f'(z)|^2\, dx\, dy\right)^{1/2} \le \phi(r), \quad \textit{for all } r \textit{ sufficiently close to } 1.$$

Proof of Theorem 3. Just as in the proof of Theorem 1, we may assume without loss of generality that the function ϕ also satisfies (5) and (6), and it suffices to prove that there exists a non-normal analytic function f in Δ satisfying

$$\left(\iint_{|z|<r} |f'(z)|^2\, dx\, dy\right)^{1/2} \le C\phi(r), \quad \text{for all } r \text{ sufficiently close to } 1. \tag{37}$$

Let f be the function defined in the proof of Theorem 1. Since B is a Blaschke product, we easily see that, for $0 < r < 1$,

$$\begin{aligned}(38)\qquad &\left(\iint_{|z|<r} |f'(z)|^2\,dx\,dy\right)^{1/2}\\ &\le \left(\iint_{|z|<r} |F'(z)|^2\,dx\,dy\right)^{1/2} + M_\infty(r,F)\left(\iint_{|z|<r} |B'(z)|^2\,dx\,dy\right)^{1/2}.\end{aligned}$$

Using arguments similar to those used in the proof of Theorem 1, we can prove that

$$(39)\qquad \left(\iint_{|z|<r} |F'(z)|^2\,dx\,dy\right)^{1/2} \le C\sum_{j=N}^{\infty} \frac{\omega(\delta_j)\delta_j}{1-r+\omega(\delta_j)\delta_j}, \quad 0<r<1,$$

and

$$(40)\qquad \left(\iint_{|z|<r} |B'(z)|^2\,dx\,dy\right)^{1/2} \le C\sum_{j=N}^{\infty} \frac{\omega(\delta_j)\delta_j}{1-r+\omega(\delta_j)\delta_j}, \quad 0<r<1.$$

Notice that the right hand side of (39) coincides with the last term of (29) and then, just as in the proof of (31), we obtain

$$(41)\qquad \left(\iint_{|z|<r_{n+1}} |F'(z)|^2\,dx\,dy\right)^{1/2} \le C\frac{\omega(\delta_n)}{\delta_n}, \quad n\ge N.$$

On the other hand, (40) and (22) show that

(42)

$$(1-r_n)\left(\iint_{|z|<r_{n+1}} |B'(z)|^2\,dx\,dy\right)^{1/2} \le C\sum_{j=N}^{\infty} \omega(\delta_j)\left[\frac{\delta_n\delta_j}{\delta_{n+1}+\omega(\delta_j)\delta_j}\right], \quad n\ge N.$$

Notice that the right hand side of (42) and the right hand side of (23) coincide and then, arguing as in the proof of (26), we obtain

$$(43)\qquad \left(\iint_{|z|<r_{n+1}} |B'(z)|^2\,dx\,dy\right)^{1/2} \le C\frac{\omega(\delta_n)}{\delta_n}, \quad n\ge N.$$

Then, using (38), (41), (28) and (43) and having in mind (11), the definitions of δ_n and r_n and (8), we obtain

$$(44)\qquad \left(\iint_{|z|<r_{n+1}} |f'(z)|^2\,dx\,dy\right)^{1/2} \le C\phi(r_n), \quad n\ge N.$$

Finally, since $\left(\iint_{|z|<r} |f'(z)|^2\,dx\,dy\right)^{1/2}$ and $\phi(r)$ are increasing functions of r, arguing as in the proof of (35), we see that (44) implies

$$\left(\iint_{|z|<r} |f'(z)|^2\,dx\,dy\right)^{1/2} \le C\phi(r), \quad r_N\le r<1.$$

This proves (37) and, since we already know that f is not a normal function, finishes the proof. □

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