EVERY NONREFLEXIVE SUBSPACE OF $L_1[0, 1]$ FAILS THE FIXED POINT PROPERTY

P. N. DOWLING AND C. J. LENNARD

(Communicated by Dale Alspach)

Abstract. The main result of this paper is that every nonreflexive subspace $Y$ of $L_1[0, 1]$ fails the fixed point property for closed, bounded, convex subsets $C$ of $Y$ and nonexpansive (or contractive) mappings on $C$. Combined with a theorem of Maurey we get that for subspaces $Y$ of $L_1[0, 1]$, $Y$ is reflexive if and only if $Y$ has the fixed point property. For general Banach spaces the question as to whether reflexivity implies the fixed point property and the converse question are both still open.

Introduction

We introduce the notion of an asymptotically isometric copy of $\ell_1$ and use it to show that every nonreflexive subspace of $L_1[0, 1]$ fails the fixed point property for nonexpansive mappings, proving the converse of a theorem of Maurey [M]. In particular, the Hardy space $H^1$ on the unit circle must fail to have the fixed point property, which contrasts with Maurey’s result in [M] that $H^1$ has the weak (and weak-star) fixed point property.

We only deal with the failure of the fixed point property (FPP) in this paper. The failure of the weak FPP for the Banach space $(L_1[0, 1], \|\cdot\|_1)$ was discovered by Alspach [A]. This is still (apart from its superspaces) the only Banach space known to fail the weak FPP. On the other hand the ultrapower techniques of Maurey [M] have been extended to prove the weak FPP in many spaces. Examples of such spaces are: $(c_0, \|\cdot\|_\infty)$ ([M]), the Tsirelson space of Figiel and Johnson (Elton et al. [ELOS]) and every Banach space with an unconditional basis, constant $< (\sqrt{33} - 3)/2$, ([Lin]).

We thank Brailey Sims, Mark Smith, Barry Turett and Bill Johnson for helpful discussions, and the referee of the original version of this paper for helpful suggestions. The second author acknowledges the support of a University of Pittsburgh F.A.S. Research Grant.
0. Preliminaries

Recall that \( \ell_1 \) is the Banach space of all scalar sequences \( x = (x_n)_{n=1}^{\infty} \) for which \( \|x\|_1 := \sum_{n=1}^{\infty} |x_n| < \infty \). \( L_1[0,1] \) is the usual space of Lebesgue integrable functions (where almost everywhere equal functions are identified), with its usual norm.

Let \((X, \| \cdot \|_X)\) be a Banach space. We say that \((X, \| \cdot \|_X)\) has the fixed point property (FPP) if given any non-empty, closed, bounded and convex subset \( C \) of \( X \), every nonexpansive mapping \( T : C \to C \) has a fixed point. Here \( T \) is nonexpansive if \( \|Tx - Ty\|_X \leq \|x - y\|_X \) for all \( x, y \in C \). Moreover, \( T \) is a contraction if \( \|Tx - Ty\|_X < \|x - y\|_X \) for every \( x, y \in C \) with \( x \neq y \). If \( X \) is a dual space, isometrically isomorphic to \( Y^* \) for some Banach space \( Y \), then \((X, \| \cdot \|_X)\) has the weak-star fixed point property (with respect to \( Y \)) if given a non-empty, weak-star compact, convex set \( C \) in \( X \), every nonexpansive mapping on \( C \) has a fixed point. The weak fixed point property is defined analogously.

1. All Nonreflexive Subspaces of \( L_1[0,1] \) Fail the FPP

1.1 Definition. We say that a Banach space \((X, \| \cdot \|_X)\) is asymptotically isometric to \( \ell_1 \) if it has a normalized Schauder basis \((x_n)_{n=1}^{\infty}\) such that for some sequence \((\lambda_n)_{n=1}^{\infty}\) in \((0, \infty)\) increasing to 1, we have that

\[
(\spadesuit) \quad \sum_{n=1}^{\infty} \lambda_n |t_n| \leq \sum_{n=1}^{\infty} t_n x_n
\]

for all \( x = \sum_{n=1}^{\infty} t_n x_n \in X \).

Note that whenever \((X, \| \cdot \|_X)\) contains a normalized sequence \((x_n)_{n=1}^{\infty}\) satisfying (\(\spadesuit\)), then the closed linear span of \((x_n)_{n=1}^{\infty}\) is an asymptotically isometric copy of \( \ell_1 \).

1.2 Theorem. Let \((Y, \| \cdot \|_Y)\) be a Banach space containing an asymptotically isometric copy of \( \ell_1 \). Then \((Y, \| \cdot \|_Y)\) fails the fixed point property for closed, bounded, convex sets in \( Y \) and nonexpansive (or contractive) maps on them.

Proof. Let \((x_n)_{n=1}^{\infty}\) in \( Y \) and \((\lambda_n)_{n=1}^{\infty}\) satisfy (\(\spadesuit\)) above. Now fix a sequence \((\mu_n)_{n=1}^{\infty}\) satisfying \( \mu_n > \mu_{n+1} \) for all \( n \in \mathbb{N} \), with \( \mu_n \to r > 0 \). Each \( \mu_{n+1}/\mu_n \in (0, 1) \), so that by passing to corresponding subsequences of \((x_n)_{n=1}^{\infty}\) and \((\lambda_n)_{n=1}^{\infty}\) (if necessary), we may ensure that

\[
\lambda_n > \frac{\mu_{n+1}}{\mu_n}, \quad \text{for all } n \in \mathbb{N}.
\]

Now define \( e_n := \mu_n x_n \), for all \( n \in \mathbb{N} \), and let

\[
K := \left\{ \sum_{n \in \mathbb{N}} \alpha_n e_n : \text{each } \alpha_n \geq 0 \text{ and } \sum_{n \in \mathbb{N}} \alpha_n = 1 \right\}.
\]

Clearly, \( K \) is closed and convex in \( Y \). \( K \) is bounded since \( \mu_n \to r \in (0, \infty) \). Define \( T : K \to K \) to be the right shift map; i.e.

\[
T \left( \sum_{n \in \mathbb{N}} \alpha_n e_n \right) := \sum_{n \in \mathbb{N}} \alpha_n e_{n+1}.
\]
Of course, $T$ is fixed point free on $K$. Finally, we show that $T$ is contractive on $K$.

Fix $z := \sum_{n \in \mathbb{N}} \alpha_n e_n$ and $w := \sum_{n \in \mathbb{N}} \beta_n e_n$ in $K$, with $z \neq w$. Then,

$$
\|Tz - Tw\|_Y = \left\| \sum_{n \in \mathbb{N}} (\alpha_n - \beta_n) e_{n+1} \right\|_Y \leq \sum_{n \in \mathbb{N}} |\alpha_n - \beta_n| \|e_{n+1}\|_Y
= \sum_{n \in \mathbb{N}} |\alpha_n - \beta_n| \mu_{n+1} < \sum_{n \in \mathbb{N}} |\alpha_n - \beta_n| \lambda_n \mu_n
\leq \left\| \sum_{n \in \mathbb{N}} (\alpha_n - \beta_n) \mu_n x_n \right\|_Y = \|z - w\|_Y .
$$

\[\square\]

Immediately we have the following corollary.

1.3 Corollary. Let $(X, \| \cdot \|_X)$ be a Banach space and $Y$ be a subspace of $X$ such that there exists a sequence $(v_n)_{n=1}^\infty$ in $Y$, a sequence $(u_n)_{n=1}^\infty$ in $X$ and a null sequence $(\gamma_n)_{n=1}^\infty$ in $(0, \infty)$ with the following properties.

(i) $\left\| \sum_{n=1}^N t_n u_n \right\|_X = \sum_{n=1}^N |t_n|$, for all scalar sequences $t_1, \ldots, t_N$ and $N \in \mathbb{N}$.

(ii) $\|u_n - v_n\|_X < \gamma_n$, for all $n \in \mathbb{N}$.

Then $(Y, \| \cdot \|_X)$ fails the fixed point property for closed, bounded, convex sets in $Y$ and nonexpansive (or contractive) mappings on them.

Proof. Without loss of generality, each $\gamma_n < 1$ and $(v_n)_{n=1}^\infty$ is normalized. Then $(v_n)_{n=1}^\infty$ spans an asymptotically isometric copy of $\ell_1$ in $(Y, \| \cdot \|_X)$, with the $\lambda_n$’s in inequality (♠) above given by $\lambda_n := 1 - \gamma_n$, for all $n \in \mathbb{N}$. \[\square\]

1.4 Theorem. Every nonreflexive subspace $Y$ of $L_1[0, 1]$, with its usual norm, fails the fixed point property for closed, bounded, convex sets in $Y$ and nonexpansive (or contractive) mappings on them. In particular, this is true for $Y := H^1(T)$, the usual Hardy space on the unit circle $T$.

Proof. By the proof of the Kadec-Pelczynski theorem [KP] (or see [D, Chapter VII]), for $X := L_1[0, 1]$ with its usual norm, sequences $(v_n)_{n=1}^\infty$ in $Y$, $(u_n)_{n=1}^\infty$ in $X$ and $(\gamma_n)_{n=1}^{\infty}$ in $(0, \infty)$ exist that satisfy the hypotheses of Corollary 1.3 above. \[\square\]

Combining 1.4 with Maurey’s theorem [M] allows us to state the fact below.

1.5 Theorem. Let $Y$ be a subspace of $L_1[0, 1]$ with its usual norm. Then the following are equivalent.

(i) $Y$ is reflexive.

(ii) $Y$ has the fixed point property.

2. Notes and remarks

The basic problem that is still open is: “If $X$ is a Banach space isomorphic to $\ell_1$, does $X$ fail the FPP?” Our results only provide a partial answer because there do exist Banach spaces $X$ isomorphic to $\ell_1$, that contain no asymptotically isometric copies of $\ell_1$. These are described in the recent paper of Dowling et al. [DJLT]. In contrast, in Dowling et al. [DLT] the authors show that the spaces $\ell_\infty$ and $\ell_1(\Gamma)$,
with $\Gamma$ uncountable, cannot be equivalently renormed to have the FPP. Indeed, all such renormings contain asymptotically isometric copies of $\ell_1$. This leads to the fact that for a broad class of Orlicz spaces with the Orlicz norm, reflexivity is equivalent to the FPP. Moreover, in Dodds et al. [DDDL], it is shown that every nonreflexive subspace of the trace class $C_1$ or the predual $M_*$ of a von Neumann algebra $M$ with a faithful, normal, finite trace $\tau$ contains an asymptotically isometric copy of $\ell_1$. Further, in Carothers et al. [CDL] the analogous result for nonreflexive subspaces of the Lorentz function space $L_{w,1}(0, \infty)$ is established. Indeed, for subspaces of $C_1$ and $L_{w,1}(0, \infty)$ with a strictly decreasing weight function $w$, the analogue of Theorem 1.5 is true (see [DDDL, CDL]). The situation where $\ell_1$ is replaced by $c_0$ is also considered in [DJLT] and [DLT].

The ideas herein were partially inspired by an example of Lim [Lim]. Smyth [S] has extended the approach based on Lim’s example to show that the dual of every space $C(\Omega)$, where $\Omega$ is an infinite compact Hausdorff space, fails the weak-star fixed point property with an affine contraction. In particular, $\ell_1$ fails the weak-star fixed point property with respect to its predual $c$ (the space of all convergent sequences) with a contractive, affine map.

References


[DDDL] P.G. Dodds, T.K. Dodds, P.N. Dowling and C.J. Lennard, Asymptotically isometric copies of $\ell_1$ in nonreflexive subspaces of symmetric operator spaces, in preparation.


Department of Mathematics and Statistics, Miami University, Oxford, Ohio 45056

Department of Mathematics and Statistics, University of Pittsburgh, Pittsburgh, Pennsylvania 15260