CONGRUENCES ON "CHARACTER" VALUES OF PERMUTATION SUMMANDS

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Abstract. A class of congruences on "character" values $\Phi_L$ of a permutation summand $L$ are exhibited, from which follows the connectedness of the prime ideal spectrum of the Grothendieck ring of permutation summands.

Let $G$ be a finite group and $A$ the ring of integers in a number field $K$. An $AG$-lattice is called a permutation lattice if it has an $A$-basis, necessarily finite, which is permuted by the action of $G$. It will be called a permutation summand (for $G$ over $A$), if it is a direct summand, as $AG$-module, of a permutation lattice.

The Grothendieck ring $\Omega_A(G)$ of the category of all permutation summands for $G$ over $A$ has been studied in [3], via a sort of numerical character $\Phi_L$ of a permutation summand $L$. The construction of $\Phi_L$ is reviewed in the first paragraph of the proof below.

In this note we exhibit a class of congruences on the values of $\Phi_L$ which are strong enough to imply the connectedness of the prime ideal spectrum of $\Omega_A(G)$.

The corresponding result for the character ring $R_K(G)$ was established in [2], where Lemma 7 gives analogous congruences on character values. For the Burnside ring $\Omega(G)$ of finite $G$-sets, the connectedness fails [1], because there are too few congruences on the number of fixed points of $G$-sets.

The function $\Phi_L$ takes values in the ring $A'$ of integers of some sufficiently large number field, for instance $K(\zeta|G|)$, and is defined on triples $(H, b, p')$ of $G$ over $A$. Here $p'$ is a non-zero prime ideal of $A'$ so that if $p$ is the unique prime number in $p'$ then $H$ is a $p$-hypoelementary subgroup of $G$ and $b$ is a generator of $H/O_p(H)$ where $O_p(H)$ is the largest normal $p$-subgroup of $H$.

Congruences. For any prime number $q$, we have

$\Phi_L(H, b, p') \equiv \Phi_L(O^q(H), b_q, p') \mod q'$

where $O^q(H)$ is the smallest normal subgroup of $H$ with $H/O^q(H)$ a $q$-group, $b_q$ is the $q'$-part of the element $b$, and $q'$ is any prime ideal above $q$.

Proof. Notations are consistent with those used in [3]. Let $i_{p'}: A' \to A'_p$ be the inclusion of $A'$ in its completion at $p'$, and let $p = p' \cap A$. Denote the $A_pG$-module $A_p \otimes A L$ by $M$ for simplicity. Since $H$ is $p$-hypoelementary, $O_p(H)$ is the normal $p$-Sylow subgroup of $H$. Decompose the restriction $M_H$ of $M$ to $H$ as $M_H \simeq M' \oplus M''$, where the vertices of the indecomposable $A_pH$-summands of $M'$ are $O_p(H)$, and

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the vertices of $M''$ are proper subgroups of $O_p(H)$. By the definition of $\Phi_L$ (cf. [3] (2.1)), we have

$$i_p'\Phi_L(H, b, p') = \text{trace of } b \text{ acting on } M'.$$

If the action of $b$ on $M'$ has eigenvalues $\lambda_1, ..., \lambda_r$ in $A_q'$, then $\Phi_L(H, b, p') = \sum_i \xi_i$, where $\xi_i$ is the preimage of $\lambda_i$ under $i_p'$. We will call this the pretrace of $b$ on $M'$ for convenience.

Denote $O_p(O^q(H)) = O_p(H) \cap O^q(H)$ by $Q$, and further decompose $M''$ as $M'' \cong M''_1 \oplus M''_2$, where the vertices of the indecomposable $A_pH$-summands of $M''_1$ contain $Q$ and those of $M''_2$ do not.

Since every indecomposable $A_pH$-summand of $M' \oplus M''$ has vertex $P$ between $Q$ and $O_p(H)$ from the above decomposition of $M_H$, it is an $A_pH$-summand of $\text{ind}_Q^{O^q(H)}(A_p)$ by [3](1.1). Its restriction to $O^q(H)$ is then an $A_pO^q(H)$-summand of $\text{ind}_Q^{O^q(H)}(A_p)$ by Mackey decomposition, hence has vertex $Q$. Every indecomposable summand of the restriction $(M''_2)_{O^q(H)}$ has vertex properly contained in $Q$ as the vertex can only drop after restriction. Therefore from the above decomposition of $M_H$, the restriction of $M$ to $O^q(H)$ has the decomposition $M_{O^q(H)} \cong (M' \oplus M''_1)_{O^q(H)} \oplus (M''_2)_{O^q(H)}$, where the vertices of the indecomposable $A_pO^q(H)$-summands of $(M' \oplus M''_1)_{O^q(H)}$ are $Q$, and the vertices of $(M''_2)_{O^q(H)}$ are proper subgroups of $Q$. Again by the definition of $\Phi_L$, applied to $(O^q(H), b_{q'}, p')$, we obtain

$$\Phi_L(O^q(H), b_{q'}, p') = \text{pretrace of } b_{q'} \text{ acting on } M' \oplus M''_1.$$

Now the congruence follows from

Claim. i) pretrace of $b$ on $M' \equiv \text{pretrace of } b_{q'} \text{ on } M' \mod q'$;

ii) pretrace of $b_{q'}$ on $M''_1 \equiv 0 \mod q'$.

Proof of Claim. i) If $m$ is a sufficiently large power of $q$, we have $b^m = b_{q'}^m$, and the eigenvalues of $b^m$ on $M'$ have preimages $\xi_1^m, ..., \xi_r^m$ under $i_p'$. Hence

$$(\text{pretrace of } b)^m = (\sum_i \xi_i)^m \equiv \sum_i \xi_i^m = \text{pretrace of } b^m \mod q'$$

and, for the same reason,

$$\text{(pretrace of } b_{q'})^m \equiv \text{pretrace of } b_{q'}^m \mod q'.$$

Combining gives

$$(\text{pretrace of } b)^m \equiv (\text{pretrace of } b_{q'})^m \mod q'$$

from which i) follows.

ii) We may assume $M''_1 \neq 0$. Then $Q \subseteq O_p(H)$, hence $p$ must be equal to $q$, and $H/Q = (O_p(H)/Q) \times (O^q(H)/Q)$ is nilpotent.

Since $Q$ acts trivially on $M''_1$ by [3](1.1), $M''_1$ can be considered as an $A_pH/Q$-module. By [4]§2, this module has the structure

$$M''_1 \cong \sum_j N_j \otimes_{A_p} \text{ind}_{D_j}^{O^q(H)/Q}(A_p)$$

for some $A_pO^q(H)/Q$-lattices $N_j$ and some $p$-subgroups $D_j$ of $O_p(H)/Q$. These $D_j$ are actually the vertices of $M''_1$, hence are properly contained in $O_p(H)/Q$.\[\text{\(\square\)}\]
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If the eigenvalues of \( b_q \) on \( N_j \) have preimages \( \xi_1^{(j)}, \ldots, \xi_{r_j}^{(j)} \) under \( i_p \), then the eigenvalues of \( b_q' \) on \( N_j \otimes_A \mu_1^{O_p(H)/Q} (A_p) \) have preimages \( \xi_1^{(j)}, \ldots, \xi_{r_j}^{(j)} \) each repeated \( |O_p(H)/Q : D_j| \) times. Thus

\[
\text{pretrace of } b_q' \text{ on } M_1'' = \sum_j |O_p(H)/Q : D_j| \sum_i \xi_i^{(j)} \equiv 0 \mod pA',
\]
as required. This completes the proof of the claim, hence of the congruence. \( \square \)

We want to examine the prime ideal spectrum \( \text{Spec}(\Omega_A(G)) \) of the commutative ring \( \Omega_A(G) \). Let \( T_G(A) \) be the set of triples \((H, b, p')\), and \((A')^T_G(A)\) the ring of all maps on triples with values in \( A' \). Since the ring homomorphism \( \Phi : \Omega_A(G) \to \Omega_A(G) \) has a nilpotent kernel \([3]\), and \( \Omega_A(G) \) is a subring of \((A')^T_G(A)\) with finite \( \mathbb{Z}\)-rank, it induces the surjection

\[
\text{Spec}((A')^T_G(A)) \xrightarrow{\text{going-down}} \text{Spec}(\text{im}\Phi) \xrightarrow{\Phi^{-1}} \text{Spec}(\Omega_A(G)).
\]

\textbf{Lemma.} With above notation, then

1. \( P_{0,T} \subset P_{q',T}; \)
2. If \( q' \) is a maximal ideal of \( A' \) above a prime number \( q \) and \( T = (H, b, p') \) is a triple, we denote the triple \((O^q(H), b_{q'}, p')\) by \( T^q \). Then \( P_{q',T} \cap T^q = P_{q',T^q}. \)

\textbf{Proof.} (1) is clear.

(2) By the congruences we have \( \Phi_x(T) \equiv T^q \mod q' \) \( x \in \Omega_A(G) \). Thus \( x \in P_{q',T} \iff \Phi_x(T) \in q' \iff \Phi_x(T^q) \in q' \iff x \in P_{q',T^q}. \) \( \square \)

\textbf{Corollary.} \( \text{Spec}(\Omega_A(G)) \) is connected.

\textbf{Proof.} Let \( C \) be the connected component of the point \( P_{0,(1)} \) in \( \text{Spec}(\Omega_A(G)) \) where \( (1) \) is the cyclic triple (cf. [3] §3) of the trivial subgroup. By (1) of the Lemma, the closure \( \overline{P_{0,T}} \) contains \( P_{q',T} \) for all \( q' \). So it suffices to show that \( C \) contains \( \overline{P_{0,T}} \) for every triple \( T \). We proceed by induction on the order of the subgroup \( H \) appearing in the triple \( T = (H, b, p') \).

If \( H \) is trivial this follows by the definition of \( C \) so we suppose \( H \) is non-trivial. Choose a prime number \( q \) so \( O^q(H) \subseteq H \), and a prime ideal \( q' \) of \( A' \) containing \( q \). By (2) and (1) of the Lemma we have \( P_{q',T} = P_{q',T^q} \), hence \( \overline{P_{0,T}} \cap \overline{P_{0,T^q}} \) is not empty, and \( \overline{P_{0,T}} \cup \overline{P_{0,T^q}} \) is connected. But \( \overline{P_{0,T^q}} \subseteq C \) by the induction hypothesis, and therefore \( \overline{P_{0,T}} \subseteq C \). \( \square \)

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