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THE INDEX NUMBER OF AN R-SPACE: AN EXTENSION OF A RESULT OF M. TAKEUCHI'S

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Abstract. M. Takeuchi proved the following nice result: The "two-number" of a symmetric R-space is equal to the sum of the Betti numbers of the space with coefficients in Z_2. In the present paper an extension of this result is given for general R-spaces.

1. Introduction

In [Ch-N] a numerical invariant associated to compact Riemannian manifolds is introduced. This invariant, called the "2-number", seems very difficult to compute except for compact symmetric spaces. Chen and Nagano obtained the 2-number for a great deal of them and at the same time showed the existence of many interesting connections between this number and topological invariants of the manifold. For a compact connected symmetric space M this number, denoted $\#_2(M)$, is defined as the maximal possible cardinality of the subsets $A \subset M$ with the property that for each $x \in A$ the symmetry s_x of M at x fixes every point of A. It is not hard to see that $\#_2(M)$ is finite.

In [Ch-N, p. 274] the authors mention a beautiful result obtained by M. Takeuchi which appeared later in [T] and which is the main motivation of the present article. The result, where $\beta_i(M, Z_2)$ denotes the ith Betti number of M with coefficients in Z_2, is the following.

Theorem 1 (Takeuchi)**.** *Let M be a symmetric R-space. Then*

$$\#_2(M) = \sum_{i \geq 0} \beta_i(M, Z_2). \quad \square$$

The difficulties of computing $\#_2(M)$ for anything different from a symmetric space motivated the paper [S], where an extension of the invariant of Chen-Nagano was introduced for flag manifolds. As is well known, they have always a k-symmetric structure (see [K] and references therein) for some $k \geq 2$ and one may define the "k-number" $\#_k(M)$ of a complex flag manifold, [S, p. 1238], as the maximal possible cardinality of the subsets $A_k \subset M$ with the property that for each $x \in A_k$ the symmetry θ_x fixes every point of A_k. If $k = 2$, i.e., M is a Hermitian symmetric space, then clearly $\#_k(M)$ is just the 2-number of M. Now corollary (3.6) of [S] can be stated as

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Theorem 2. *If M is a complex flag manifold, then for any of its k-symmetric structures*

$$\#_k(M) = \sum_{i\geq 0} \beta_i(M, Z_2). \quad \square$$

Since complex flag manifolds are also R-spaces (see the next section), we see that there are two families of R-spaces, namely symmetric R-spaces and complex flag manifolds, where essentially the same result is true. Then one cannot help wondering whether there is a general theorem for R-spaces having Theorems 1 and 2 as particular cases. Naturally the main difficulty for this is that, in general, R-spaces have no "symmetries".

The objective of this paper is to present a version of this general theorem (see Theorem 4 below). The paper is organized as follows. In section 2 we recall the definition of R-space (sometimes called orbit of an s-representation) and prove a fact which in some sense is implicit in certain definitions of R-space. Namely every R-space can be isometrically imbedded in a complex flag manifold (see Proposition 3). We have not seen this fact explicitly mentioned in the literature. This leads us to the natural definition of the *index of an R-space* and that of its *index number*. The main theorem is stated at the end of section 2 and proved in section 3. Section 4 contains an appendix.

2. Basic facts and main result

Let us recall the definition of an R-space. Let $\mathcal{G}$ be a real semisimple Lie algebra without compact factors, $\mathcal{K}$ a maximal compactly imbedded subalgebra of $\mathcal{G}$ and $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$ the Cartan decomposition of $\mathcal{G}$ relative to $\mathcal{K}$. If we denote by B the Killing form of $\mathcal{G}$, then $\mathcal{P}$ can be considered a Euclidean space with the inner product induced by the restriction of B to $\mathcal{P}$. Let $G = \mathrm{Int}(\mathcal{G})$ be the group of inner automorphisms of $\mathcal{G}$. The Lie algebra of G may be identified with $\mathcal{G}$. Let K be the analytic subgroup of G corresponding to $\mathcal{K}$; then K is compact and acts on $\mathcal{P}$ as an isometry group. By definition, an R-space M is the orbit of a non-zero vector $E \in \mathcal{P}$ by K, i.e., $M = \mathrm{Ad}(K)E$. If E satisfies the relation $(\mathrm{ad}(E))^3 = \mathrm{ad}(E)$ in $\mathcal{G}$, then the R-space M is symmetric and it is called a *symmetric R-space* (see for instance [F]). We consider in M the Riemannian metric induced from the inner product on $\mathcal{P}$.

R-spaces may also be defined as homogeneous spaces of the form G/P where G is a real semisimple Lie group without compact factors and P is a parabolic subgroup. Here $G/P \equiv K/K \cap P$ which is a K-orbit on $\mathcal{P}$. This seems to be a more standard definition; many authors call them *real flag manifolds*.

Every complex flag manifold may be considered an R-space. In fact let U be a compact connected semisimple centerless Lie group and $\mathcal{U}$ its Lie algebra. The complex flag manifolds of U are the orbits of the adjoint action of U on $\mathcal{U}$. Take $M = \mathrm{Ad}(U)Y, Y \neq 0$ in $\mathcal{U}$. Let $\mathcal{G} = \mathcal{U}_c = \mathcal{U} \oplus i\mathcal{U}$ then this is a Cartan decomposition of the realification $\mathcal{G}_R$ of $\mathcal{G}$ [H] and we may consider M as the orbit of iY in $i\mathcal{U}$ by the adjoint action of U.

It is well known (see for instance [B], [W], [S1]) that the integral homology of complex flag manifolds is nonzero only in even dimensions and torsion free. For this reason their Betti numbers are the same for any choice of coefficient field.

Proposition 3. *Let M be an arbitrary R-space. Then there exists a complex flag manifold M_c such that M is isometrically imbedded in M_c. If M happens to be a symmetric R-space, then M_c is a Hermitian symmetric space and the isometric imbedding is totally geodesic. If M is already a complex flag manifold, then $M_c = M$.*

Proof. Associated to M we have the data indicated above, namely $\mathcal{G}$, semisimple without compact factors, $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$ a Cartan decomposition, $K \subset G = \mathrm{Int}(\mathcal{G})$, B the Killing form and $E \in \mathcal{P}$ such that $M = \mathrm{Ad}_G(K)E$. Let $a \subset \mathcal{P}$ be a maximal abelian subspace with $E \in a$. We may extend a to a Cartan subalgebra $\mathcal{H} = t \oplus a$ of $\mathcal{G}$ by taking as t a Cartan subalgebra of the centralizer of a in $\mathcal{K}$. Let $\mathcal{G}_c$ be the complexification of $\mathcal{G}$ and σ the conjugation of $\mathcal{G}_c$ with respect to $\mathcal{G}$. Since $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$ is a Cartan decomposition, there exists a compact real form $\mathcal{G}_u$ of $\mathcal{G}_c$ such that

$$\begin{gathered}\sigma\mathcal{G}_u \subset \mathcal{G}_u,\\ \mathcal{K} = \mathcal{G} \cap \mathcal{G}_u,\\ \mathcal{P} = \mathcal{G} \cap i\mathcal{G}_u,\\ \mathcal{G}_u = \mathcal{K} \oplus i\mathcal{P}.\end{gathered}$$

Let G_c be the simply connected semisimple Lie group associated to $\mathcal{G}_c$. Let G_1 and G_u be the analytic subgroups of G_c corresponding to the subalgebras $\mathcal{G}$ and $\mathcal{G}_u$ respectively. By [H, p. 152, 4(ii)] they are both closed in G_c and by [H, p. 257] G_u is also simply connected. Let K_1 be the analytic subgroup of G_c corresponding to $\mathcal{K}$. The group K_1 is compact and clearly $K_1 \subset G_1 \cap G_u$. Now $\mathrm{Ad}_{G_1} : G_1 \to G$ is a surjective analytic homomorphism and $K = \mathrm{Ad}_{G_1}(K_1)$. The manifold $\mathrm{Ad}_{G_1}(K_1)E$ is clearly M.

On the other hand, the orbit $\mathrm{Ad}_{G_1}(K_1)iE \subset i\mathcal{P}$ is again our space M since the representations of K_1 on $\mathcal{P}$ and $i\mathcal{P}$ are equivalent. By [H, p. 180] if $X, Y \in \mathcal{P}$

$$-B_u(iX, iY) = -B_c(iX, iY) = B_c(X, Y) = B(X, Y),$$

and so if we put on $i\mathcal{P}$ the Euclidean metric induced by $-B_u$, then M and $\mathrm{Ad}_{G_1}(K_1)iE$ are isometric.

Let M_c be the orbit of iE by $\mathrm{Ad}_{G_u}(G_u)$. It is obviously a complex flag manifold. Now consider on M_c the Riemannian metric induced by the inner product on $\mathcal{G}_u$ defined by $(-B_u)$. It is clear that M is isometrically imbedded in M_c.

If M is a symmetric R-space, then, by definition, our original vector $E \in \mathcal{P}$ satisfies $\mathrm{ad}(E)^3 = \mathrm{ad}(E)$ in $\mathcal{G}$. Then in $\mathcal{G}_u$ the vector iE satisfies $\mathrm{ad}(iE)^3 = -\mathrm{ad}(iE)$ and it is not hard to see that under this condition, the orbit $\mathrm{Ad}_{G_u}(G_u)(iE)$ is a Hermitian symmetric space and $T_{(iE)}(M) \subset T_{(iE)}(M_c)$ is a Lie triple system.

If M happens to be a complex flag manifold, then we may write it as $M = U/L$ where U is a compact semisimple simply connected Lie group and L the centralizer of a torus in U. If $\mathcal{U}$ denotes the Lie algebra of U we define $\mathcal{G}_c = \mathcal{U} \oplus i\mathcal{U} = \mathcal{U}_c$. If we consider the realification $(\mathcal{G}_c)^R$ of $\mathcal{G}_c$ we have that $(\mathcal{G}_c)^R = \mathcal{U} \oplus i\mathcal{U} = \mathcal{K} \oplus \mathcal{P}$ is a Cartan decomposition [H, p. 185]. To construct the imbedding of M we take some element $E \in z(\mathcal{L}) \subset \mathcal{L} = \mathrm{Lie}(L)$, such that $\{\exp tE : t \in R\}^- = Z(L)$ (the center of L) and consider again the orbit of iE in $i\mathcal{U}$. Now by repeating the above construction in this case, we get the complexification of $(\mathcal{G}_c)^R$, namely $[(\mathcal{G}_c)^R]_c = \mathcal{G}_c \oplus \mathcal{G}_c$ and its compact real form is just $[(\mathcal{G}_c)^R]_u = \mathcal{U} \oplus \mathcal{U}$. Now by keeping track of our original $E \in z(\mathcal{L})$, $iE \in i\mathcal{U}$ and $i(iE) = -E \in \mathcal{U}$, we see that

the complex flag manifold M_c is the adjoint orbit of $-E = (0, -E) \in \mathcal{U} \oplus \mathcal{U}$ by $U \times U$, i.e., $M_c = \frac{U \times U}{U \times L} = U/L = M$ and therefore nothing new is obtained in this case. This completes the proof of Proposition 3. $\square$

Let us consider now the flag manifold M_c. It is well known (see for instance [J, p. 135, A]) that there exists a natural number $n_o = n_o(M_c) \geq 2$ such that for each $n \geq n_o$, M_c is an *n-symmetric manifold*; see for instance [K]. This takes us naturally to the definition of the *index of the R-space* M. We define Index(M) *as the first prime number p greater than or equal to $n_o(M_c)$.*

Let $\{\theta_x : x \in M_c\}$ be the p-symmetric structure on M_c. We are ready to define the *index number* of the R-space M, denoted by $\#_I(M)$, as follows. $\#_I(M)$ *is the maximal possible cardinality of the subsets $A_p \cap M$ where $A_p \subset M_c$ has the property that for each $x \in A_p$ the symmetry θ_x* (of order p of M_c) *fixes every point of A_p.*

We must point out that if M is a symmetric R-space, then by Proposition 3 Index$(M) = 2$ while if M is a complex flag manifold we have that its *Index* is the first prime number greater than or equal to $n_o(M)$. Obviously in the latter case, $\#_I(M) = \#_p(M)$ as defined in [S, p. 1238]. In the former case, it is not obvious that $\#_I(M) = \#_2(M)$, but we shall see however that this is indeed true.

Now we may write down our mentioned extension of Theorem 1.

Theorem 4. *Let M be an R-space. Then*

$$\#_I(M) = \sum_{i \geq 0} \beta_i(M, Z_2). \quad \square$$

Notice that this shows in particular that for symmetric R-spaces we have $\#_I(M) = \#_2(M)$. The proof of the theorem is contained in the next section.

3. Proof of the main result

Let M be our R-space, M_c its complex flag manifold and $p = \text{Index}(M)$. As was indicated in [S, p. 1238], it is easy to see that $\#_p(M_c)$ is finite. Let $A \subset M_c$ be a subset which satisfies the following property:

(P) *For each $x \in A$ the symmetry θ_x fixes every point of A.*

If the cardinality of A is precisely $\#_p(M_c)$ we shall say that this subset is a $\#_p$-set of M_c.

Lemma 5. *Let $A \subset M_c$ be a $\#_p$-set. Then there exists a maximal torus $T_o \subset G_u$ such that A is the fixed point set $F(T_o, M_c)$ of the torus T_o in M_c. Furthermore, if $A_1 \subset M_c$ is a subset which satisfies the property (P), then there exists a $\#_p$-set A such that $A_1 \subset A$.*

Proof. Consider, for each $a \in A$, the isotropy subgroup $(G_u)_a$ of G_u at a and denote $Q = \bigcap_{a \in A} (G_u)_a$. The group Q is *connected* and of *maximal rank* in G_u (see the Appendix). For any Lie group H let $Z(H)$ denote its center. $Z(Q)$ is contained in all the maximal tori of Q. Let T_o be a maximal torus in Q; it is then a maximal torus in G_u. Let $A_o = F(T_o, M_c)$. It is well known that A_o is finite.

Notice now that $A \subset A_o$. In fact, for each $a \in A$, we know $\theta_a \in Z[(G_u)_a] \subset T_o$ and therefore $g\theta_a = \theta_a g$, $\forall g \in T_o$, and since θ_a is a symmetry and T_o is connected we must have $g(a) = a$, $\forall g \in T_o$.

On the other hand if $x \in A_o$, then θ_x fixes every point of A_o because $T_o \subset (G_u)_x$ and $\theta_x \in Z[(G_u)_x] \subset T_o$. So if A_o contains A properly, then A cannot be a $\#_p$-set. Then $A = A_o$.

Let A_1 be a subset of M_c which satisfies (P) and consider again the subgroup $Q_1 = \bigcap_{a \in A_1} (G_u)_a$, *connected* and of *maximal rank* in G_u. If T_o is a maximal torus in Q_1, then $A_1 \subset A_o = F(T_o, M_c)$. This concludes the proof of the lemma. □

Corollary 6. *Let $A \subset M_c$ be a $\#_p$-set. Then there exists a Cartan subalgebra $\mathcal{H}_{uo} \subset \mathcal{G}_u$ such that $A = \mathcal{H}_{uo} \cap M_c$.* □

The converse of this corollary is clear: given any Cartan subalgebra $\mathcal{H}_u$ in $\mathcal{G}_u$, $\mathcal{H}_u \cap M_c$ is a $\#_p$-set.

Let $ia \subset i\mathcal{P}$ be a maximal abelian subspace. We may complete ia to a Cartan subalgebra $\mathcal{H}_u = t \oplus ia \subset \mathcal{G}_u$ by taking as t, in the usual fashion, a Cartan subalgebra of the centralizer of ia in $\mathcal{K}$ [M].

By extension of the name usually given in the non-compact case, we call this Cartan subalgebra of $\mathcal{G}_u$ *minimally compact*.

Since the cardinality of the subsets $A \subset M_c$ with the property (P) cannot be greater than $\#_p(M_c)$, we see that $\#(A \cap M) \leq \#_p(M_c)$ for each set A with this property, where $\#(X)$ indicates the cardinality of X. If A is a subset of M_c with the property (P) and such that $\#(A \cap M)$ is maximal, then there is a $\#_p$-set A' in M_c such that

$$\#(A \cap M) = \#(A' \cap M),$$

then if $\#(A \cap M)$ is maximal we may think that A is a $\#_p$-set in M_c.

Lemma 7. *Let $A \subset M_c$ be a $\#_p$-set. Then there exists a minimally compact Cartan subalgebra $\mathcal{H}_u \subset \mathcal{G}_u$ such that $A' = \mathcal{H}_u \cap M_c$ satisfies $A \cap M \subseteq A' \cap M$.*

Proof. Let $A \subset M_c$ be a $\#_p$-set. By Lemma 5 there exists a Cartan subalgebra $\mathcal{H}_{uo} \subset \mathcal{G}_u$ such that $A = \mathcal{H}_{uo} \cap M_c$. Now

$$M \cap A = M \cap (M_c \cap \mathcal{H}_{uo}) = M \cap \mathcal{H}_{uo} = M \cap (\mathcal{H}_{uo} \cap i\mathcal{P}).$$

Let $ib = (\mathcal{H}_{uo} \cap i\mathcal{P})$ (this may be $\{0\}$). We may extend ib to a maximal abelian subspace $ia \supseteq ib$ in $i\mathcal{P}$ and construct with ia a minimally compact Cartan subalgebra $\mathcal{H}_u$. By construction $M \cap A \subset M \cap \mathcal{H}_u$. Then $\mathcal{H}_u$ has the required property. □

Lemma 8. *If $\mathcal{H}_u$ is minimally compact and $A = \mathcal{H}_u \cap M_c$, then $\#(A \cap M)$ is maximal.*

Proof. Let us assume that there exists a minimally compact Cartan subalgebra $\mathcal{H}_{uo}$ such that $A_o = \mathcal{H}_{uo} \cap M_c$ and $\#(A_o \cap M)$ is not maximal, i.e., there exists a subset $A_1 \subset M_c$ which satisfies (P) and

$$\#(A_o \cap M) < \#(A_1 \cap M). \tag{1}$$

By Lemma 5 given A_1 there exists a $\#_p$-set A_2 in M_c such that $A_1 \subset A_2$ in M_c. Then for A_2 we have

$$\#(A_1 \cap M) < \#(A_2 \cap M).$$

By Lemma 7 there is a minimally compact Cartan subalgebra $\mathcal{H}_{u3}$ such that if we define $A_3 = \mathcal{H}_{u3} \cap M_c$, then $(A_2 \cap M) \subset (A_3 \cap M)$, and then

$$\#(A_2 \cap M) \leq \#(A_3 \cap M).$$

Now write $\mathcal{H}_{uj} = t_j \oplus ia_j$ for $j = 0, 3$ and consider

$$M \cap A_o = M \cap (\mathcal{H}_{uo} \cap I\mathcal{P}) = M \cap ia_o = B_o,$$
$$M \cap A_3 = M \cap (\mathcal{H}_{u3} \cap i\mathcal{P}) = M \cap ia_3 = B_3.$$

Since ia_o and ia_3 are conjugate by an element $q \in K$ by [H, Ch. V: 6, 6.3], $\#(B_o) = \#(B_3)$. This contradicts inequality (1) and the lemma follows. □

Now let $\mathcal{H}_u$ be a fixed minimally compact Cartan subalgebra of $\mathcal{G}_u$ containing iE and let $N_{G_u}(\mathcal{H}_u)$ denote the normalizer of $\mathcal{H}_u$ in the compact simply connected Lie group G_u. As in Proposition 3 denote by σ the conjugation of $\mathcal{G}_c$ with respect to $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$ and by τ the conjugation with respect to $\mathcal{G}_u = \mathcal{K} \oplus i\mathcal{P}$. Notice that $\mathcal{H}_u$ is σ-invariant. Let us denote by $[N_{G_u}(\mathcal{H}_u)]_\sigma$ the subgroup of those elements which commute with σ on $\mathcal{H}_u$. This is the same as the subgroup of those elements commuting with $\theta = \sigma\tau$ since τ is the identity on $\mathcal{G}_u$. Clearly the orbit $N_{G_u}(\mathcal{H}_u)(iE)$ is $M_c \cap \mathcal{H}_u$ and hence a $\#_p$-set in M_c. Let $N_{K_1}(ia)$ denote the normalizer of ia in the group K_1.

Lemma 9. $N_{G_u}(\mathcal{H}_u)(iE) \cap M = [N_{G_u}(\mathcal{H}_u)]_\sigma(iE) = N_{K_1}(ia)(iE)$.

Proof. First we show that $N_{G_u}(\mathcal{H}_u)(iE) \cap M \subset [N_{G_u}(\mathcal{H}_u)]_\sigma(iE)$. Take $g \in N_{G_u}(\mathcal{H}_u)$ such that $\mathrm{Ad}(g)(iE) \in M$. Then there exists $h \in K_1$ such that $\mathrm{Ad}(h)(iE) = \mathrm{Ad}(g)(iE)$. Since $\mathrm{Ad}(g)(iE) \in \mathcal{H}_u$ and $M \subset i\mathcal{P}$, we have $\mathrm{Ad}(g)(iE) \in \mathcal{H}_u \cap i\mathcal{P} = ia$ and hence $\mathrm{Ad}(h)(iE) \in ia$. By [H, p. 285, (2.2)] there exists $s \in W(G_u, K_1)$ (notation [H, p. 284]) such that $s.(iE) = \mathrm{Ad}(h)(iE)$. Now there exists, by [H, p. 325, (8.10)], $g_1 \in [N_{G_u}(\mathcal{H}_u)]_\sigma$ such that $\mathrm{Ad}(g_1)(iE) = s.(iE)$ and this proves the indicated inclusion. To prove the other one we notice that it is enough to show that $[N_{G_u}(\mathcal{H}_u)]_\sigma(iE) \subset M$ and this is clearly a by-product of the second equality. Then to finish our proof we just have to show $[N_{G_u}(\mathcal{H}_u)]_\sigma(iE) = N_{K_1}(ia)(iE)$.

Clearly $[N_{G_u}(\mathcal{H}_u)]_\sigma(iE) \subset N_{G_u}(ia)(iE)$ (the normalizer of ia in G_u). Now take $g \in N_{G_u}(ia)$, by [H, p. 324, (8.8)] we have $g = uv$ where $u \in N_{K_1}(ia)$ and $v \in A_*$ (notation in [H, p. 324]).

Remark. Notice that G_u is simply connected and therefore [H, p. 324, (8.8)] can be applied.

Then $\mathrm{Ad}(g)(iE) = \mathrm{Ad}(u)(iE)$ and this proves

$$[N_{G_u}(\mathcal{H}_u)]_\sigma(iE) \subset N_{G_u}(ia)(iE) \subset N_{K_1}(ia)(iE).$$

Now to prove the other inclusion take $h \in N_{K_1}(ia)$. We have $\mathrm{Ad}(h)t = t'$ and in general $t \neq t'$. But t and t' are both Cartan subalgebras of $M = \mathrm{Lie}(N_{K_1}(ia)) = \mathrm{Lie}(Z_{K_1}(ia))$ (these are M and M' in the notation of [H, p. 284]) and therefore by the Theorem of conjugation [H, p. 248, (6.4) (ii)] there exists $w \in [Z_{K_1}(ia)]_e$, the connected component of e in $[Z_{K_1}(ia)]$, such that $\mathrm{Ad}(w)t' = t$. Let us now define $g = wh$; it is clear that g normalizes both t and ia. Hence g normalizes $\mathcal{H}_u$ and commutes with σ there and therefore $g \in [N_{G_u}(\mathcal{H}_u)]_\sigma$.

Since $\mathrm{Ad}(g)(iE) = \mathrm{Ad}(h)(iE)$, we have $N_{K_1}(ia)(iE) \subset [N_{G_u}(\mathcal{H}_u)]_\sigma(iE)$ and this completes the proof of the lemma. □

It is clear now that Lemma 9 together with [T-K, p. 205, (2.1)] yield the identity

$$\#((N_{G_u}(\mathcal{H}_u)(iE)) \cap M) = \sum_{i \geq 0} \beta_i(M, Z_2).$$

Theorem 4 now follows from this equality and Lemma 8. □

4. Appendix

Here we give a proof of the following fact used in Lemma 5.

Lemma 10. *The group Q (defined in the proof of Lemma 5) is connected and has maximal rank in G_u.*

Proof. Let $A = \{a_1, \dots, a_r\}$, $r \geq 1$, be the $\#_p$-set in M_c considered in Lemma 5. To simplify notation we write $K_j = (G_u)_{a_j}$ and let $\mathcal{K}_j$ denote their respective Lie subalgebras in $\mathcal{G}_u$. The subgroups K_j (which are conjugate to each other) are centralizers of tori in G_u.

For each $j = 1, \dots, r$, let us decompose $\mathcal{G}_u = \mathcal{K}_j \oplus \mathcal{M}_j$ orthogonally with respect to $(-B_u)$ and consider the following observation.

$$(*) \qquad \textit{For each } j = 1, \dots, r, A \subset \mathcal{K}_j.$$

In fact this is obvious for a_j but it is also true for a_t for $t \neq j$ because we may write $a_t = v_t + h_t$ with $v_t \in \mathcal{K}_j$ and $h_t \in \mathcal{M}_j$. Since $h_t \in \mathcal{M}_j = T_{a_j}(M_c)$ and $\theta_{a_j}(a_t) = a_t$ (A satisfies property (P)) we must have $h_t = 0$ because θ_{a_j} is a symmetry at a_j. Then $a_t \in \mathcal{K}_j$.

Let us now take $a_2 \in \mathcal{K}_1$. Its orbit by the adjoint action of K_1 on its Lie algebra $\mathcal{K}_1$ is $K_1(a_2) = K_1/(K_1 \cap K_2)$ and since this is a complex flag manifold in $\mathcal{K}_1$ we have that $(K_1 \cap K_2)$ is connected and has maximal rank in K_1 (hence in G_u).

To proceed by induction let us assume that we have shown that the subgroup $K_1 \cap K_2 \cap \cdots \cap K_j$ is connected and of maximal rank in G_u and consider a_{j+1} in A. By $(*)$ we have $a_{j+1} \in (\mathcal{K}_1 \cap \mathcal{K}_2 \cap \cdots \cap \mathcal{K}_j)$ and hence its orbit by the adjoint action of the group $(K_1 \cap K_2 \cap \cdots \cap K_j)$ in the Lie algebra $(\mathcal{K}_1 \cap \mathcal{K}_2 \cap \cdots \cap \mathcal{K}_j)$ is a complex flag manifold. Namely

$$(K_1 \cap K_2 \cap \cdots \cap K_j)(a_{j+1}) = \frac{(K_1 \cap K_2 \cap \cdots \cap K_j)}{(K_1 \cap K_2 \cap \cdots \cap K_j \cap K_{j+1})},$$

and hence $K_1 \cap K_2 \cap \cdots \cap K_{j+1}$ is connected and of maximal rank in $K_1 \cap K_2 \cap \cdots \cap K_j$ and therefore in G_u. Then Q is connected and of maximal rank in G_u. $\square$

In the proof of Lemma 10 we did not use the fact that A is a $\#_p$-set, only that it satisfies property (P).

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