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PEŁCZYŃSKI'S PROPERTY (V*) FOR SYMMETRIC OPERATOR SPACES

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Abstract. We show that if a rearrangement invariant Banach function space E on the positive semi-axis contains no subspace isomorphic to c_0 then the corresponding space $E(\mathcal{M})$ of τ-measurable operators, affiliated with an arbitrary semifinite von-Neumann algebra $\mathcal{M}$ equipped with a distinguished faithful, normal and semifinite trace τ, has Pełczyński's property (V*).

1. Introduction

Let X be a Banach space. We say that X has Pełczyński's property (V*) if any subset $K \subset X$ such that $\lim_{n\to\infty} \sup_{x\in K} x_n^*(x) = 0$ for every weakly unconditionally Cauchy series (w.u.c.) $\sum_{n=1}^{\infty} x_n^*$ in X^*, is relatively weakly compact. This property was introduced by Pełczyński in [9]. The most notable examples of Banach spaces that have property (V*) are L^1-spaces and it was generalized by Saab and Saab in [13] and independently by Emmanuele in [4] that Banach lattices that do not contain any copy of c_0 have property (V*). On the other hand, one can deduce from a result of Pfitzner in [10] (see also [11]) that non-commutative L^1-spaces have property (V*).

It is the intention of the present note to give a common generalization of these results in the setting of symmetric spaces of measurable operators (see definition below). Let $\mathcal{M}$ be a semifinite von-Neumann algebra equipped with a distinguished faithful normal semifinite trace τ. Our principal result shows that if E is a rearrangement invariant function space on $(0,\infty)$ that does not contain any copy of c_0 then the corresponding symmetric space of τ-measurable operators $E(\mathcal{M})$ has property (V*). This was motivated by [3] where it was shown that such a space is weakly sequentially complete (a property weaker than property (V*)). Non-commutative symmetric spaces associated with semifinite von-Neumann algebras have been considered by several authors. Some Banach space properties of these spaces can be found in [1], [3], [16].

We refer to [15] and [8] for general information concerning von-Neumann algebras as well as basic notions of non-commutative integration, to [2] and [7] for Banach space theory.

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2. Preliminaries

Let $\mathcal{M}$ be a semifinite von-Neumann algebra acting on a Hilbert space H. Let τ be a faithful semifinite normal trace on $\mathcal{M}$. A densely-defined closed operator x, affiliated with $\mathcal{M}$, is called τ-measurable (in the sense of [8]) if for each $\varepsilon > 0$, there exists a projection $e \in \mathcal{M}$ such that $xe \in \mathcal{M}$ and $\tau(1-e) < \varepsilon$. The set $\widetilde{\mathcal{M}}$ of all τ-measurable operators is a $*$-algebra with respect to the strong sum, the strong product, and the adjoint operation [8]. The rearrangement of an operator $x \in \widetilde{\mathcal{M}}$ is the function $\mu(x)$ defined by

$$\mu_t(x) = \inf\{\|xe\| : e \text{ projection } \in \mathcal{M}\ ,\ \tau(1-e) \leq t\}\ , \qquad t \geq 0.$$

The function $t \to \mu_t(x)$ from $(0,\infty)$ to $[0,\infty)$ is non-increasing, continuous from the right and $\lim_{t\to 0^+} \mu_t(x) = \|x\|\ \forall\ x \in \mathcal{M}$. For complete detailed study of $\mu_{(\cdot)}$, we refer to [5].

Unless stated otherwise, E will always denote an order continuous rearrangement invariant Banach function space on $(0,\infty)$ (in the sense of [7]). We define the symmetric operator space $E(\mathcal{M})$ of measurable operators by setting

$$E(\mathcal{M}) = \{x \in \widetilde{\mathcal{M}} : \mu(x) \in E\}$$

and

$$\|x\|_{E(\mathcal{M})} = \|\mu(x)\|_E\ , \qquad x \in E(\mathcal{M})\ .$$

It is well known that $E(\mathcal{M})$ is a Banach space and if $E = L^p(0,\infty)$ $(1 \leq p < \infty)$, then $E(\mathcal{M})$ coincides with the usual non-commutative L^p-space associated with the semifinite von Neumann algebra $\mathcal{M}$. The space $E(\mathcal{M})$ is often referred to as the non-commutative version of the function space E.

We will now recall some facts about property (V*).

Definition 1. Let X be a Banach space . A series $\sum_{n=1}^\infty x_n$ is said to be weakly unconditionally Cauchy (w.u.c.) if for every $x^* \in X^*$, the series $\sum_{n=1}^\infty |x^*(x_n)|$ is convergent.

There are many criteria for a series to be a w.u.c. series (see [2]).

The following property was introduced by Pełczyński in [9]:

Definition 2. A Banach space X is said to have property (V*) if a bounded subset K of X is relatively weakly compact whenever $\lim\limits_{n\to\infty} \sup\limits_{x\in K} |{x_n}^*(x)| = 0$ for every w.u.c. series $\sum_{n=1}^\infty {x_n}^*$ in X^*.

Perhaps the main importance of property (V*) is that it provides good information about existence of a complemented copy of ℓ^1. In fact we have the following characterization: A Banach space X has property (V*) if and only if it is weakly sequentially complete and every sequence that is equivalent to ℓ^1 has a subsequence that generates a complemented copy of ℓ^1. For more information and examples of spaces with property (V*), we refer to [4], [6], [12] and [14].

3. Main Theorem

Theorem 1. *Let E be a symmetric function space on $(0,\infty)$ and $(\mathcal{M},\tau)$ be a semifinite von-Neumann algebra. Then $E(\mathcal{M})$ has property* (V*) *if and only if E does not contain any copy of c_0.*

Proof. We shall assume without loss of generality that E is a separable function space. In this case, $E(\mathcal{M})^*$ is completely described in [3] as follows:

$$E(\mathcal{M})^* = \left\{x \in \widetilde{\mathcal{M}} : \|x\|_{E(\mathcal{M})^*} = \sup\{\tau(|xy|);\ y \in E(\mathcal{M}),\ \|y\|_{E(\mathcal{M})} \le 1\} < \infty\right\}\ ,$$

and the action of $x \in E(\mathcal{M})^*$ on $y \in E(\mathcal{M})$ is $\tau(xy*)$.

Recall also that $L^1(\mathcal{M},\tau) \cap \mathcal{M} \subset E(\mathcal{M}) \subset L^1(\mathcal{M},\tau) + \mathcal{M}$.

Assume first that τ is a finite trace. For simplicity, we assume without loss of generality that $\tau(1) = 1$. Let $K \subset E(\mathcal{M})$ be a (V*)-set, i.e., $\lim\limits_{n\to\infty} \sup_{a\in K} x_n^*(a) = 0$ for every w.u.c. series $\sum_{n=1}^{\infty} x_n^*$ in $E(\mathcal{M})^*$. Our goal is to show that K is relatively weakly compact in $E(\mathcal{M})$.

Since $\tau(1) = 1$, we have $\mathcal{M} \subset L^1(\mathcal{M},\tau)$ and therefore $\mathcal{M} \subset E(\mathcal{M}) \subseteq L^1(\mathcal{M},\tau)$ with the inclusions being continuous.

The fact that $L^1(\mathcal{M},\tau)$ has property (V*) (see [10]) implies that K is relatively weakly compact in $L^1(\mathcal{M},\tau)$. Hence, there exists a sequence $(a_n)_n$ in K that converges weakly to a in $L^1(\mathcal{M},\tau)$. We claim that $(a_n)_n$ is weakly Cauchy in $E(\mathcal{M},\tau)$.

Let $h \in E(\mathcal{M},\tau)^*$. Since h is a measurable operator, $\forall\varepsilon > 0$, there exists a projection $e \in \mathcal{M}$ with $\tau(1-e) < \varepsilon$ and $he \in \mathcal{M}$.

For each $n \in \mathbb{N}$, let $\varepsilon = \frac{1}{2^n}$ and choose e_n a projection in $\mathcal{M}$ with $\tau(1-e_n) < \frac{1}{2^n}$ and $he_n \in \mathcal{M}$. We define the following sequence of projections:

$$\begin{aligned} f_1 &= e_1, \\ f_2 &= e_2 \wedge (1 - f_1), \\ &\vdots \\ f_n &= e_n \wedge (1 - f_1 - f_2 - \cdots - f_{n-1}). \end{aligned}$$

Clearly the family $(f_n)_{n\ge1}$ consists of pairwise disjoint projections that satisfy $1 - f_1 - f_2 - \cdots - f_n \le 1 - e_n \quad \forall n \in \mathbb{N}$. Moreover since $he_n \in \mathcal{M}$, the operator $hf_n \in \mathcal{M}$.

Lemma 2. *The series $\sum_{n=1}^{\infty} hf_n$ is a w.u.c. series in $E(\mathcal{M},\tau)^*$ and $h =$ weak*-$\sum_{n=1}^{\infty} hf_n$.*

To prove the lemma, let σ be a finite subset of $\mathbb{N}$. We have

$$\Big\|\sum_{n\in\sigma} he_n\Big\| = \Big\|h\Big(\sum_{n\in\sigma} e_n\Big)\Big\| \le \|h\|_{E(\mathcal{M},\tau)^*}\ .$$

So $\sum_{n=1}^{\infty} hf_n$ is a w.u.c. series. Moreover if $a \in E(\mathcal{M},\tau)$, the operator $ha \in L^1(\mathcal{M},\tau)$ and

$$\begin{aligned} |\langle\sum_{n=1}^{m} hf_n - h, a\rangle| &= |\langle h(1 - \sum_{n=1}^{m} f_n), a\rangle| \\ &= |\langle 1 - \sum_{n=1}^{m} f_n, ah^*\rangle| \\ &\le \tau\left((1 - \sum_{n=1}^{m} f_n)|ah^*|\right). \end{aligned}$$

Since $ah^* \in L^1(\mathcal{M},\tau)$ and the projections $\theta_m = 1-\sum_{n=1}^m f_n$ satisfy $\tau(\theta_m) \leq 1/2^m$ $\forall m \in \mathbb{N}$, we get that $\lim_{m\to\infty} \tau(\theta_m|ah^*|) = 0$ which shows that $\sum_{n=1}^\infty \langle hf_n, a\rangle = \langle h, a\rangle$. The lemma is proved.

Now since K is a (V*)-subset of $E(\mathcal{M},\tau)$, the series $\sum_{n=1}^\infty hf_n$ converges unconditionally uniformly on K. If not, one can find $\delta > 0$, $p_1 < p_2 < \cdots < p_n < \cdots$ such that for every $n \geq 1$,

$$\sup_{a\in K}\left(\sum_{j=p_n+1}^{p_{n+1}} \langle hf_j, a\rangle\right) > \delta\ .$$

For each $n \in \mathbb{N}$, let $b_n = \sum_{j=p_n+1}^{p_{n+1}} hf_j$. The series $\sum_{n=1}^\infty b_n$ is also a w.u.c. series in $E(\mathcal{M},\tau)^*$ and $\sup_{a\in K}\langle b_n, a\rangle > \delta\ \ \forall\, n \in \mathbb{N}$, thus contradicting the fact that K is a (V*)-subset.

Fix $\varepsilon > 0$; there exists $m > 0$ such that for all $n \geq 1$,

$$\Big|\sum_{j=m+1}^\infty \langle hf_j, a_n\rangle\Big| < \frac{\varepsilon}{3}\ .$$

Let $\mathcal{E} = \sum_{j=1}^m hf_j$, by the construction of $(f_j)_j$, the operator $\mathcal{E} \in \mathcal{M}$.

Now since $(a_n)_n$ is weakly convergent in $L^1(\mathcal{M},\tau)$, the sequence $(\mathcal{E}(a_n))_{n\geq 1}$ is convergent so there exists $N \in \mathbb{N}$ such that if $p, q > N$,

$$|\mathcal{E}(a_p) - \mathcal{E}(a_q)| < \frac{\varepsilon}{3}\ .$$

Hence for $p, q > N$,

$$\begin{aligned} |h(a_p) - h(a_q)| &\leq \Big|\sum_{j=1}^\infty hf_j(a_p) - hf_j(a_q)\Big| \\ &\leq |\mathcal{E}(a_p) - \mathcal{E}(a_q)| + \Big|\sum_{j=m+1}^\infty hf_j(a_p)\Big| + \Big|\sum_{j=m+1}^\infty hf_j(a_q)\Big|. \end{aligned}$$

So $|h(a_p) - h(a_q)| < \varepsilon$ which shows that $(a_n)_n$ is weakly Cauchy in $E(\mathcal{M},\tau)$ and since $E(\mathcal{M},\tau)$ is weakly sequentially complete, the sequence $(a_n)_n$ is weakly convergent so $E(\mathcal{M},\tau)$ has property (V*). This ends the proof for the case where the trace τ is finite.

For the general case, let $\mathcal{M}$ be a semifinite von-Neumann algebra with a distinguished faithful, normal semifinite trace τ.

Choose a mutually orthogonal family $(e_i)_{i\in I}$ of projections in $\mathcal{M}$ with $\sum_{i\in I} e_i = 1$ for the strong operator topology and $\tau(e_i) < \infty$ for all $i \in I$.

Let K be a (V*)-subset of $E(\mathcal{M},\tau)$ and fix $(a_n)_{n\geq 1}$ a sequence in K. Using a similar argument as in [16], one can get an at most countable subset $(e_k)_{k\in\mathbb{N}}$ of $(e_i)_{i\in I}$ such that for each e_i outside of $(e_k)_{k\in\mathbb{N}}$, $e_i a_n = a_n e_i = 0$ $\forall n \in \mathbb{N}$. Let $e = \sum_{k\in\mathbb{N}} e_k$. We have $ea_n = a_n e = a_n$ $\forall n \in \mathbb{N}$. Replacing $\mathcal{M}$ by $e\mathcal{M}e$ and τ by its restriction on $e\mathcal{M}e$, we may assume that $e = 1$.

For $l \geq 1$ and $k \geq 1$, let $e_{lk} = e_l \vee e_k$. It is clear that $\tau(e_{lk}) < \infty$ and the sequence $(e_l a_n e_k)_{n\geq 1}$ is a (V*)-subset of $E(e_{lk}\mathcal{M}e_{lk}, \tau_{lk})$ where τ_{lk} is the restriction of τ on $e_{lk}\mathcal{M}e_{lk}$. By the previous case, the space $E(e_{lk}\mathcal{M}e_{lk}, \tau_{lk})$ has property (V*) so there exists a subsequence $(a_{n_j})_{n\geq 1}$ of $(a_n)_{n\geq 1}$ so that $(e_l a_{n_j} e_k)_j$ converges weakly to $\varphi_{lk} \in E(e_{lk}\mathcal{M}e_{lk}, \tau_{lk})$. Using diagonal argument, one can assume that this is true for all $l \geq 1$ and $k \geq 1$ (i.e., the subsequence $(a_{n_j})_j$ is independent of l and k).

Note that for each $l \geq 1$ and $k \geq 1$, $\varphi_{lk} \in E(\mathcal{M},\tau)$. We claim that the series $\sum_{l\geq 1}\sum_{k\geq 1}\varphi_{lk}$ is convergent in $E(\mathcal{M},\tau)$ and if we let $a = \sum_{l\geq 1}\sum_{k\geq 1}\varphi_{lk}$ then $(a_{n_j})_j$ converges weakly to the operator a in $E(\mathcal{M},\tau)$. To see this, we will show first that for each $l \geq 1$, the series $\sum_{k\geq 1}\varphi_{lk}$ is unconditionally convergent. For that notice that since $E(\mathcal{M},\tau)$ does not contain c_0, it is enough to check that the series $\sum_{k=1}^{\infty}\varphi_{lk}$ is a w.u.c. series. Using the same argument as in the proof of Lemma 2, one can conclude that for each $b \in E(\mathcal{M},\tau)$, the series $\sum_{k=1}^{\infty} e_l b e_k$ is a w.u.c. series in $E(\mathcal{M},\tau)$. In fact for each finite subset $P \subseteq \mathbb{N}$,

$$\Big\|\sum_{k\in P} e_l b e_k\Big\| \leq \|e_l b\|_{E(\mathcal{M},\tau)}.$$

Now for any finite subset P of $\mathbb{N}$,

$$\begin{aligned}\Big\|\sum_{k\in P}\varphi_{lk}\Big\| &\leq \varliminf_{j\to\infty}\Big\|\sum_{k\in P} e_l a_{n_j} e_k\Big\|\\ &\leq \varlimsup_{j\to\infty}\|e_l a_{n_j}\|\\ &\leq \varlimsup_{j\to\infty}\|a_{n_j}\| = C.\end{aligned}$$

The constant C being independent of P shows that $\sum_{k=1}^{\infty}\varphi_{lk}$ is a w.u.c. series.

For each $l \geq 1$, let $b_l = \sum_{k=1}^{\infty}\varphi_{lk}$. The operator b_l is well defined and the sequence $(e_l a_{n_j})_j$ converges to b_l weakly in $E(\mathcal{M},\tau)$. For this recall that the sequence $(a_{n_j})_j$ is a (V*)-set and so is the sequence $(e_l a_{n_j})_j$. Fix $y \in E(\mathcal{M},\tau)^*$. The series $\sum_{k=1}^{\infty} y e_k$ is a w.u.c. series in $E(\mathcal{M},\tau)^*$ and converges to y for the weak*-topology (Lemma 2.).

For $\varepsilon > 0$, $\exists\ m \in \mathbb{N}$ so that $\forall\ j \geq 1$,

$$\Big|\sum_{k=m+1}^{\infty}\langle y e_k,\ e_l a_{n_j} - b_l\rangle\Big| < \frac{\varepsilon}{2}\ .$$

On the other hand,

$$\begin{aligned}\lim_{j\to\infty}\sum_{k=1}^{m}\langle y e_k,\ e_l a_{n_j} - b_l\rangle &= \lim_{j\to\infty}\sum_{k=1}^{m}\langle y, e_l a_{n_j} e_k - b_l e_k\rangle\\ &= \lim_{j\to\infty}\sum_{k=1}^{m}\langle y, e_l a_{n_j} e_k - \varphi_{lk}\rangle = 0.\end{aligned}$$

Hence $\exists\ N \in \mathbb{N}$ so that if $j \geq N$,

$$\Big|\sum_{k=1}^{m}\langle y e_k,\ e_l a_{n_j} - b_l\rangle\Big| < \frac{\varepsilon}{2}$$

and therefore for $j \geq N$,

$$\begin{aligned}|\langle y,\ e_l a_{n_j} - b_l\rangle| &\leq \Big|\sum_{k=1}^{m}\langle y e_k,\ e_l a_{n_j} - b_l\rangle\Big| + \Big|\sum_{k=m+1}^{\infty}\langle y e_k,\ e_l a_{n_j} - b_l\rangle\Big|\\ &\leq \varepsilon\end{aligned}$$

which proves that $\lim_{j\to\infty}\langle y, e_l a_{n_j} - b_l\rangle = 0$ so $(e_l a_{n_j})_j$ converges weakly to b_l in $E(\mathcal{M},\tau)$.

To complete the proof, we repeat the same argument above for the indeces l. First, one can show as above that the series $\sum_{l\geq 1} b_l$ is unconditionally convergent and since $(e_l a_{n_j})_j$ converges weakly to b_l for every $l \geq 1$, we then deduce that $(a_{n_j})_j$ converges weakly to $a = \sum_{l\geq 1} b_l$. Details are left to the reader.

Since the sequence $(a_n)_n$ was chosen arbitrary in K, we conclude that the set K is relatively weakly compact. $\square$

Remark. The above theorem shows in particular that if E is a symmetric sequence space that does not contain any copy c_0 then the associated unitary matrix space C_E has property (V*). As examples, the Lorentz-Schatten classes $C_{p,1}$ $(1 \leq p < \infty)$ have property (V*).

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