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SIMPLE CONNECTEDNESS OF PROJECTIVE VARIETIES

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Abstract. A Lefschetz type theorem is proven relating the algebraic fundamental group of a smooth projective variety X to the algebraic fundamental group of a subvariety set theoretically defined by $\leq \dim(X) - 2$ forms.

In this paper we prove a generalization of Grothendieck's Lefschetz theorem for complete intersections (SGA2 XII 3.5). Our result is:

Theorem 1. *Suppose that k is a field, W is a smooth, geometrically connected subvariety of $\mathbf{P}_k^m$ of dimension n and $Z \subset W$ is a closed subscheme set theoretically defined by the vanishing of r forms of $\mathbf{P}_k^m$ on W.*

1. *If $r \leq n-1$ then Z is geometrically connected and there is a surjection $\pi_1(Z) \to \pi_1(W)$.*
2. *If $r \leq n-2$, then $\pi_1(Z) \cong \pi_1(W)$.*

Corollary 2. *Suppose that k is a field and $Z \subset \mathbf{P}_k^n$ is a closed subscheme set theoretically defined by r forms.*

1. *If $r \leq n-1$ then Z is geometrically connected.*
2. *If $r \leq n-2$, then $\pi_1(Z) \cong \mathrm{Gal}(\overline{k}/k)$ where $\overline{k}$ is an algebraic closure of k.*

The corresponding theorem for the topological fundamental group of a complex projective variety follows from Hamm [H] and the Theorem of II 1.2 in [GM]. Their proofs use different methods (Morse theory) and do not extend to positive characteristic.

$\pi_1(X)$ will denote the algebraic fundamental group of a scheme X. If k is a field, $\overline{k}$ will denote an algebraic closure of k.

Simple connectedness

Proof of the first half of 1. of Theorem 1. Suppose that $r \leq n-1$. Let K be an extension field of k. Let (A, m) be the local ring of the homogeneous coordinate ring of $W \otimes_k K$, $\hat{A}$ be completion at m. Suppose that Z is set theoretically defined by r forms $f_1, \ldots, f_r \in A$. $r \leq (n+1) - 2$ implies $\mathrm{spec}(\hat{A}/(f_1, \ldots, f_r)) - \hat{m}$ is connected by Corollary 4 to Theorem 1 [F], since $\hat{A}$ is a domain. Hence Z is geometrically connected.

Definition 3. A morphism $f : X \to Y$ is separable if f is flat and for any $y \in Y$, $X \times_Y k(y)$ is geometrically reduced over $k(y)$.

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Lemma 4. *Suppose that Y is a noetherian scheme, $f : X \to Y$ is a finite type morphism. Let*

$$A = \{y \in Y \mid X_y \textit{ is geometrically reduced over } k(y)\}.$$

Then

1. *A is a constructible subset of Y.*
2. *If $g : Y' \to Y$ is a morphism then*

$$g^{-1}(A) = \{y' \in Y' \mid X \times_Y Y'_{y'} \textit{ is geometrically reduced over } k(y')\}.$$

Proof. This is EGA IV.9.7.7 and IV.9.2.2 (iv).

Lemma 5. *Suppose that Y is a noetherian scheme and $Z \subset Y$ is a constructible subset. Let $\overline{Z}$ be the closure of Z in Y. If Z contains no generic points of codimension one irreducible subschemes of Y, then $\overline{Z}$ has codimension ≥ 2 in Y.*

Proof. This follows from the fact proved in EGA $0_{III}9.2.3$ that a constructible subset of an irreducible subset W of Y is dense in W if and only if it contains a nonempty open subset.

We need to generalize to morphisms which are not separable the exact homotopy sequence for proper separable morphisms of SGA1 X 1.4 and Theorem 6.3.2.1 [M].

Theorem 6. *Suppose that Y is a connected regular scheme, X is normal, $f : X \to Y$ is a proper morphism such that f is separable in codimension one (in Y) and $f_*\mathcal{O}_X \cong \mathcal{O}_Y$. Let $y \in Y$ be the generic point, $\overline{X_y} = X \times_Y \overline{k(y)}$. Then there is a natural right exact sequence*

$$\pi_1(\overline{X_y}) \xrightarrow{\Phi} \pi_1(X) \xrightarrow{\Psi} \pi_1(Y) \to 0.$$

Proof. The proofs that Ψ is surjective and $\Psi \circ \Phi = 0$ are exactly as in the proof of Theorem 6.3.2.1 [M]. We must prove that Image$(\Phi) \supset$ kernel(Ψ). By the criterion of 5.2.4 [M], we must show that if $g : X' \to X$ is a connected étale cover of X and the base change $\overline{g} : X' \times_X \overline{X_y} \to \overline{X_y}$ has a section σ over $\overline{X_y}$, then there exists a connected étale cover Y'/Y such that $X' \cong X \times_Y Y'$.

Suppose that $g : X' \to X$ is such a morphism. $f \circ g$ is proper and separable in codimension one by Lemma 6.3.2.2 [M]. Let $X' \xrightarrow{h'} Y' \to Y$ be the stein factorization of $f \circ g$. By Theorem 6.2.1 [M] $Y' \to Y$ is étale in codimension one. Y' is normal since X' is. By purity of branch locus for regular schemes (c.f. SGA1 X 3.1) $Y' \to Y$ is étale.

It remains to show that the natural map $\alpha : X' \to X \times_Y Y'$ is an isomorphism. This is shown exactly as on pages 115-116 in the proof of Theorem 6.3.2.1 [M].

Corollary 7. *Suppose that Y is a connected regular excellent scheme, X is normal, $f : X \to Y$ is a proper morphism such that f is separable in codimension 1, and $f_*\mathcal{O}_X = \mathcal{O}_Y$. Let $z_0 \in Y$ be a point and $z_1 \in Y$ be the generic point. Let $\overline{X}_0 = X \times_Y \overline{k(z_0)}$, $\overline{X}_1 = X \times_Y \overline{k(z_1)}$. Then there is a natural surjection*

$$\pi_1(\overline{X_1}) \to \pi_1(\overline{X_0}).$$

Proof. Let $A = \mathcal{O}_{Y,z_0}$, $\hat{A}$ be the completion of A at its maximal ideal, $Y' = \operatorname{spec}(\hat{A})$. Let z_1' be the generic point and z_0' be the closed point of Y', $X' = X \times_Y Y'$, with natural morphism $f' : X' \to Y'$. f' is proper and $f'_*\mathcal{O}_{X'} = \mathcal{O}_{Y'}$. X' is normal since X is excellent. If $P \subset \hat{A}$ is a height one prime then $\operatorname{ht}(P \cap A) \leq 1$. By Lemma

4 f' is separable in codimension 1. Let $\overline{X_0'} = X' \times_{Y'} \overline{k(z_0')}$, $\overline{X_1'} = X' \times_{Y'} \overline{k(z_1')}$. $\pi_1(\overline{X_0'}) = \pi_1(\overline{X_0})$ and $\pi_1(\overline{X_1'}) \cong \pi_1(\overline{X_1})$ by Proposition 7.3.2 [M]. By careful choice of basepoints we have a commutative diagram:

$$\begin{array}{ccccccccc} & & \pi_1(\overline{X_1'}) & \to & \pi_1(X') & \to & \pi_1(Y') & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & \pi_1(\overline{X_0'}) & \to & \pi_1(X') & \to & \pi_1(Y') & \to & 0 \end{array}$$

The top row is right exact by Theorem 6. The bottom row is exact by SGA1 X 2.2 or Theorem 8.1.1 [M]. The two vertical maps are the identity. Hence there is a surjection $\pi_1(\overline{X_1}) \to \pi_1(\overline{X_0})$.

Let notation be as in the statement of Theorem 1. Let $I_W \subset k[x_0, \dots x_m]$ be the homogeneous ideal of W. Suppose that $r \leq n-1$ and k is algebraically closed. There exist integers d_1 and d_2, where we can assume that

$$H^1(\mathbf{P}_k^m, \mathcal{O}(d_1) \otimes \tilde{I_W}) = H^1(\mathbf{P}_k^m, \mathcal{O}(d_2) \otimes \tilde{I_W}) = 0$$

and we can take d_2 arbitrarily large relative to d_1 such that Z is defined set theoretically by the vanishing of $r-1$ forms $f_1, \dots, f_{r-1}$ of degree d_1 and a form f_r of degree d_2. Let $t_1 = h^0(W, \mathcal{O}_W(d_1))$, $t_2 = h^0(W, \mathcal{O}_W(d_2))$. Let $a_1^I, \dots, a_{r-1}^I, b_r^J$ be $(r-1)t_1 + t_2$ indeterminates, where I indexes a basis $\sigma_1, \dots, \sigma_{t_1}$ of $H^0(W, \mathcal{O}_W(d_1))$ and J indexes a basis $\tau_1, \dots, \tau_{t_2}$ of $H^0(W, \mathcal{O}_W(d_2))$. Let

$$F_1 = \sum_{I=1}^{t_1} a_1^I \sigma_I, \dots, F_{r-1} = \sum_{I=1}^{t_1} a_{r-1}^I \sigma_I, F_r = \sum_{J=1}^{t_2} b_r^J \tau_J.$$

$$F_1, \dots, F_r \in (k[x_0, \dots, x_m]/I_W)[a_1^I, \dots, a_{r-1}^I, b_r^J].$$

Let $Y = \operatorname{Proj}(k[a_1^I, \dots, a_{r-1}^I, b_r^J])$,

$$X = V(F_1, \dots, F_r) \subset Y \times W,$$

be the subscheme determined by $F_1, \dots, F_r$. There is a natural projective morphism $f : X \to Y$. Let $p \in Y$ be the closed point such that $(X_p)_{red} \cong Z_{red}$.

Proposition 8. *Let notation be as in the above paragraph.*

1. *X is smooth over k.*
2. *$f_* \mathcal{O}_X = \mathcal{O}_Y$.*
3. *Let $E = \{y \in Y \mid X_y$ is not geometrically reduced over $k(y)\}$, $\overline{E}$ be the closure of E in Y. Then* $\operatorname{codim}_Y(\overline{E}) \geq 2$.
4. *Let $F = \{y \in Y \mid$ there exists $x \in f^{-1}(y)$ such that $\mathcal{O}_{X,x}$ is not flat over $\mathcal{O}_{Y,y}\}$. Then* $\operatorname{codim}_Y(F) \geq 2$.

Note That E is constructible and F is closed.

Proof. X is smooth over k by the Jacobian criterion.

Let $X \to Y' \to Y$ be the Stein factorization of f. $Y' \to Y$ is dominant, finite and Y' is normal. By Bertini's theorem (cf. II 8.18 [Ha]), there exists a dense open subset U of Y such that if $q \in Y$ is a closed point, then $X_q \subset W$ is a smooth irreducible complete intersection of dimension ≥ 1. Hence $Y' \to Y$ is generically 1-1. Thus $Y' = Y$ by Zariski's Main Theorem, and $f_* \mathcal{O}_X = \mathcal{O}_Y$.

Given $\overline{a_i^I} \in k$ (with at least one $\overline{a_i^I} \neq 0$), let

$$V(\overline{a_i^I}) = \operatorname{Proj}(k[a_1^I, \dots, a_{(r-1)}^I, b_r^J]/(\overline{a_i^I} a_k^K - \overline{a_k^K} a_i^I)) \subset Y.$$

Each $V(\overline{a_i^I}) \cong \mathbf{P}^{t_2}$. The union of $V(\overline{a_i^I})$ over all choices of $\overline{a_i^I}$ is Y. Let

$$D = \operatorname{Proj}(k[a_1^I, \dots, a_{(r-1)}^I, b_r^J]/(a_1^I, \dots, a_{(r-1)}^I)) \cong \operatorname{Proj}(k[b_r^J]).$$

Let $\overline{F}_1, \dots, \overline{F}_{r-1} \in k[x_0, \dots, x_m]/I_W$ be the corresponding specializations of $F_1, \dots, F_{r-1}$ over $(a_i^I) \to (\overline{a_i^I})$.

We can choose $\overline{a_i^I}$ such that if Γ is a codimension one integral component of $F \cup \overline{E}$ then $\Gamma \cap V(\overline{a_i^I})$ is not contained in D and if T is the subscheme of W determined by the vanishing of $\overline{F_1}, \dots, \overline{F}_{r-1}$, then T is a smooth subvariety of W of dimension $n-(r-1) \geq 2$.

Let $V = V(\overline{a_i^I}) - D$. V parametrizes the intersections of T with the zero locus of sections of $H^0(W, \mathcal{O}_W(d_2))$. Hence we have a natural identification

$$V = \mathbf{V}(S(H^0(W, \mathcal{O}_W(d_2))^*)),$$

where $*$ denotes dual k vector space.

To show that $\operatorname{codim}_Y(\overline{E}) \geq 2$ and $\operatorname{codim}_Y(F) \geq 2$, it suffices by our construction of V to show that $\operatorname{codim}_V(V \cap \overline{E}) \geq 2$ and $\operatorname{codim}_V(V \cap F) \geq 2$.

Let $I_T = (\overline{F}_1, \dots, \overline{F}_{r-1}) + I_W \subset k[x_0, \dots, x_m]$. I_T is the homogeneous ideal of T. Recall that for fixed T, we are free to choose d_2 arbitrarily large. Since $\mathcal{O}(1)$ is ample, $\dim T \geq 2$, we can choose d_2 so that

i) $H^1(W, \mathcal{O}_W(d_2) \otimes \tilde{I_T}) = 0$.
ii) $h^0(W, \mathcal{O}_W(d_2) \otimes \tilde{I_T}) < h^0(W, \mathcal{O}_W(d_2)) - 1$.
iii) If D, C are nonzero effective divisors on T such that $\mathcal{O}_T(D+C) \cong \mathcal{O}_T(d_2)$, then

$$h^0(T, \mathcal{O}_T(C)) < h^0(T, \mathcal{O}_T(d_2)) - \dim \operatorname{Pic}(T) - 1.$$

Assertions i) and ii) follow from Serre's vanishing theorem, and since $h^0(W, \mathcal{O}_W(d))$ is a polynomial in d of degree n for large d and $h^0(T, \mathcal{O}_T(d))$ is a polynomial in d of degree $n-(r-1) \geq 2$ for large d.

Now we will verify iii). Let $s = \dim \operatorname{Pic}(T) + 2$. If d_2 is sufficiently large, $\mathcal{O}_T(d_2)$ has the property that if $p_1, \dots, p_s$ are any distinct closed points in T, then

$$h^0(T, \mathcal{O}_T(d_2) \otimes \mathcal{O}_T(-p_1 - \dots - p_s)) = h^0(T, \mathcal{O}_T(d_2)) - s.$$

If D, C are as in iii), and $p_1, \dots, p_s$ are distinct closed points of D, then

$$\begin{aligned} h^0(T, \mathcal{O}_T(C)) &= h^0(T, \mathcal{O}_T(d_2)) \otimes \mathcal{O}_T(-D)) \\ &\leq h^0(T, \mathcal{O}_T(d_2) \otimes \mathcal{O}_T(-p_1 - \dots - p_s)) \\ &< h^0(T, \mathcal{O}_T(d_2)) - \dim \operatorname{Pic}(T) - 1 \end{aligned}$$

and iii) holds.

By i) we have a natural exact sequence

$$(1) \qquad 0 \to H^0(W, \mathcal{O}_W(d_2) \otimes \tilde{I_T}) \to H^0(W, \mathcal{O}_W(d_2)) \to H^0(T, \mathcal{O}_T(d_2)) \to 0.$$

Let $H \to \mathbf{P}(H^0(T, \mathcal{O}_T(d_2))^*)$ be the universal family parametrizing the subschemes of T given by vanishing of sections of $H^0(T, \mathcal{O}_T(d_2))$. Let

$$V' = \mathbf{V}(H^0(W, \mathcal{O}_W(d_2))^*) - \mathbf{V}(H^0(W, \mathcal{O}_W(d_2) \otimes \tilde{I_T})^*).$$

That is, V' is the complement of $\mathbf{V}(H^0(W, \mathcal{O}_W(d_2) \otimes \tilde{I_T})^*)$ in

$$V = \mathbf{V}(H^0(W, \mathcal{O}_W(d_2))^*).$$

(1) gives a natural surjection

$$\lambda : V' \to \mathbf{P}(H^0(T, \mathcal{O}_T(d_2))^*)$$

such that $X_{V'} = \lambda^*(H)$.

We have $V \cap F = \mathbf{V}(H^0(W, \mathcal{O}_W(d_2) \otimes I_{\tilde{T}})^*)$, so that $\mathrm{codim}_V(V \cap F) \geq 2$ by ii) and assertion 4) follows.

If $\zeta \in V$ is the generic point of a codimension 1 subvariety of V, then $\zeta \in V'$ by ii), and the closure of $\lambda(\zeta)$ has codimension ≤ 1 in $\mathbf{P}(H^0(T, \mathcal{O}_T(d_2))^*)$. Let $\alpha = \lambda(\zeta)$. By Lemma 4 X_ζ is geometrically reduced over $k(\zeta)$ if and only if H_α is geometrically reduced over $k(\alpha)$.

Let B' be the closure of α in $\mathbf{P}(H^0(T, \mathcal{O}_T(d_2))^*)$. There is a finite radicial morphism $\tau : B \to B'$ such that if β is the generic point of B, $(H_\beta)_{red}$ is geometrically reduced over $k(\beta)$ (cf. IV.4.6.6 EGA). If H_β is not reduced, then there exists a dense open $U \subset B$ and a flat map $(H_B)_{red} \times_B U \to U$ such that the fibers over closed points of U are pairwise distinct subschemes of T, each given by the vanishing of a section of $H^0(T, \mathcal{O}_T(C))$ for some effective divisor C on T with $h^0(T, \mathcal{O}_T(d_2) \otimes \mathcal{O}_T(-C)) > 0$, and where each fiber has a common Hilbert polynomial P. Let Hilb^P be the component of the Hilbert scheme of T of subschemes with the Hilbert polynomial P.

There exists an immersion $U \to \mathrm{Hilb}^P$ such that $(H_B)_{red} \times_B U$ is the pullback of the universal family over Hilb^P.

There is a morphism $\gamma : \mathrm{Hilb}^P \to \mathrm{Pic}(T)$ where the fiber containing the point corresponding to the subscheme C is $\mathbf{P}(H^0(T, \mathcal{O}_T(C))^*)$. By iii),

$$\mathrm{dim}\mathrm{Hilb}^P \leq \dim \mathbf{P}(H^0(T, \mathcal{O}_T(d_2))^*) - 2.$$

Hence $\dim(B') = \dim(U) \leq \dim \mathbf{P}(H^0(T, \mathcal{O}_T(d_2))^*) - 2$. This shows that $\lambda(\beta) = \alpha$ has codimension > 1 in $\mathbf{P}(H^0(T, \mathcal{O}_T(d_2))^*)$, a contradiction, so that X_ζ is geometrically reduced over $k(\zeta)$. $\mathrm{codim}_V(V \cap \overline{E}) \geq 2$ so that $\mathrm{codim}_Y(\overline{E}) \geq 2$.

Proof of Theorem 1. First suppose that k is algebraically closed. Consider the map $f : X \to Y$ defined before Proposition 8. By Proposition 8 the assumptions of Corollary 7 are satisfied. Hence there is a surjection $\pi_1(\overline{X_1}) \to \pi_1(X_p) = \pi_1(Z)$. $X_1 \subset W \otimes_k k(a_i^I, a_r^J)$ is a smooth irreducible complete intersection of dimension ≥ 1 under the assumptions of 1) and of dimension ≥ 2 under the assumptions of 2). The composite map

$$\pi_1(\overline{X}_1) \to \pi_1(Z) \to \pi_1(W)$$

is a surjection under the assumptions of 1) and is an isomorphism under the assumptions of 2) by SGA2 XII 3.5 and Proposition 7.3.2 [M]. If k is not algebraically closed the conclusions of 1) and 2) now hold by SGA1 IX 6.1 or Theorem 8.1.1 [M].

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