

SERRE-DUALITY FOR TAILS(A)

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ABSTRACT. A version of Serre-duality is proved for Artin's non-commutative projective schemes.

0. INTRODUCTION

Let k be a field and A a left noetherian \mathbb{N} -graded k -algebra. If A is commutative, it describes a geometric object, the projective k -scheme $\text{Proj}(A)$. Most of the geometry on $\text{Proj}(A)$ really is about sheaves of coherent or quasi-coherent modules on $\text{Proj}(A)$; thus, the abelian categories of such sheaves play prominent roles in algebraic geometry.

After dropping the condition that A should be commutative, it is possible to define abelian categories generalizing the categories of (quasi-)coherent sheaves. These categories, $\text{tails}(A)$ and $\text{Tails}(A)$, are obtained as the quotients

$$\begin{aligned} \text{tails}(A) &= \text{grmod}(A)/\text{tors}(A), \\ \text{Tails}(A) &= \text{GrMod}(A)/\text{Tors}(A); \end{aligned}$$

here $\text{grmod}(A)$ is the category of finitely generated graded left A -modules and graded homomorphisms, and $\text{tors}(A)$ is the dense subcategory of torsion-modules; $\text{GrMod}(A)$ and $\text{Tors}(A)$ are the corresponding objects without the finiteness condition. $\text{tails}(A)$ is a full subcategory of $\text{Tails}(A)$. Details of the construction can be found in [1], and generalities on quotient-categories in [3].

Using the philosophy that $\text{tails}(A)$ and $\text{Tails}(A)$ generalize notions from (commutative) geometry, it is natural to ask if they support generalizations of various well-known geometrical constructions. In particular, it is natural to look for a duality-theory corresponding to Serre-duality, which is one of the most important technical tools in modern commutative geometry.

Depending on how “good” the k -scheme X is, classical Serre-duality yields functorial isomorphisms

$$H^n(X, \mathcal{F})' \cong \text{Ext}_X^{d-n}(\mathcal{F}, \omega^\circ)$$

for a range of n -values. Here d is the dimension of X , and \mathcal{F} is a quasi-coherent sheaf on X . The prime denotes dualization with respect to k , and Ext_X is the usual Ext-functor on the abelian category of quasi-coherent sheaves, while H^n is the n 'th

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right derived functor of the global section-functor, Γ . The sheaf ω° is a “dualizing object”.

In more advanced versions of the theory it is possible, by replacing ω° by a complex of sheaves (“the dualizing complex”), to prove the duality-formula above with fewer assumptions; one then has to use a hyper-Ext instead of the usual Ext.

In their paper [7], A. Yekutieli and J. Zhang construct a theory of Serre-duality for tails(A). This paper does the same thing but on Tails(A) and with fewer conditions on A ; also, the theory presented here is hypercohomological, yielding Serre-duality, corollary 3.5, in the version with a dualizing complex as mentioned above. Classical Serre-duality, corollary 3.6, is obtained as a special case. The paper follows hypercohomological ideas from the commutative theory presented in [6], and uses the Brown Adjoint Functor Theorem, [6, thm. 4.1, p. 223]. The main results are corollary 3.3 and its special cases, corollaries 3.5 and 3.6.

A bit about the notation: throughout the paper, k is a field and A is a left-noetherian \mathbb{N} -graded k -algebra. We have the canonical projection-functor,

$$\pi : \text{GrMod}(A) \longrightarrow \text{Tails}(A),$$

and its right-adjoint,

$$\omega : \text{Tails}(A) \longrightarrow \text{GrMod}(A).$$

$\pi\omega$ is equivalent to the identity-functor, cf. [3, III.2, prop. 3, p. 371]. Objects in Tails(A) are denoted by calligraphic letters, and $\mathcal{A} = \pi A$; twisting in Tails(A) is denoted by (). We call

$$\Gamma(-) = \text{Hom}_{\text{Tails}(A)}(\mathcal{A}, -) : \text{Tails}(A) \longrightarrow \text{Vect}(k)$$

the section-functor. The n 'th right-derived functor of Γ is denoted H^n , and the cohomological dimension of Γ is denoted $d = \text{cd}(\text{Tails}(A))$.

An isomorphism in any of the categories that come into play will be symbolized by “ $\xrightarrow{\cong}$ ”. If \mathfrak{A} is an abelian category, complexes of \mathfrak{A} -objects are equipped with a “ $*$ ” as superscript, and Hom^* is complex-Hom. $K(\mathfrak{A})$ is the category of complexes of \mathfrak{A} -objects and homotopy-classes of chain-maps, while $D(\mathfrak{A})$ is the derived category of \mathfrak{A} (these are categories of unbounded complexes). The symbol h^n will denote the n 'th cohomology-functor on these categories; H^n is reserved for the n 'th derived functor of Γ . A quasi-isomorphism from one complex to another is symbolized by “ $\xrightarrow{\sim}$ ”, and twisting of complexes is denoted by []. If $T : K(\mathfrak{A}) \longrightarrow K(\mathfrak{B})$ is a triangulated functor, its derived functor is denoted $\underline{RT} : D(\mathfrak{A}) \longrightarrow D(\mathfrak{B})$ (but may not exist!) We write $D(\mathcal{A})$ resp. $D(k)$ for the derived categories $D(\text{Tails}(A))$ resp. $D(\text{Vect}(k))$.

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1. SOME FACTS ABOUT TAILS(A)

This section proves a number of easy but useful properties, mostly of a homological nature, about Tails(A).

Proposition 1.1. *The category Tails(A) has all small direct limits.*

Proof. Since $\text{Tails}(A)$ is abelian it has all cokernels and thus all coequalizers. Now π is a left-adjoint, so it preserves direct limits. In particular, it preserves small direct sums, and since any object in $\text{Tails}(A)$ has the form πM for some $M \in \text{GrMod}(A)$, cf. [3, III.1, p. 365], the functor π provides $\text{Tails}(A)$ with small direct sums. But a category with coequalizers and small direct sums has all small direct limits, cf. [5, V.2, cor. 2, p. 109]. \square

Lemma 1.2. *In $\text{Tails}(A)$ small direct sums preserve exactness.*

Proof. Given a small system of exact sequences in $\text{Tails}(A)$,

$$0 \rightarrow \mathcal{M}'_i \longrightarrow \mathcal{M}_i \longrightarrow \mathcal{M}''_i \rightarrow 0.$$

According to [3, III.1, cor. 1, p. 368], each sequence can be obtained as

$$0 \rightarrow \pi \mathcal{M}'_i \xrightarrow{\pi(\mu'_i)} \pi \mathcal{M}_i \xrightarrow{\pi(\mu_i)} \pi \mathcal{M}''_i \rightarrow 0,$$

where the sequences of \mathcal{M}_i 's are short-exact in $\text{GrMod}(A)$. Now the sequence

$$0 \rightarrow \bigoplus_i \mathcal{M}'_i \longrightarrow \bigoplus_i \mathcal{M}_i \longrightarrow \bigoplus_i \mathcal{M}''_i \rightarrow 0$$

is clearly short-exact, but applying the exact functor π yields the direct sum of the \mathcal{M}_i -sequences, so this sum is exact. \square

Corollary 1.3. *Taking (co-)homology of $\text{Tails}(A)$ -complexes commutes with taking small direct sums of such complexes.*

Lemma 1.4. *In $\text{Tails}(A)$ the direct sum of a small family of injective objects is again an injective object.*

Proof. A is left-noetherian, so the corresponding statement for $\text{GrMod}(A)$ is true. Now let $\{\mathcal{E}_i\}_{i \in I}$ be a small family of injectives from $\text{Tails}(A)$. Then for each i we know that $\mathcal{E}_i = \pi \omega \mathcal{E}_i$ and that $\omega \mathcal{E}_i$ is a graded-injective and torsionfree graded module, cf. [1, prop. 7.1(1), p. 270]. The sum $\bigoplus_i \omega \mathcal{E}_i$ is thus also graded-injective and torsionfree, and applying π yields an injective object of $\text{Tails}(A)$, cf. [1, prop. 7.1(3), p. 270] — but

$$\pi \bigoplus_i \omega \mathcal{E}_i \cong \bigoplus_i \pi \omega \mathcal{E}_i \cong \bigoplus_i \mathcal{E}_i.$$

\square

Corollary 1.5. *Let \mathfrak{A} be an abelian category and let $T : \text{Tails}(A) \rightarrow \mathfrak{A}$ be an additive functor which commutes with small direct sums. Then each right-derived functor $R^i T$ also commutes with small direct sums.*

Proof. An immediate consequence of lemma 1.4 and corollary 1.3. \square

Corollary 1.6. *If $\mathcal{M} \in \text{tails}(A)$, then each functor*

$$\text{Ext}_{\text{Tails}(A)}^i(\mathcal{M}, -) : \text{Tails}(A) \longrightarrow \text{Vect}(k)$$

commutes with small direct sums.

Proof. Because of corollary 1.5 it is enough to see that $\text{Hom}_{\text{Tails}(A)}(\mathcal{M}, -)$ commutes with small direct sums, but this is a simple consequence of the fact that if M is in $\text{grmod}(A)$, then the functor

$$\text{Hom}_{\text{GrMod}(A)}(M, -) : \text{GrMod}(A) \longrightarrow \text{Vect}(k)$$

commutes with small direct sums. \square

2. HYPERCOHOMOLOGY FOR TAILS(A)

Here we study the hypercohomology of $\text{Tails}(A)$. Recall that d denotes the cohomological dimension of the functor $\Gamma(-) = \text{Hom}_{\text{Tails}(A)}(\mathcal{A}, -)$. All the results below involving Γ presuppose $d < \infty$.

Proposition 2.1. *The categories $D(\mathcal{A})$ and $D(k)$ have small direct sums, and if $T(-) = \text{Hom}_{\text{Tails}(A)}(\mathcal{A}(m), -)$ and $d < \infty$ then the derived functor*

$$\underline{\underline{R}}T : D(\mathcal{A}) \longrightarrow D(k)$$

exists and preserves small direct sums. In particular this is true for $\underline{\underline{R}}\Gamma$.

Proof. Small direct sums in the categories in question are obtained as the ordinary direct sums of the relevant complexes, cf. [2, sect. 1, pp. 211-213].

Twisting by a fixed amount is an auto-equivalence of the category $\text{Tails}(A)$, cf. [1, p. 236], so for $\mathcal{M}, \mathcal{N} \in \text{Tails}(A)$ and $m \in \mathbb{Z}$ we have

$$\begin{aligned} \text{Hom}_{\text{Tails}(A)}(\mathcal{M}(m), \mathcal{N}) &\cong \text{Hom}_{\text{Tails}(A)}(\mathcal{M}(m)(-m), \mathcal{N}(-m)) \\ &\cong \text{Hom}_{\text{Tails}(A)}(\mathcal{M}, \mathcal{N}(-m)). \end{aligned}$$

If S denotes twisting by $-m$ we see that $T \cong \Gamma \circ S$, and S is exact and preserves injectives, so

$$R^n T \cong H^n \circ S.$$

Since Γ has cohomological dimension d , so does T , i.e. the derived functors $R^n T$ vanish precisely from number $d+1$ onwards. Since $\text{Tails}(A)$ has enough injectives, this is sufficient by [4, cor. I.5.3(γ), p. 56] to secure the existence of $\underline{\underline{R}}T$.

Finally, suppose that $\mathcal{C}^* \in D(\mathcal{A})$ has the resolution $\mathcal{C}^* \xrightarrow{\sim} \mathcal{Q}^*$, meaning that the arrow is a quasi-isomorphism. Suppose also that \mathcal{Q}^* consists of T -acyclic objects; such a resolution exists by [4, proof of cor. I.5.3, p. 58]. Then

$$\underline{\underline{R}}T(\mathcal{C}^*) = T(\mathcal{Q}^*),$$

cf. [4, proof of thm. I.5.1, p. 54]. Because of corollary 1.3, if we have resolutions $\mathcal{C}_i^* \xrightarrow{\sim} \mathcal{Q}_i^*$ of a small family of complexes, $\bigoplus_i \mathcal{C}_i^* \xrightarrow{\sim} \bigoplus_i \mathcal{Q}_i^*$ is also a resolution, and because of corollary 1.6, if all the \mathcal{Q}_i^* 's consist of T -acyclics, so will $\bigoplus_i \mathcal{Q}_i^*$. But then

$$\underline{\underline{R}}T\left(\bigoplus_i \mathcal{C}_i^*\right) = T\left(\bigoplus_i \mathcal{Q}_i^*\right) \cong \bigoplus_i T(\mathcal{Q}_i^*) = \bigoplus_i \underline{\underline{R}}T(\mathcal{C}_i^*).$$

□

Lemma 2.2. *Let $\mathcal{C}^* \in D(\mathcal{A})$ have $h^n \mathcal{C}^* \neq 0$. Then for any $M \in \mathbb{N}$ there exists an $m \geq M$ such that*

$$\text{Hom}_{D(\mathcal{A})}(\mathcal{A}(-m)[-n], \mathcal{C}^*) \neq 0.$$

Proof. \mathcal{C}^* is a complex over $\text{Tails}(A)$, so $C^* = \omega \mathcal{C}^*$ is a complex over $\text{GrMod}(A)$ and $C^* \cong \pi \omega \mathcal{C}^* = \pi C^*$. Since π is exact and $h^n C^* \neq 0$, we see that $h^n C^*$ cannot be torsion. So $h^n C^*$ has homogeneous non-torsion elements of degree larger than M for any $M \in \mathbb{N}$. We can thus construct

$$A(-m) \longrightarrow Z^n C^* \longrightarrow h^n C^*$$

with non-torsion image in $h^n C^*$ for some $m \geq M$, yielding a chain-map

$$A(-m)[-n] \longrightarrow C^*$$

whose induced map in cohomology has non-torsion image. But then $\mathcal{A}(-m)[-n] \rightarrow \mathcal{C}^*$ does not induce the zero-map on cohomology, whence

$$\mathrm{Hom}_{D(\mathcal{A})}(\mathcal{A}(-m)[-n], \mathcal{C}^*) \neq 0.$$

□

Lemma 2.3. *Let \mathfrak{A} be an abelian k -linear category with enough injectives, let $A \in \mathfrak{A}$, put $T(-) = \mathrm{Hom}_{\mathfrak{A}}(A, -)$ and suppose that $R^{N+1}T = R^{N+2}T = \dots = 0$ for some $N \in \mathbb{N}$. Then the derived functor*

$$\underline{\underline{RT}} : D(\mathfrak{A}) \longrightarrow D(k)$$

exists and satisfies

$$h^0 \underline{\underline{RT}}(-) \xrightarrow{\cong} \mathrm{Hom}_{D(\mathfrak{A})}(A, -).$$

Proof. Existence of $\underline{\underline{RT}}$ follows from [4, cor. I.5.3(γ), p. 56].

The isomorphism will be constructed as follows: any \mathfrak{A} -complex is quasi-isomorphic to a complex of T -acyclic objects, so it will be sufficient to come up with isomorphisms

$$\varphi : h^0 \underline{\underline{RT}}(Q^*) \xrightarrow{\cong} \mathrm{Hom}_{D(\mathfrak{A})}(A, Q^*)$$

for Q^* 's consisting of T -acyclics. Now, $h^0 \underline{\underline{RT}}(Q^*) = h^0 \mathrm{Hom}_{\mathfrak{A}}(A, Q^*)$ is the k -space of homotopy-classes of chain-maps from A to Q^* . We may thus define φ by letting it send a class represented by $a : A \rightarrow Q^*$ to the $D(\mathfrak{A})$ -morphism represented by the diagram

$$\begin{array}{ccc} & Q^* & \\ a \nearrow & \swarrow & \\ A & & Q^*. \end{array}$$

It remains to be proved that this really gives an isomorphism.

First a general remark: suppose that Q^* is an acyclic complex of T -acyclic objects of \mathfrak{A} . Then TQ^* is also acyclic, since we have $TQ^* = \underline{\underline{RT}}Q^*$ and the latter object is isomorphic to zero in $D(k)$, because Q^* is acyclic and therefore isomorphic to zero in $D(\mathfrak{A})$. Thus

$$\mathrm{Hom}_{K(\mathfrak{A})}(A, Q^*) \cong h^0 \mathrm{Hom}_{\mathfrak{A}}(A, Q^*) = h^0 T(Q^*) = 0.$$

Now let $Q^* \xrightarrow{\cong} Q'^*$ be a quasi-isomorphism of complexes of T -acyclics and complete it with the mapping-cone to a distinguished triangle,

$$\begin{array}{ccc} Q^* & \xrightarrow{\cong} & Q'^* \\ \kappa \searrow & & \swarrow \\ & & Q''^*. \end{array}$$

The mapping-cone is an acyclic complex of T -acyclics, so the long-exact sequence induced by the triangle reads

$$\begin{aligned} 0 &= \mathrm{Hom}_{K(\mathfrak{A})}(A, Q''^*(-1)) \\ &\downarrow \\ \mathrm{Hom}_{K(\mathfrak{A})}(A, Q^*) &\longrightarrow \mathrm{Hom}_{K(\mathfrak{A})}(A, Q'^*) \longrightarrow \mathrm{Hom}_{K(\mathfrak{A})}(A, Q''^*) = 0, \end{aligned}$$

whence $\text{Hom}_{K(\mathfrak{A})}(A, Q^*) \xrightarrow{\cong} \text{Hom}_{K(\mathfrak{A})}(A, Q'^*)$ is an isomorphism.

φ is injective: Assume that $\varphi(a) = 0$. By [4, proof of prop. I.3.1, p. 30], this means that we can choose $q : Q^* \xrightarrow{\cong} B^*$ such that qa is null-homotopic. But choose $b : B^* \xrightarrow{\cong} Q'^*$ with Q'^* consisting of T -acyclics. We then have the diagram

$$\begin{array}{ccccc}
& & Q'^* & & \\
& \swarrow & \uparrow \simeq & \searrow & \\
& b & B^* & q & \\
& \downarrow bqa & \uparrow qa & \downarrow & \\
A & & Q^* & & Q^* \\
& \nearrow a & \searrow & \nearrow & \\
& & Q^* & &
\end{array}$$

bqa is null-homotopic because qa is, but since bq is a quasi-isomorphism, by the general remark above, we must have a null-homotopic.

φ is surjective: Suppose that we are given a morphism α in $\text{Hom}_{D(\mathfrak{A})}(A, Q^*)$. It is represented by a diagram

$$\begin{array}{ccc}
& B^* & \\
a' \nearrow & \uparrow \simeq & \searrow q \\
A & & Q^*.
\end{array}$$

We can construct a resolution $b : B^* \xrightarrow{\cong} Q'^*$ where Q'^* consists of T -acyclics. But then bq is a quasi-isomorphism from Q^* to Q'^* , so by the general remark above we can find $a : A \rightarrow Q^*$ such that ba' and bqa are homotopy-equivalent. It is easy to see that the class of this a is mapped to the given morphism, α , by φ . \square

Proposition 2.4. *Suppose that $d < \infty$. Then the set $\{\mathcal{A}(m)[n]\}_{m,n \in \mathbb{Z}}$ is generating for $D(\mathcal{A})$, i.e., if $\text{Hom}_{D(\mathcal{A})}(\mathcal{A}(m)[n], C^*) = 0$ for all m and n , then $C^* = 0$. Also, the set consists of compact $D(\mathcal{A})$ -objects, i.e., each $\text{Hom}_{D(\mathcal{A})}(\mathcal{A}(m)[n], -)$ commutes with small direct sums. Thus, since the triangulated category $D(\mathcal{A})$ has small direct sums, it is compactly generated, cf. [6, def. 1.7, p. 211].*

Proof. The first claim follows directly from lemma 2.2, since an acyclic complex is isomorphic to the zero-complex in the derived category $D(\mathcal{A})$.

The second claim is an equally direct consequence of lemma 2.3: by [6, observation after def. 1.6, p. 210] it will suffice to prove that each $\mathcal{A}(m)$ is compact, and if we set $T(-) = \text{Hom}_{\text{Tails}(\mathcal{A})}(\mathcal{A}(m), -)$ we know by proposition 2.1 that $\underline{\underline{R}}T$ exists and preserves small direct sums. But then

$$\text{Hom}_{D(\mathcal{A})}(\mathcal{A}(m), -) \cong h^0 \underline{\underline{R}}T(-)$$

also preserves small direct sums. \square

3. NON-COMMUTATIVE SERRE-DUALITY

This section contains two general results, theorem 3.1 and corollary 3.3, and their corollaries, 3.5 (hypercohomological Serre-duality) and 3.6 (classical Serre-duality).

Throughout this section we assume that the cohomological dimension, d , of Γ is finite.

Theorem 3.1. *The derived functor $\underline{\underline{R}\Gamma}$ has a right-adjoint,*

$$G : D(k) \longrightarrow D(\mathcal{A}).$$

Proof. This is an immediate consequence of the Brown Adjoint Functor Theorem, cf. [6, thm. 4.1, p. 223], since we have already proved propositions 2.1 and 2.4. \square

Definition 3.2. *We set*

$$\omega^* = G(k), \quad \omega^\circ = h^{-d}(\omega^*).$$

ω^* will be called the dualizing complex.

Corollary 3.3. *There is an isomorphism*

$$\mathrm{Hom}_{D(\mathcal{A})}(\mathcal{C}^*, \omega^*) \cong (h^0 \underline{\underline{R}\Gamma} \mathcal{C}^*)'$$

natural in $\mathcal{C}^* \in D(\mathcal{A})$; as in the introduction, the prime denotes dualization with respect to k .

Proof. Theorem 3.1 states that there is a natural isomorphism

$$\mathrm{Hom}_{D(\mathcal{A})}(\mathcal{C}^*, G\mathcal{V}^*) \cong \mathrm{Hom}_{D(k)}(\underline{\underline{R}\Gamma} \mathcal{C}^*, \mathcal{V}^*)$$

in $\mathcal{C}^* \in D(\mathcal{A})$ and $\mathcal{V}^* \in D(k)$. In particular, with $\mathcal{V}^* = k$,

$$\mathrm{Hom}_{D(\mathcal{A})}(\mathcal{C}^*, \omega^*) \cong \mathrm{Hom}_{D(k)}(\underline{\underline{R}\Gamma} \mathcal{C}^*, k).$$

But k is an injective resolution of itself, so

$$\mathrm{Hom}_{D(k)}(\underline{\underline{R}\Gamma} \mathcal{C}^*, k) \cong h^0 \mathrm{Hom}^*(\underline{\underline{R}\Gamma} \mathcal{C}^*, k) \cong (h^0 \underline{\underline{R}\Gamma} \mathcal{C}^*)'.$$

\square

Lemma 3.4. *The only non-vanishing cohomology of ω^* is in degrees $-d, -d + 1, \dots, 0$.*

Proof. The duality-isomorphism yields

$$\begin{aligned} \mathrm{Hom}_{D(\mathcal{A})}(\mathcal{A}(-m)[-n], \omega^*) &\cong (h^0 \underline{\underline{R}\Gamma} \mathcal{A}(-m)[-n])' \\ &= (R^{-n} \Gamma \mathcal{A}(-m))' = H^{-n}(\mathcal{A}(-m))', \end{aligned}$$

so the lemma's statement follows directly from lemma 2.2. \square

Lemma 3.4 implies that

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \omega^{-d-1} & \longrightarrow & \omega^{-d} & \longrightarrow & \omega^{-d+1} \longrightarrow \cdots \longrightarrow \omega^0 \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \parallel \\ \cdots & \longrightarrow & 0 & \longrightarrow & \omega^{-d}/B^{-d}(\omega^*) & \longrightarrow & \omega^{-d+1} \longrightarrow \cdots \longrightarrow \omega^0 \longrightarrow \cdots \end{array}$$

is a quasi-isomorphism, whence we may assume that ω^* has $\omega^{-d-r} = 0$ for every $r > 0$. As a consequence, by [4, lemma I.4.6, p. 42] we may assume that ω^* consists of injective Tails(A)-objects. These assumptions will be made henceforth.

Corollary 3.5 (Hypercohomological Serre-duality). *The dualizing complex, ω^* , consisting of injective Tails(A)-objects has the property that*

$$H^n(\mathcal{M})' \cong h^{-n} \mathrm{Hom}_{\mathrm{Tails}(A)}(\mathcal{M}, \omega^*) = \mathrm{Ext}_{\mathrm{Tails}(A)}^{-n}(\mathcal{M}, \omega^*)$$

functorially in $\mathcal{M} \in \mathrm{Tails}(A)$ (the Ext appearing here is hyper-Ext over Tails(A)).

Proof. Simple computations tell us that

$$h^0 \underline{R}\Gamma(\mathcal{M}[n]) = h^n \underline{R}\Gamma\mathcal{M} = H^n(\mathcal{M})$$

and

$$\begin{aligned} \mathrm{Hom}_{D(A)}(\mathcal{M}[n], \omega^*) &\cong h^0 \mathrm{Hom}^*(\mathcal{M}[n], \omega^*) \\ &= h^{-n} \mathrm{Hom}^*(\mathcal{M}, \omega^*) = h^{-n} \mathrm{Hom}_{\mathrm{Tails}(A)}(\mathcal{M}, \omega^*); \end{aligned}$$

now use corollary 3.3. \square

Corollary 3.6 (Classical Serre-duality). *Let $r \in \mathbb{N}$ and assume that*

$$H^{d-1}(\mathcal{A}(-m)) = H^{d-2}(\mathcal{A}(-m)) = \cdots = H^{d-r}(\mathcal{A}(-m)) = 0$$

for all $m \gg 0$ (the condition is vacuous for $r = 0$). Then for $n = d, d-1, \dots, d-r$ we have

$$H^n(\mathcal{M})' \cong \mathrm{Ext}_{\mathrm{Tails}(A)}^{d-n}(\mathcal{M}, \omega^\circ)$$

functorially in $\mathcal{M} \in \mathrm{Tails}(A)$ (the Ext appearing here is the classical Ext over $\mathrm{Tails}(A)$).

Proof. By corollary 3.3 we have

$$\mathrm{Hom}_{D(A)}(\mathcal{A}(-m)[d-n], \omega^*) \cong h^0 \underline{R}\Gamma(\mathcal{A}(-m)[d-n])' = H^{d-n}(\mathcal{A}(-m))',$$

so via lemma 2.2, the assumption on the cohomology of the $\mathcal{A}(-m)$'s implies that

$$h^{-d+1}\omega^* = h^{-d+2}\omega^* = \cdots = h^{-d+r}\omega^* = 0.$$

We see that ω^* is the beginning of ω° 's injective resolution, twisted with $[d]$, whence by corollary 3.5

$$H^n(\mathcal{M})' \cong h^{-n} \mathrm{Hom}_{\mathrm{Tails}(A)}(\mathcal{M}, \omega^*) = \mathrm{Ext}_{\mathrm{Tails}(A)}^{d-n}(\mathcal{M}, \omega^\circ).$$

\square

Note that the case $r = 0$ yields an isomorphism

$$H^d(\mathcal{M})' \cong \mathrm{Hom}_{\mathrm{Tails}(A)}(\mathcal{M}, \omega^\circ),$$

natural in $\mathcal{M} \in \mathrm{Tails}(A)$, without any conditions on A apart from the standing ones, that A is a left noetherian \mathbb{N} -graded k -algebra with $d < \infty$.

REFERENCES

- [1] M. Artin and J. J. Zhang, *Noncommutative projective schemes*, Adv. Math., **109** (1994) 228–287. MR **96a**:14004
- [2] M. Bökstedt and A. Neeman, *Homotopy limits in triangulated categories*, Comp. Math., **86** (1993) 209–234. MR **94f**:18008
- [3] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France, **90** (1962) 323–448. MR **38**:1144
- [4] R. Hartshorne, *Residues and duality*, Lecture Notes in Math., vol. 20, Springer, Berlin, 1966. MR **36**:5145
- [5] S. MacLane, *Categories for the working mathematician*, Springer Graduate Texts in Math., vol. 5, Springer, Berlin, 1971. MR **50**:7275
- [6] A. Neeman, *The Grothendieck duality theorem via Bousfield's techniques and Brown representability*, J. Amer. Math. Soc., **9** (1996), 205–236. MR **96c**:18006
- [7] A. Yekutieli and J. J. Zhang, *Serre duality for noncommutative projective schemes*, Proc. Amer. Math. Soc., this issue.

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