A UNIQUE CONTINUATION THEOREM
FOR THE SCHRÖDINGER EQUATION
WITH SINGULAR MAGNETIC FIELD

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Abstract. We show a unique continuation theorem for the Schrödinger equation \((\frac{1}{i} \nabla - A)^2 u + Vu = 0\) with singular coefficients \(A\) and \(V\).

1. Main results

In this paper we show a unique continuation theorem for the Schrödinger operator \(H = (\frac{1}{i} \nabla - A)^2 + V\) with singular magnetic field. In fact we shall establish, under some assumptions on \(A\) and \(V\), the following estimate:

\[ \int_{B_r(x_0)} |u|^2 dx \leq C \int_{B_r(x_0)} |u|^2 dx. \]

The latter holds for \(r > 0\) and \(x_0 \in \Omega\) with \(B_r(x_0) \subset \Omega\) and for solutions \(u \in H^2_{\text{loc}}(\Omega)\) of

\( Hu = (\frac{1}{i} \nabla - A)^2 u + Vu = 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^n, \)

where \(n \geq 3, \Omega\) is a domain, \(A(x) = (A_j(x))_{j=1}^n\) is real-valued and \(V(x) = V^R(x) + iV^I(x)\) is complex-valued. This estimate implies a strong unique continuation theorem (Corollary 1.1). The proof uses a Rellich’s type identity and the variational method (originally due to Garofalo and Lin, see e.g., [GL1], [GL2], [Ku1]), which does not need Carleman type estimates. We emphasize that our method requires neither higher integrability nor pointwise estimates for \(A\) compared with previous results [BKRS], [H], [Wo1], [So], [GL1], [GL2], [Ku1].

Throughout this paper we use the notation

\[ \nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}), \quad V^+(x) = \max(V(x), 0), \quad V^-(x) = \max(-V(x), 0), \]

\[ \mathbf{B} = (b_{jk})_{j,k=1}^n, \quad b_{jk}(x) = \frac{\partial A_j}{\partial x_k} - \frac{\partial A_k}{\partial x_j}, \]

\[ D_j = \frac{1}{i} \frac{\partial}{\partial x_j} - A_j, \quad D_j^* = -\frac{1}{i} \frac{\partial}{\partial x_j} - A_j, \]

\[ H^m_{\text{loc}}(\Omega) = \{ u \in L^2_{\text{loc}}(\Omega); \quad D^\alpha u \in L^2_{\text{loc}}(\Omega), |\alpha| \leq m \}. \]
Note that $\overline{D_j u} = D_j \overline{\pi}$ for a complex-valued function $u$.

To state our main results, first we recall the definition of the Kato class $K_{\text{loc}}^n(\Omega)$.

**Definition 1.1.** We say $f \in L^1_{\text{loc}}(\Omega)$ belongs to the Kato class $K_{\text{loc}}^n(\Omega)$ if
\[
\lim_{r \to 0} \eta(r; f\chi_{\Omega_r}) = 0
\]
for every compact subdomain $\Omega_r$ of $\Omega$. Here
\[
\eta(r; f\chi_{\Omega_r}) = \sup_{x \in \mathbb{R}^n} \int_{|x-y|<r \cap \Omega_r} \frac{|f(y)|}{|x-y|^n} \, dy.
\]

The assumptions on $A$ and $V$ in this paper are the following.

**Assumption (A):** Let $x_0 \in \Omega$ be fixed.

1. $A \in L^4_{\text{loc}}(\Omega)$, $\nabla A \in L^2_{\text{loc}}(\Omega)$, $|A|^2 \in K_{\text{loc}}^n(\Omega)$, $\{(|x-x_0||B||)^2 \in K_{\text{loc}}^n(\Omega)$;

2. $V \in K_{\text{loc}}^n(\Omega)$, $(|x-x_0||V||)^2 \in K_{\text{loc}}^n(\Omega)$, $(2V^R + (x-x_0) \cdot \nabla V^R)^- = K_{\text{loc}}^n(\Omega)$.

Taking an arbitrary compact subdomain $\Omega_o \subset \Omega$ such that $x_o \in \Omega_o$, we will use the notation
\[
\theta_o(r) = \eta_o(r; (2V^R + (x-x_o) \cdot \nabla V^R)^-)
\]
and
\[
\eta_o(r; ((x-x_o)V^I)^2) + \eta_o(r; (|x-x_o||B||)^2)^{1/2},
\]
where $\eta_o(r; f) = \eta(r; f\chi_{\Omega_o})$.

**Theorem 1.1.** Suppose Assumption (A) and $\int_0^{r_0} \frac{\theta_a(r)}{r} \, dr < +\infty$ for some $r_0 > 0$. Let $u \in H^2_{\text{loc}}(\Omega)$ be a solution of (1). Then there exist constants $r_*, C > 0$ such that
\[
\int_{B_{2r}(x_o)} |u|^2 \, dx \leq C \int_{B_r(x_o)} |u|^2 \, dx
\]
for every $0 < r < r_*$.

When we do not assume $\int_0^{r_0} \frac{\theta_a(r)}{r} \, dr < +\infty$, we have

**Theorem 1.2.** Suppose Assumption (A) and let $u \in H^2_{\text{loc}}(\Omega)$ be a solution of (1). Then, for each $r_1 \in (0, r_*)$, there exist constants $C > 0$ and $L(r_1) > 0$ such that
\[
\int_{B_{2r}(x_o)} |u|^2 \, dx \leq C \exp\left(\frac{L(r_1)}{rC^2(1)}\right) \int_{B_{r}(x_o)} |u|^2 \, dx
\]
for every $r_{1/2} > r > 0$.

The constant $r_*$ is determined by the condition that $\eta(r; (V^R)^-) \leq \frac{1}{2C(n)}$ hold for all $r \in (0, r_*)$, where $C(n)$ is a constant depending only on $n$ (Lemma 2.3). The constants $C$ and $L(r_1)$ do not depend on $r$, but depend on $u$. It is well known that these yield unique continuation theorems (e.g., [GL1], [GL2], [Ku1]).

**Corollary 1.1.** Suppose Assumption (A) and the condition $\int_0^{r_0} \frac{\theta_a(r)}{r} \, dr < +\infty$, $r_0 > 0$, for every $x_o \in \Omega$. Then $H$ has SUCP (strong unique continuation property); if $u \in H^2_{\text{loc}}(\Omega)$ is a solution of (1) and satisfies, for some $x_o \in \Omega$ and for every $m > 0$,
\[
\int_{B_r(x_o)} |u|^2 \, dx = O(r^m) \quad (r \to 0),
\]
then $u \equiv 0$ in $\Omega$. 

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Corollary 1.2. Suppose Assumption (A) for every \( x_o \in \Omega \). Then \( H \) has a unique continuation property; if \( u \in H^2_{loc}(\Omega) \) is a solution of (1) and satisfies, for some \( x_o \in \Omega \) and \( A, \alpha > 0 \),
\[
\int_{B_r(x_o)} |u|^2 \, dx = O(\exp(-\frac{A}{r^\alpha})) \quad (r \to 0),
\]
then \( u \equiv 0 \) in \( \Omega \).

In particular, \( H \) has WUCP (weak unique continuation property); if \( u \) vanishes on a subdomain \( \Omega' \) of \( \Omega \), then \( u \equiv 0 \) in \( \Omega \).

Since \( H u = -\Delta u + 2iA \cdot \nabla u + i(\text{div} A)u + |A|^2 u + Vu \), putting \( b = 2iA, W = i(\text{div} A) + |A|^2 + V \), we can rewrite the equation (1) in the following way:
\[
H u = -\Delta u + b \cdot \nabla u + W u.
\]

Although we can apply results of [BKRS], [Wo1], [Wo2], [So], [H] [GL2], [Ku1] etc. under suitable conditions on \( b \) and \( W \), our theorems cannot be covered by these previous results. First, previous results require a stronger condition on \( \text{div} A \) even for WUCP. For instance, [Wo2] require \( W \in L^{n/2}_{loc} \) for WUCP and hence \( \text{div} A \in L^{n/2}_{loc} \) etc. On the other hand, our method only requires \( \text{div} A \in L^{2}_{loc} \) for \( \text{div} A \) and, instead of that, \( (|x - x_o||B|)^2 \in K^1_{\text{loc}}(\Omega) \) (or \( (|x - x_o||B|)^2 \in F^1_{\text{loc}}(\Omega) \)) for the magnetic field \( B \) (see Example 1.1). Secondly, to obtain SUCP [BKRS], [H] require \( |A| \in L^q_{loc} \) \( q > \frac{3n - 2}{2} \) (for related results see also [Wo1]) and [So], [GL2], [Ku1] require a pointwise estimate
\[
|A(x)| \leq \frac{f(|x - x_o|)}{|x - x_o|}, \quad \int_0^r \frac{f(t)}{t} \, dt < +\infty
\]
for \( A \), but our method requires neither higher integrability nor pointwise estimates.

Remark 1.1. These theorems also hold even if we replace the class \( K_{\text{loc}}^1(\Omega) \) in Assumption (A) by the more general one \( Q^1_{\text{loc}}(\Omega) = K_{\text{loc}}^1(\Omega) + F^1_{\text{loc}}(\Omega) \), \( 1 < t \leq n/2 \), where \( F^1_{\text{loc}}(\Omega) \) is the Fefferman-Phong class. However, in this case we must assume an additional condition
\[
\lim_{r \to 0} \sup_{x \in \Omega} \| (V^R)^{-} \|_{Q^1_{\text{loc}}(\Omega)} \leq \epsilon_o
\]
for sufficiently small \( \epsilon_o > 0 \) and take \( H^2_{loc}(\Omega) \cap L^\infty_{loc}(\Omega) \) as a solution class. See [Ku1].

Remark 1.2. A sufficient condition to assure \( \int_0^r \frac{\theta_o(r)}{r} \, dr < +\infty \) is that
\[
2V^R + (x - x_o) \cdot \nabla V^R, ((x - x_o)V^I)^2, ((x - x_o)|B|)^2 \in K_{\delta, \text{loc}}(\Omega) (\subseteq K_{\text{loc}}^{\delta}(\Omega))
\]
hold for some \( \delta > 0 \), where we say \( f \in K_{\delta, \text{loc}}(\Omega) \) if \( f \) satisfies, for every compact subdomain \( \Omega_o \) of \( \Omega \),
\[
\lim_{r \to 0} \sup_{x \in \Omega_o} \int_{|x - y| < r} \int_{|x - y| < r} \frac{|f(y)|}{|x - y|^{n-2+\delta}} \, dy = 0.
\]

Remark 1.3. By using the approximation argument in [Ku1] we can show unique continuation theorems similar to the theorems above even for \( H^1_{loc} \) solutions.

Finally, we must remark that Kalf [Ka] proved WUCP under the assumptions \( A \in H^1_{loc}(\Omega), (V^R)^2 \in K_{\text{loc}}(\Omega) (V^I \equiv 0) \), but his method cannot be applied to SUCP. For WUCP our results complement his result; compare that one needs
Note that (7)

Example 1.2. Let $x_o = O, B_1 = B_1(O)$ and

$$V(x) = \frac{1}{|x|^l \log |x|^m}, \quad m, l > 0.$$ 

Then we have

$$V \in K_n^{\text{loc}}(B_1) \iff (m > 1, l = 2) \text{ or } (m \in \mathbb{R}, l < 2),$$

(5)

$$V^2 \in K_n^{\text{loc}}(B_1) \iff (m > 1/2, l = 1) \text{ or } (m \in \mathbb{R}, l < 1),$$

$$2V + x \cdot \nabla V \in K_n^{\text{loc}}(B_1) \iff (m > 0, l = 2) \text{ or } (m \in \mathbb{R}, l < 2).$$

So in this example the condition $V^2 \in K_n^{\text{loc}}(\Omega)$ is stronger than $(2V + x \cdot \nabla V)^- \in K_n^{\text{loc}}(\Omega)$.

Example 1.3. Let $N \in \mathbb{N}$ and $R = (R_1, \cdots, R_N) \in \mathbb{R}^{3N}$ be fixed and

$$V(x) = \sum_{j=1}^{N} \frac{1}{|x_j - R_j|}, \quad x = (x_1, \cdots, x_N) \in \mathbb{R}^{3N}, \quad x_j \in \mathbb{R}.$$ 

Then $V^2 \notin K_n^{\text{loc}}$, but $2V + (x - x_o) \cdot \nabla V \in K_n^{\text{loc}}$ for every $x_o \in \mathbb{R}^{3N}$.

2. Proof of the Theorems

We may assume $x_o$ is the origin $O$ and write $B_r = B_r(O)$. For the sake of simplicity, we also use $\Omega, \eta(r; f)$ and $\theta(r)$ instead of $\Omega_o, \eta(r; f\chi_{\Omega_o})$ and $\theta_o(r)$, respectively.

Let $u \in H_0^{\text{loc}}(\Omega)$ be a solution of (1) and put

$$I(r) = \int_{B_r} |Du|^2 + V^R |u|^2 \, dx = \int_{B_r} |(\nabla - iA)u|^2 + V^R |u|^2 \, dx, \quad (6)$$

$$H(r) = \int_{\partial B_r} |u|^2 \, dS, \quad N(r) = \frac{rI(r)}{H(r)}. \quad (7)$$

Note that $H = \left(\frac{1}{2} \nabla - A\right)^2 + V = \sum_{j=1}^{n} D_j D_j + V$. Our argument is based on the following identity.

Lemma 2.1. Suppose Assumption (A) (for $A$). Then $u \in H_0^{\text{loc}}(\Omega) \cap L^\infty(\Omega)$ satisfies

$$\text{Im} \left( \int_{B_r} (x \cdot Du) \sum_{j=1}^{n} D_j D_j u \, dx \right)$$

$$= \frac{r}{2} \int_{\partial B_r} |Du|^2 \, dS - \left( \frac{n-2}{2} \right) \int_{B_r} |Du|^2 \, dx$$

$$- r \int_{\partial B_r} |n \cdot Du|^2 \, dS + \text{Re} \left( \sum_{j,k=1}^{n} b_{jk}(x) x_j D_k u \, dx \right)$$

(8)

for every $r > 0$ with $B_r \subset \Omega$. 

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Proof. This is a kind of Rellich’s identity. The detailed computation can be seen in [EK]. So we omit it.

We remark that under the assumption $|A|^2, V \in K_{loc}^1(\Omega)$ a solution $u \in H_{loc}^1(\Omega)$ is locally bounded (see [Ku2]). So we can apply Lemma 2.1 for solutions $u \in H_{loc}^2(\Omega)$ of (1). This identity implies

**Lemma 2.2.** Suppose Assumption (A). Then for a.e. $r \in (0, R_o)$

$$I'(r) = \frac{n-2}{r} I(r) + 2 \int_{\partial B_r} |n \cdot Du|^2 dS$$

$$- n \frac{2}{r} \int_{B_r} V R |u|^2 dx + \int_{\partial B_r} V R |u|^2 dS$$

$$- \frac{2}{r} Re(\sum_{j,k=1}^n \int_{B_r} b_{jk}(x)x_j(D_k u)\overline{u} dx)$$

(9)

holds, where $R_o = \max\{r; B_r \subset \Omega\}$.

Proof. Note that

$$I'(r) = \int_{\partial B_r} |Du|^2 + VR |u|^2 dS \quad a.e. \ r.$$  

Lemma 2.1 and this identity imply (9).

Since $x \cdot D^* u = x \cdot D^* \overline{u} = i(x \cdot \nabla \overline{u}) - (x \cdot A) \overline{u}$, we have

$$Im(\int_{B_r} (x \cdot D^* u) V u dx) = Re(\int_{B_r} (x \cdot \nabla \overline{u}) V R u dx) + Im(\int_{B_r} (x \cdot D^* \overline{u}) V I u dx).$$

Put

$$J(V^R; u; r) = \int_{\partial B_r} VR |u|^2 dS - n \frac{2}{r} \int_{B_r} VR |u|^2 dx$$

$$- \frac{1}{r} \int_{B_r} (x \cdot \nabla(|u|^2)) VR dx.$$  

(10)

By integration by parts we have

$$J(V^R; u; r) \geq - \frac{1}{r} \int_{B_r} (2VR + x \cdot \nabla VR^- |u|^2 dx.$$  

Then from this observation and Lemma 2.2 we obtain the following estimate:

$$I'(r) \geq \frac{n-2}{r} I(r) + 2 \int_{\partial B_r} |n \cdot Du|^2 dS$$

$$- \frac{1}{r} \int_{B_r} (2VR + x \cdot \nabla VR^- |u|^2 dx +$$

$$- \frac{2}{r} \int_{B_r} (|x||V^R| + |x||B|)|Du||u| dx$$

(11)

for a.e. $r \in (0, R_o)$. To control the last two terms we need the following inequality.
Lemma 2.3. Let $W \in K_n^{\text{loc}}(\Omega)$ and $u \in H^1_{\text{loc}}(\Omega)$. Then there exists a constant $C(n)$ such that

\[(12) \quad \int_{B_r} |W| |u|^2 \, dx \leq C(n) \eta(r; W \chi_{\Omega_n})(\int_{B_r} |Du|^2 \, dx + \frac{1}{r} \int_{\partial B_r} |u|^2 \, dS)\]

for $B_r \subset \Omega, \Omega^c \subset \Omega$.

Proof. Since $|u| \in H^1_{\text{loc}}(\Omega)$, we know that

\[(13) \quad \int_{B_r} |W| |u|^2 \, dx \leq C(n) \eta(r; W \chi_{\Omega_n})(\int_{B_r} |\nabla u|^2 \, dx + \frac{1}{r} \int_{\partial B_r} |u|^2 \, dS)\]

holds (FGL). Noting $|\nabla u| \leq |Du|$ a.e., we obtain the desired inequality. \hfill \Box

Lemma 2.4. There exists a constant $r_*>0$ such that $H(r) > 0$ for every $r \in (0, r_*)$ unless $u \equiv 0$ in $B_{r_*}$.

Proof. Note that the constant $r_*>0$ is determined by the condition $\eta^R(r) \leq \frac{1}{2C(n)}$ for $r \in (0, r_*)$, where $\eta^R(r) = \eta(r; (V^R)^-) \chi_{\Omega_n}$ and $C(n)$ is the constant in Lemma 2.3. Since $V^R \in K_n^{\text{loc}}(\Omega)$, the constant $r_*>0$ satisfying this condition exists.

Suppose $H(r_o) = 0$ for some $r_o \in (0, r_*)$. Noting a simple computation yields

\[(14) \quad I(r) = \text{Re}(\int_{\partial B_r} u \overline{u} \, dS) = \text{Re}(\int_{\partial B_r} (n \cdot Du) \overline{u} \, dS),\]

we have $I(r_o) = 0$. Since Lemma 2.3 implies

\begin{align*}
I(r) &\geq \int_{B_r} |Du|^2 \, dx - \int_{B_r} (V^R)^- |u|^2 \, dx \\
&\geq \int_{B_r} |Du|^2 \, dx - C(n) \eta^R(r)(\frac{H(r)}{r} + \int_{B_r} |Du|^2 \, dx),
\end{align*}

the choice of $r_*$ yields

\[(16) \quad 0 = I(r_o) \geq \frac{1}{2} \int_{B_{r_o}} |Du|^2 \, dx.\]

Hence we obtain $|Du(x)| = 0$ a.e. $x \in B_{r_o}$. Since $|\nabla u| \leq |Du|$ a.e., $|u|$ is constant in $B_{r_o}$. $H(r_o) = 0$ implies $|u| \equiv 0$ in $B_{r_o}$. The conclusion can be obtained by an argument similar to the one in [Ku1, Theorem 1.5]. \hfill \Box

Hence we may assume $H(r) > 0$ for every $r \in (0, r_*)$.

Lemma 2.5. There exists an absolute constant $C_0 > 0$ such that

\[(17) \quad \int_{B_r} |Du|^2 \, dx \leq C_0 I(r)\]

for every $r \in \Gamma = \{r \in (0, r_*); N(r) > 1\}$.

Proof. Since $\frac{H(r)}{r} < I(r)$ for $r \in \Gamma$, (15) implies

$$\int_{B_r} |Du|^2 \, dx \leq \frac{1 + C(n) \eta^R(r)}{1 - C(n) \eta^R(r)} I(r) \leq 3I(r)$$

for all $r \in \Gamma$. This concludes the desired estimate. \hfill \Box
Using (11) and Lemmas 2.3, 2.5 we obtain, for a.e. \( r \in \Gamma, \)

\[
I'(r) \geq \frac{n-2}{r} I(r) + 2 \int_{\partial B_r} \left| n \cdot Du \right|^2 dS - C \frac{\theta(r)}{r} I(r),
\]

\[
\theta(r) \eta(r; (2V^R + x \cdot \nabla V^R)^-) + \eta(r; \langle |x|^2 \rangle^{1/2}) + \eta(r; (\langle |x| B \rangle)^{1/2}).
\]

On the other hand, we have the following identity:

\[
H'(r) = \frac{n-1}{r} H(r) + 2 \text{Re} \left( \int_{\partial B_r} u_{\rho} n dS \right), \quad u_{\rho} = \frac{x}{r} \cdot \nabla u.
\]

Hence (14) and (19) imply

\[
H'(r) = \frac{n-1}{r} H(r) + 2 I(r),
\]

\[
\frac{d}{dr} \left( \log \frac{H(r)}{r^{n-1}} \right) = \frac{2 I(r)}{H(r)} = \frac{2 N(r)}{r}.
\]

Therefore, for a.e. \( r \in \Gamma \) we obtain

\[
\frac{N'(r)}{N(r)} = \frac{1}{r} + \frac{I'(r)}{I(r)} - \frac{H'(r)}{H(r)} \geq 2 \left( \frac{\int_{\partial B_r} \left| n \cdot Du \right|^2 dS}{I(r)} - \frac{I(r)}{H(r)} \right) - C \frac{\theta(r)}{r}
\]

\[
\geq -C \frac{\theta(r)}{r}.
\]

In the last inequality we used Schwarz’s inequality. This differential inequality yields the following growth estimate for \( N(r) \).

**Theorem 2.1.** (i) If \( \int_0^{r_o} \frac{\theta(r)}{r} \, dr < +\infty \) for some \( r_o > 0 \), then

\[
N(r) \leq \max(1, N(r_o)) \exp\left(C \int_0^{r_o} \frac{\theta(r)}{r} \, dr \right)
\]

for every \( r \in (0, r_o) \).

(ii) In general, for each \( r_1 \in (0, r_o) \) there exist constants \( C, L(r_1) > 0 \) such that

\[
N(r) \leq \frac{L(r_1)}{r^{C \theta(r_1)}},
\]

for every \( r \in (0, r_1/2) \).

This theorem implies Theorems 1.1 and 1.2 by a standard argument. For the details see [Ku1], [GL2].

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REFERENCES


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