A UNIQUE CONTINUATION THEOREM 
FOR THE SCHRÖDINGER EQUATION 
WITH SINGULAR MAGNETIC FIELD

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Abstract. We show a unique continuation theorem for the Schrödinger equation \((\frac{1}{i}\nabla - A)^2u + Vu = 0\) with singular coefficients \(A\) and \(V\).

1. Main results

In this paper we show a unique continuation theorem for the Schrödinger operator \(H = (\frac{1}{i}\nabla - A)^2 + V\) with singular magnetic field. In fact we shall establish, under some assumptions on \(A\) and \(V\), the following estimate:

\[
\int_{B_{2r}(x_0)} |u|^2 \, dx \leq C \int_{B_r(x_0)} |u|^2 \, dx.
\]

The latter holds for \(r > 0\) and \(x_0 \in \Omega\) with \(B_{2r}(x_0) \subset \Omega\) and for solutions \(u \in H^2_{\text{loc}}(\Omega)\) of

\[
Hu = (\frac{1}{i}\nabla - A)^2u + Vu = 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^n,
\]

where \(n \geq 3, \Omega\) is a domain, \(A(x) = (A_j(x))_{j=1}^n\) is real-valued and \(V(x) = V^R(x) + iV^I(x)\) is complex-valued. This estimate implies a strong unique continuation theorem (Corollary 1.1). The proof uses a Rellich’s type identity and the variational method (originally due to Garofalo and Lin, see e.g., [GL1], [GL2], [Ku1]), which does not need Carleman type estimates. We emphasize that our method requires neither higher integrability nor pointwise estimates for \(A\) compared with previous results [BKRS], [H], [Wo1], [So], [GL1], [GL2], [Ku1].

Throughout this paper we use the notation

\[
\nabla = (\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}), \quad V^+(x) = \max(V(x), 0), \quad V^-(x) = \max(-V(x), 0),
\]

\[
B = (b_{jk})_{j,k=1}^n, \quad b_{jk}(x) = \frac{\partial A_j}{\partial x_k} - \frac{\partial A_k}{\partial x_j},
\]

\[
D_j = \frac{1}{i} \frac{\partial}{\partial x_j} - A_j, \quad D_j^* = -\frac{1}{i} \frac{\partial}{\partial x_j} - A_j,
\]

\[
H^m_{\text{loc}}(\Omega) = \{u \in L^2_{\text{loc}}(\Omega); \quad D^\alpha u \in L^2_{\text{loc}}(\Omega), |\alpha| \leq m\}.
\]
Note that $\overline{D_j u} = D_j^* u$ for a complex-valued function $u$.

To state our main results, first we recall the definition of the Kato class $K_n^{\text{loc}}(\Omega)$.

**Definition 1.1.** We say $f \in L_1^{\text{loc}}(\Omega)$ belongs to the Kato class $K_n^{\text{loc}}(\Omega)$ if
\[
\lim_{r \to 0} \eta(r; f_{\chi_{\Omega_r}}) = 0 \quad \text{for every compact subdomain } \Omega_r \text{ of } \Omega.
\]
Here
\[
\eta(r; f_{\chi_{\Omega_r}}) = \sup_{x \in \mathbb{R}^n} \int_{(|x-y|<r) \cap \Omega_r} \frac{|f(y)|}{|x-y|^n} \, dy.
\]

The assumptions on $A$ and $V$ in this paper are the following.

**Assumption (A):** Let $x_0 \in \Omega$ be fixed.

1. $A \in L_1^4(\Omega)$, $\nabla A \in L_2^{\text{loc}}(\Omega)$, $|A|^2 \in K_1^{\text{loc}}(\Omega)$, $(|x-x_0||B|)^2 \in K_1^{\text{loc}}(\Omega)$;
2. $V \in K_2^{\text{loc}}(\Omega)$, $(|x-x_0||V|)^2 \in K_1^{\text{loc}}(\Omega)$, $(2V^R + (x-x_0) \cdot \nabla V^R) \in K_1^{\text{loc}}(\Omega)$.

Taking an arbitrary compact subdomain $\Omega_0 \subset \Omega$ such that $x_0 \in \Omega_0$, we will use the notation
\[
\theta_o(r) = \eta_o(r; (2V^R + (x-x_0) \cdot \nabla V^R)) + \eta_o(r; ((x-x_0)|B|^2)^{1/2})^{1/2},
\]
where $\eta_o(r; f) = \eta(r; f_{\chi_{\Omega_r}})$.

**Theorem 1.1.** Suppose Assumption (A) and $\int_0^{r_o} \frac{\theta_o(r)}{r} \, dr < +\infty$ for some $r_o > 0$. Let $u \in H_1^{\text{loc}}(\Omega)$ be a solution of (1). Then there exist constants $r_*, C > 0$ such that
\[
(2) \quad \int_{B_{2r}(x_0)} |u|^2 \, dx \leq C \int_{B_{r}(x_0)} |u|^2 \, dx
\]
for every $0 < r < r_*$.

When we do not assume $\int_0^{r_o} \frac{\theta_o(r)}{r} \, dr < +\infty$, we have

**Theorem 1.2.** Suppose Assumption (A) and let $u \in H_1^{\text{loc}}(\Omega)$ be a solution of (1). Then, for each $r_1 \in (0, r_*)$, there exist constants $C > 0$ and $L(r_1) > 0$ such that
\[
(3) \quad \int_{B_{2r}(x_0)} |u|^2 \, dx \leq C \exp\left(\frac{L(r_1)}{r C \theta_o(r_1)}\right) \int_{B_{r}(x_0)} |u|^2 \, dx
\]
for every $r_1/2 > r > 0$.

The constant $r_*$ is determined by the condition that $\eta(r; (V^R)^{-}) \leq \frac{1}{2C(n)}$ hold for all $r \in (0, r_*),$ where $C(n)$ is a constant depending only on $n$ (Lemma 2.3). The constants $C$ and $L(r_1)$ do not depend on $r$, but depend on $u$. It is well known that these yield unique continuation theorems (e.g., [GL1], [GL2], [Ku1]).

**Corollary 1.1.** Suppose Assumption (A) and the condition $\int_0^{r_o} \frac{\theta_o(r)}{r} \, dr < +\infty$, $r_0 > 0$, for every $x_0 \in \Omega$. Then $H$ has SUCP (strong unique continuation property); if $u \in H_1^{\text{loc}}(\Omega)$ is a solution of (1) and satisfies, for some $x_0 \in \Omega$ and for every $m > 0$,
\[
\int_{B_r(x_0)} |u|^2 \, dx = O(r^m) \quad (r \to 0),
\]
then $u \equiv 0$ in $\Omega$. 
**Corollary 1.2.** Suppose Assumption (A) for every \( x_o \in \Omega \). Then \( H \) has a unique continuation property; if \( u \in H^2_{\text{loc}}(\Omega) \) is a solution of (1) and satisfies, for some \( x_o \in \Omega \) and \( A, \alpha > 0 \),
\[
\int_{B_r(x_o)} |u|^2 \, dx = O(\exp(-\frac{A}{r^\alpha})) \quad (r \to 0),
\]
then \( u \equiv 0 \) in \( \Omega \).

In particular, \( H \) has WUCP (weak unique continuation property); if \( u \) vanishes on a subdomain \( \Omega' \) of \( \Omega \), then \( u \equiv 0 \) in \( \Omega \).

Since \( Hu = -\Delta u + 2iA \cdot \nabla u + i(|\text{div}A|)u + |A|^2u + Vu \), putting \( b = 2iA, W = i(\text{div}A) + |A|^2 + V \), we can rewrite the equation (1) in the following way:
\[
(4) \quad Hu = -\Delta u + b \cdot \nabla u + Wu.
\]
Although we can apply results of [BKRS], [Wo1], [Wo2], [So], [H] [GL2], [Ku1] etc. under suitable conditions on \( b \) and \( W \), our theorems cannot be covered by these previous results. First, previous results require a stronger condition on \( \text{div}A \) even for WUCP. For instance, [Wo2] require \( W \in L^2_{\text{loc}} \) for WUCP and hence \( \text{div}A \in L^2_{\text{loc}} \) etc. On the other hand, our method only requires \( \text{div}A \in L^2_{\text{loc}} \) for \( \text{div}A \) and, instead of that, \( (|x - x_o||B|)^2 \in K_{n,\delta}^{\text{loc}}(\Omega) \) (or \( (|x - x_o||B|)^2 \in F_{t,\delta}^{\text{loc}}(\Omega) \)) for the magnetic field \( B \) (see Example 1.1). Secondly, to obtain SUCP [BKRS], [H] require \( |A| \in L^q_{\text{loc}}, q > \frac{3n-2}{2} \) (for related results see also [Wo1]) and [So], [GL2], [Ku1] require a pointwise estimate
\[
|A(x)| \leq \frac{f(|x - x_o|)}{|x - x_o|}, \quad \int_0^{r_o} \frac{f(t)}{t} \, dt < +\infty
\]
for \( A \), but our method requires neither higher integrability nor pointwise estimates.

**Remark 1.1.** These theorems also hold even if we replace the class \( K_{n,\delta}^{\text{loc}}(\Omega) \) in Assumption (A) by the more general one \( Q_t^{\text{loc}}(\Omega) = K_{n,\delta}^{\text{loc}}(\Omega) + F_t^{\text{loc}}(\Omega), 1 < t \leq n/2 \), where \( F_t^{\text{loc}}(\Omega) \) is the Fefferman-Phong class. However, in this case we must assume an additional condition
\[
\lim_{r \to 0} \sup_{x \in \Omega} \|(V^R)^-\|_{Q_t(B_r(x) \cap \Omega)} \leq \epsilon_o
\]
for sufficiently small \( \epsilon_o > 0 \) and take \( H^2_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega) \) as a solution class. See [Ku1].

**Remark 1.2.** A sufficient condition to assure \( \int_0^{r_o} \frac{\theta_{\delta}(r)}{r} \, dr < +\infty \) is that
\[
(2V^R + (x - x_o) \cdot \nabla V^R)^-((x - x_o)V^I)^2, ((x - x_o)|B|)^2 \in K_{n,\delta}^{\text{loc}}(\Omega) \subset K_{n,\delta}^{\text{loc}}(\Omega)
\]
hold for some \( \delta > 0 \), where we say \( f \in K_{n,\delta}^{\text{loc}}(\Omega) \) if \( f \) satisfies, for every compact subdomain \( \Omega_o \) of \( \Omega \),
\[
\lim_{r \to 0} \sup_{x \in \Omega_o} \int_{|x - y| < r} \frac{|f(y)|}{|x - y|^{n-2+2\delta}} \, dy = 0.
\]

**Remark 1.3.** By using the approximation argument in [Ku1] we can show unique continuation theorems similar to the theorems above even for \( H^1_{\text{loc}} \) solutions.

Finally, we must remark that Kalf [Ka] proved WUCP under the assumptions \( A \in H^1_{\text{loc}}(\Omega), (V^R)^2 \in K_{n,\delta}^{\text{loc}}(\Omega) \) \((V^I \equiv 0)\), but his method cannot be applied to SUCP. For WUCP our results complement his result; compare that one needs
\[(V^R)^2 \in K_n^\text{loc}(\Omega)\] in his case and \((2V^R + (x - x_o) \cdot \nabla V^R)^- \in K_n^\text{loc}(\Omega)\) in our case (see Examples 1.2, 1.3).

**Example 1.1.** Let \(A_i = C \frac{x_i}{|x|}\). Then \(B \equiv 0\) and \(\nabla A \in L^2_{\text{loc}}, A \in L^4_{\text{loc}}\) when \(n \geq 5\). Note that \(|A|^2 = C^2/|x|^2 \notin K_n^\text{loc}\), but \(|A|^2 = C^2/|x|^2 \in F_n^\text{loc}\) for \(1 < t < n/2\). Hence we can apply our method to show SUCP for this \(A\) (see Remark 1.1).

**Example 1.2.** Let \(x_o = O, B_1 = B_1(O)\) and
\[V(x) = \frac{1}{|x|^l \log |x|^m}, \quad m, l > 0.\]
Then we have
\[V \in K_n^\text{loc}(B_1) \leftrightarrow (m > 1, l = 2) \text{ or } (m \in \mathbb{R}, l < 2),\]
\[V^2 \in K_n^\text{loc}(B_1) \leftrightarrow (m > 1/2, l = 1) \text{ or } (m \in \mathbb{R}, l < 2),\]
\[2V + x \cdot \nabla V \in K_n^\text{loc}(B_1) \leftrightarrow (m > 0, l = 2) \text{ or } (m \in \mathbb{R}, l < 2).\]
So in this example the condition \(V^2 \in K_n^\text{loc}(\Omega)\) is stronger than \((2V + x \cdot \nabla V)^- \in K_n^\text{loc}(\Omega)\).

**Example 1.3.** Let \(N \in \mathbb{N}\) and \(R = (R_1, \cdots, R_N) \in \mathbb{R}^{3N}\) be fixed and
\[V(x) = \sum_{j=1}^N \frac{1}{|x_j - R_j|}, \quad x = (x_1, \cdots, x_N) \in \mathbb{R}^{3N}, \quad x_j \in \mathbb{R}.\]
Then \(V^2 \notin K_n^\text{loc}\), but \(2V + (x - x_o) \cdot \nabla V \in K_n^\text{loc}\) for every \(x_o \in \mathbb{R}^{3N}\).

## 2. Proof of the Theorems

We may assume \(x_o\) is the origin \(O\) and write \(B_r = B_r(O)\). For the sake of simplicity, we also use \(\Omega, \eta(r; f)\) and \(\theta(r)\) instead of \(\Omega_o, \eta(r; f\chi_{\Omega_o})\) and \(\theta_o(r)\), respectively.

Let \(u \in H^2_{\text{loc}}(\Omega)\) be a solution of (1) and put
\[
I(r) = \int_{B_r} |Du|^2 + V^R|u|^2 \, dx = \int_{B_r} |(\nabla - iA)u|^2 + V^R|u|^2 \, dx,
\]
\[
H(r) = \int_{\partial B_r} |u|^2 \, dS, \quad N(r) = \frac{r I(r)}{H(r)}.
\]
Note that \(H = \left(\frac{1}{t} \nabla - A\right)^2 + V = \sum_{j=1}^n D_j D_j + V\). Our argument is based on the following identity.

**Lemma 2.1.** Suppose Assumption (A) (for \(A\)). Then \(u \in H^2_{\text{loc}}(\Omega) \cap L^\infty(\Omega)\) satisfies
\[
\text{Im}(\int_{B_r} (x \cdot \overline{Du}) \sum_{j=1}^n D_j D_j u \, dx) = \frac{r}{2} \int_{\partial B_r} |Du|^2 \, dS - \left(\frac{n - 2}{2}\right) \int_{B_r} |Du|^2 \, dx
\]
\[
- r \int_{\partial B_r} |n \cdot Du|^2 \, dS + \text{Re}\left(\sum_{j,k=1}^n \int_{B_r} b_{jk}(x) x_j D_k u \overline{u} \, dx\right)
\]
for every \(r > 0\) with \(B_r \subset \Omega\).
Proof. This is a kind of Rellich’s identity. The detailed computation can be seen in [EK]. So we omit it.

We remark that under the assumption $|\mathbf{A}|^2, V \in K^{\text{loc}}_n(\Omega)$ a solution $u \in H^{1\text{loc}}(\Omega)$ is locally bounded (see [Ku2]). So we can apply Lemma 2.1 for solutions $u \in H^{2\text{loc}}(\Omega)$ of (1). This identity implies

**Lemma 2.2.** Suppose Assumption (A). Then for a.e. $r \in (0, R_o)$

$$I'(r) = \frac{n-2}{r} I(r) + 2 \int_{\partial B_r} |n \cdot Du|^2 dS$$

$$- \frac{n-2}{r} \int_{B_r} V^R |u|^2 dx + \int_{\partial B_r} V^R |u|^2 dS$$

$$- \frac{2}{r} \text{Im} \left( \int_{B_r} (x \cdot \overline{Du}) V u dx \right)$$

$$- \frac{2}{r} \text{Re} \left( \sum_{j,k=1}^n \int_{B_r} b_{jk}(x)x_j(D_k u)\overline{u} dx \right)$$

(9)

holds, where $R_o = \max\{r; B_r \subset \Omega\}$.

Proof. Note that

$$I'(r) = \int_{\partial B_r} |Du|^2 + V^R |u|^2 dS \quad \text{a.e. } r.$$

Lemma 2.1 and this identity imply (9).

Since $x \cdot \overline{Du} = x \cdot D^* \overline{u} = i(x \cdot \nabla \overline{u}) - (x \cdot \mathbf{A})\overline{u}$, we have

$$\text{Im} \left( \int_{B_r} (x \cdot \overline{Du}) V u dx \right) = \text{Re} \left( \int_{B_r} (x \cdot \nabla \overline{u}) V^R u dx \right) + \text{Im} \left( \int_{B_r} (x \cdot \overline{Du}) V^I u dx \right).$$

Put

$$J(V^R; u; r) = \int_{\partial B_r} V^R |u|^2 dS - \frac{n-2}{r} \int_{B_r} V^R |u|^2 dx$$

$$- \frac{1}{r} \int_{B_r} (x \cdot \nabla(|u|^2))V^R dx.$$

(10)

By integration by parts we have

$$J(V^R; u; r) \geq - \frac{1}{r} \int_{B_r} (2V^R + x \cdot \nabla V^R) - |u|^2 dx.$$

Then from this observation and Lemma 2.2 we obtain the following estimate:

$$I'(r) \geq \frac{n-2}{r} I(r) + 2 \int_{\partial B_r} |n \cdot Du|^2 dS$$

$$- \frac{1}{r} \int_{B_r} (2V^R + x \cdot \nabla V^R) - |u|^2 dx$$

$$- \frac{2}{r} \int_{B_r} (|x||V^I| + |x||B|)|Du||u| dx$$

(11)

for a.e. $r \in (0, R_o)$. To control the last two terms we need the following inequality.
Lemma 2.3. Let \( W \in K_n^\text{loc}(\Omega) \) and \( u \in H^1_{\text{loc}}(\Omega) \). Then there exists a constant \( C(n) \) such that

\[
\int_{B_r} |W||u|^2 \, dx \leq C(n)\eta(r; W\chi_{\Omega_0})\left(\int_{B_r} |Du|^2 \, dx + \frac{1}{r} \int_{\partial B_r} |u|^2 \, dS\right)
\]

for \( B_r \subset \Omega_0, \Omega_0^c \subset \Omega \).

Proof. Since \( |u| \in H^1_{\text{loc}}(\Omega) \), we know that

\[
\int_{B_r} |W||u|^2 \, dx \leq C(n)\eta(r; W\chi_{\Omega_0})\left(\int_{B_r} |\nabla|u|^2 \, dx + \frac{1}{r} \int_{\partial B_r} |u|^2 \, dS\right)
\]

holds (FGL). Noting \( |\nabla|u|^2 \leq |Du| \) a.e., we obtain the desired inequality. \( \square \)

Lemma 2.4. There exists a constant \( r_* > 0 \) such that \( H(r) > 0 \) for every \( r \in (0, r_*) \) unless \( u \equiv 0 \) in \( B_{r_*} \).

Proof. Note that the constant \( r_* > 0 \) is determined by the condition \( \eta^R(r) \leq \frac{1}{2C(n)} \)

for \( r \in (0, r_*) \), where \( \eta^R(r) = \eta(r; (V^R)^-) \) and \( C(n) \) is the constant in Lemma 2.3. Since \( V^R \in K_n^\text{loc}(\Omega) \), the constant \( r_* > 0 \) satisfying this condition exists.

Suppose \( H(r_o) = 0 \) for some \( r_o \in (0, r_*). \) Noting a simple computation yields

\[
I(r) = Re(\int_{\partial B_r} u_n \bar{u} \, dS) = Re(\int_{\partial B_r} (n \cdot Du) \bar{u} \, dS),
\]

we have \( I(r_o) = 0 \). Since Lemma 2.3 implies

\[
I(r) \geq \int_{B_r} |Du|^2 \, dx - \int_{B_r} (V^R)^- |u|^2 \, dx
\]

\[
\geq \int_{B_r} |Du|^2 \, dx - C(n)\eta^R(r)(\frac{H(r)}{r}) + \int_{B_r} |Du|^2 \, dx,
\]

the choice of \( r_* \) yields

\[
0 = I(r_o) \geq \frac{1}{2} \int_{B_{r_o}} |Du|^2 \, dx.
\]

Hence we obtain \( |Du(x)| = 0 \) a.e. \( x \in B_{r_o}. \) Since \( |\nabla|u|^2 \leq |Du| \) a.e., \( |u| \) is constant in \( B_{r_o}. \) \( H(r_o) = 0 \) implies \( |u| \equiv 0 \) in \( B_{r_o}. \) The conclusion can be obtained by an argument similar to the one in [Ku1, Theorem 1.5]. \( \square \)

Hence we may assume \( H(r) > 0 \) for every \( r \in (0, r_*). \)

Lemma 2.5. There exists an absolute constant \( C_0 > 0 \) such that

\[
\int_{B_r} |Du|^2 \, dx \leq C_0 I(r)
\]

for every \( r \in \Gamma = \{ r \in (0, r_*); N(r) > 1 \}. \)

Proof. Since \( \frac{H(r)}{r} < I(r) \) for \( r \in \Gamma \), (15) implies

\[
\int_{B_r} |Du|^2 \, dx \leq \frac{1 + C(n)\eta^R(r)}{1 - C(n)\eta^R(r)} I(r) \leq 3I(r)
\]

for all \( r \in \Gamma. \) This concludes the desired estimate. \( \square \)
Using (11) and Lemmas 2.3, 2.5 we obtain, for a.e. \( r \in \Gamma \),

\[
I'(r) \geq \frac{n-2}{r} I(r) + 2 \int_{\partial B_r} |n \cdot Du|^2 dS - C \frac{\theta(r)}{r} I(r),
\]

\[
\theta(r) \eta(r; (2V^R + x \cdot \nabla V^R)^-) + \eta(r; (|x|V^I)^2)^{1/2} + \eta(r; (|x||B|)^2)^{1/2}.
\]

On the other hand, we have the following identity:

\[
H'(r) = \frac{n-1}{r} H(r) + 2 \text{Re} \left( \int_{\partial B_r} u_\rho \overline{\eta} dS \right), \quad u_\rho = \frac{x}{r} \cdot \nabla u.
\]

Hence (14) and (19) imply

\[
H'(r) = \frac{n-1}{r} H(r) + 2 I(r),
\]

\[
\frac{d}{dr} \left( \log \frac{H(r)}{r^{n-1}} \right) = \frac{2 I(r)}{H(r)} = \frac{2 N(r)}{r}.
\]

Therefore, for a.e. \( r \in \Gamma \) we obtain

\[
\frac{N'(r)}{N(r)} = \frac{1}{r} + \frac{I'(r)}{I(r)} - \frac{H'(r)}{H(r)} \geq 2 \left( \frac{\int_{\partial B_r} |n \cdot Du|^2 dS}{I(r)} - \frac{I(r)}{H(r)} \right) - C \frac{\theta(r)}{r},
\]

\[
\geq - C \frac{\theta(r)}{r}.
\]

In the last inequality we used Schwarz’s inequality. This differential inequality yields the following growth estimate for \( N(r) \).

**Theorem 2.1.** (i) If \( \int_0^{r_o} \frac{\theta(r)}{r} \, dr < +\infty \) for some \( r_o > 0 \), then

\[
N(r) \leq \max(1, N(r_*)) \exp\left( C \int_0^{r_*} \frac{\theta(r)}{r} \, dr \right)
\]

for every \( r \in (0, r_*) \).

(ii) In general, for each \( r_1 \in (0, r_*) \) there exist constants \( C, L(r_1) > 0 \) such that

\[
N(r) \leq \frac{L(r_1)}{r^{C \theta(r_1)}}
\]

for every \( r \in (0, r_1/2) \).

This theorem implies Theorems 1.1 and 1.2 by a standard argument. For the details see [Ku1], [GL2].

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