A UNIQUE CONTINUATION THEOREM FOR THE SCHRÖDINGER EQUATION WITH SINGULAR MAGNETIC FIELD

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Abstract. We show a unique continuation theorem for the Schrödinger equation \((\frac{1}{i} \nabla - A)^2 u + Vu = 0\) with singular coefficients \(A\) and \(V\).

1. Main results

In this paper we show a unique continuation theorem for the Schrödinger operator \(H = (\frac{1}{i} \nabla - A)^2 + V\) with singular magnetic field. In fact we shall establish, under some assumptions on \(A\) and \(V\), the following estimate:

\[
\int_{B_{2r}(x_o)} |u|^2 \, dx \leq C \int_{B_r(x_o)} |u|^2 \, dx.
\]

The latter holds for \(r > 0\) and \(x_o \in \Omega\) with \(B_{2r}(x_o) \subset \Omega\) and for solutions \(u \in H^2_{\text{loc}}(\Omega)\) of

\[
Hu = (\frac{1}{i} \nabla - A)^2 u + Vu = 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^n,
\]

where \(n \geq 3, \Omega\) is a domain, \(A(x) = (A_j(x))_{j=1}^n\) is real-valued and \(V(x) = V^R(x) + iV^I(x)\) is complex-valued. This estimate implies a strong unique continuation theorem (Corollary 1.1). The proof uses a Rellich’s type identity and the variational method (originally due to Garofalo and Lin, see e.g., [GL1], [GL2], [Ku1]), which does not need Carleman type estimates. We emphasize that our method requires neither higher integrability nor pointwise estimates for \(A\) compared with previous results [BKRS], [H], [Wo1], [So], [GL1], [GL2], [Ku1].

Throughout this paper we use the notation

\[
\nabla = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right), \quad V^+(x) = \max(V(x), 0), \quad V^-(x) = \max(-V(x), 0),
\]

\[
\mathbf{B} = (b_{jk})_{j,k=1}^n, \quad b_{jk}(x) = \frac{\partial A_j}{\partial x_k} - \frac{\partial A_k}{\partial x_j},
\]

\[
D_j = \frac{1}{i} \frac{\partial}{\partial x_j} - A_j, \quad D_j^* = -\frac{1}{i} \frac{\partial}{\partial x_j} - A_j,
\]

\[
H^m_{\text{loc}}(\Omega) = \{ u \in L^2_{\text{loc}}(\Omega); \quad D^\alpha u \in L^2_{\text{loc}}(\Omega), |\alpha| \leq m \}.
\]
Note that $D_j u = D_j^x u$ for a complex-valued function $u$.

To state our main results, first we recall the definition of the Kato class $K^n_{\text{loc}}(\Omega)$.

**Definition 1.1.** We say $f \in L^1_{\text{loc}}(\Omega)$ belongs to the Kato class $K^n_{\text{loc}}(\Omega)$ if
\[
\lim_{r \to 0} \eta(r; f|\Omega_o) = 0
\]
for every compact subdomain $\Omega_o$ of $\Omega$. Here
\[
\eta(r; f|\Omega_o) = \sup_{x \in \mathbb{R}^n} \int_{(|x-y|<r) \cap \Omega_o} \frac{|f(y)|}{|x-y|^{n-2}} \, dy.
\]

The assumptions on $A$ and $V$ in this paper are the following.

**Assumption (A):** Let $x_o \in \Omega$ be fixed.

1. $A \in L^4_{\text{loc}}(\Omega)$, $\nabla A \in L^2_{\text{loc}}(\Omega)$, $|A|^2 \in K^n_{\text{loc}}(\Omega)$, $(|x-x_o||B|)^2 \in K^n_{\text{loc}}(\Omega)$;
2. $V \in K^n_{\text{loc}}(\Omega)$, $|V^1|^2 \in K^n_{\text{loc}}(\Omega)$, $(2V^R + (x-x_o)\nabla V^R)^- \in K^n_{\text{loc}}(\Omega)$.

Taking an arbitrary compact subdomain $\Omega_o \subset \Omega$ such that $x_o \in \Omega_o$, we will use the notation
\[
\theta_o(r) = \eta_o(r; (2V^R + (x-x_o)\cdot \nabla V^R)^-) + \eta_o(r; ((x-x_o)V^1)^2)^{1/2} + \eta_o(r; ((x-x_o)|B|)^2)^{1/2},
\]
where $\eta_o(r; f) = \eta(r; f|\Omega_o)$.

**Theorem 1.1.** Suppose Assumption (A) and $\int_0^{r_o} \frac{\theta_o(r)}{r} \, dr < +\infty$ for some $r_o > 0$. Let $u \in H^2_{\text{loc}}(\Omega)$ be a solution of $(1)$. Then there exist constants $r_*, C > 0$ such that
\[
\int_{B_{2r}(x_o)} |u|^2 \, dx \leq C \int_{B_r(x_o)} |u|^2 \, dx
\]
for every $0 < r < r_*$.

When we do not assume $\int_0^{r_o} \frac{\theta_o(r)}{r} \, dr < +\infty$, we have

**Theorem 1.2.** Suppose Assumption (A) and let $u \in H^2_{\text{loc}}(\Omega)$ be a solution of $(1)$.
Then, for each $r_1 \in (0, r_*), \; \text{there exist constants } C > 0 \text{ and } L(r_1) > 0 \text{ such that}
\[
\int_{B_{2r_1}(x_o)} |u|^2 \, dx \leq C \exp\left(\frac{L(r_1)}{r C \theta_o(r_1)}\right) \int_{B_{r_1}(x_o)} |u|^2 \, dx
\]
for every $r_1/2 > r > 0$.

The constant $r_*$ is determined by the condition that $\eta(r; (V^R)^-) \leq \frac{1}{2C(n)}$ hold for all $r \in (0, r_*)$, where $C(n)$ is a constant depending only on $n$ (Lemma 2.3). The constants $C$ and $L(r_1)$ do not depend on $r$, but depend on $u$. It is well known that these yield unique continuation theorems (e.g., [GL1], [GL2], [Ku1]).

**Corollary 1.1.** Suppose Assumption (A) and the condition $\int_0^{r_o} \frac{\theta_o(r)}{r} \, dr < +\infty$, $r_o > 0$, for every $x_o \in \Omega$. Then $H$ has SUCP (strong unique continuation property); if $u \in H^2_{\text{loc}}(\Omega)$ is a solution of $(1)$ and satisfies, for some $x_o \in \Omega$ and for every $m > 0$,
\[
\int_{B_r(x_o)} |u|^2 \, dx = O(r^m) \quad (r \to 0),
\]
then $u \equiv 0$ in $\Omega$. 

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Corollary 1.2. Suppose Assumption (A) for every \( x_0 \in \Omega \). Then \( H \) has a unique continuation property: if \( u \in H^2_{\text{loc}}(\Omega) \) is a solution of (1) and satisfies, for some \( x_0 \in \Omega \) and \( A, \alpha > 0 \),

\[
\int_{B_r(x_0)} |u|^2 \, dx = O(\exp(-\frac{A}{r^\alpha})) \quad (r \to 0),
\]

then \( u \equiv 0 \) in \( \Omega \).

In particular, \( H \) has WUCP (weak unique continuation property); if \( u \) vanishes on a subdomain \( \Omega' \) of \( \Omega \), then \( u \equiv 0 \) in \( \Omega \).

Since \( Hu = -\Delta u + 2iA \cdot \nabla u + i(\text{div} A)u + |A|^2 u + Vu \), putting \( b = 2iA, W = i(\text{div} A) + |A|^2 + V \), we can rewrite the equation (1) in the following way:

\[
(4) \quad Hu = -\Delta u + b \cdot \nabla u + Wu.
\]

Although we can apply results of [BKRS], [Wo1], [Wo2], [So], [H], [GL2], [Ku1] etc. under suitable conditions on \( b \) and \( W \), our theorems cannot be covered by these previous results. First, previous results require a stronger condition on \( \text{div} A \) even for WUCP. For instance, [Wo2] require \( W \in L^{n/2}_{\text{loc}} \) for WUCP and hence \( \text{div} A \in L^{n/2}_{\text{loc}} \) etc. On the other hand, our method only requires \( \text{div} A \in L^2_{\text{loc}} \) for \( \text{div} A \) and, instead of that, \((|x - x_0||B|)^2 \in K^\text{loc}_n(\Omega) \) (or \((|x - x_0||B|)^2 \in F^\text{loc}_t(\Omega) \)) for the magnetic field \( B \) (see Example 1.1). Secondly, to obtain SUCP [BKRS], [H] require \( |A| \in L^q_{\text{loc}}, q > \frac{3n-2}{2} \) (for related results see also [Wo1]) and [So], [GL2], [Ku1] require a pointwise estimate

\[
|A(x)| \leq \frac{f(|x - x_0|)}{|x - x_0|}, \quad \int_0^{r_o} \frac{f(t)}{t} \, dt < +\infty
\]

for \( A \), but our method requires neither higher integrability nor pointwise estimates.

Remark 1.1. These theorems also hold even if we replace the class \( K^\text{loc}_n(\Omega) \) in Assumption (A) by the more general one \( Q^\text{loc}_t(\Omega) = K^\text{loc}_n(\Omega) + F^\text{loc}_t(\Omega) \), \( 1 < t < n/2 \), where \( F^\text{loc}_t(\Omega) \) is the Fefferman-Phong class. However, in this case we must assume an additional condition

\[
\limsup_{r \to 0} \| (V^R)^- \|_{Q_t(B_r(x) \cap \Omega)} \leq \epsilon_o
\]

for sufficiently small \( \epsilon_o > 0 \) and take \( H^2_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega) \) as a solution class. See [Ku1].

Remark 1.2. A sufficient condition to assure \( \int_0^{r_o} \frac{\theta_o(r)}{r} \, dr < +\infty \) is that

\[
(2V^R + (x - x_0) \cdot \nabla V^R)^-, ((x - x_0)V^I)^2, (|x - x_0||B|)^2 \in K^\text{loc}_{n,\delta}(\Omega)(\subset K^\text{loc}_n(\Omega))
\]

hold for some \( \delta > 0 \), where we say \( f \in K^\text{loc}_{n,\delta}(\Omega) \) if \( f \) satisfies, for every compact subdomain \( \Omega_o \) of \( \Omega \),

\[
\limsup_{r \to 0} \int_{(x - y)<r \cap \Omega_o} \frac{|f(y)|}{|x - y|^{n-2+2\delta}} \, dy = 0.
\]

Remark 1.3. By using the approximation argument in [Ku1] we can show unique continuation theorems similar to the theorems above even for \( H^1_{\text{loc}} \) solutions.

Finally, we must remark that Kalf [Ka] proved WUCP under the assumptions \( A \in H^1_{loc}(\Omega), (V^R)^2 \in K^\text{loc}_n(\Omega) \) \((V^I \equiv 0)\), but his method cannot be applied to SUCP. For WUCP our results complement his result; compare that one needs...
Then we have

$$V \in K_n^\text{loc}(B_1) \iff (m > 1, l = 2) \text{ or } (m \in \mathbb{R}^1, l < 2),$$

(5)

$$V^2 \in K_n^\text{loc}(B_1) \iff (m > 1/2, l = 1) \text{ or } (m \in \mathbb{R}^1, l < 1),$$

$$2V + x \cdot \nabla V \in K_n^\text{loc}(B_1) \iff (m > 0, l = 2) \text{ or } (m \in \mathbb{R}^1, l < 2).$$

So in this example the condition $V^2 \in K_n^\text{loc}(\Omega)$ is stronger than $(2V + x \cdot \nabla V)^- \in K_n^\text{loc}(\Omega)$.

**Example 1.3.** Let $N \in \mathbb{N}$ and $R = (R_1, \ldots, R_N) \in \mathbb{R}^{3N}$ be fixed and

$$V(x) = \frac{1}{\sum_{j=1}^{N} \frac{1}{|x_j - R_j|}}, \quad x = (x_1, \ldots, x_N) \in \mathbb{R}^{3N}, \quad x_j \in \mathbb{R}^3.$$

Then $V^2 \notin K_n^\text{loc}$, but $2V + (x - x_0) \cdot \nabla V \in K_n^\text{loc}$ for every $x_0 \in \mathbb{R}^{3N}$.

2. **Proof of the Theorems**

We may assume $x_o$ is the origin $O$ and write $B_r = B_r(O)$. For the sake of simplicity, we also use $\Omega$, $\eta(r; f)$ and $\theta(r)$ instead of $\Omega_o$, $\eta(r; f_{\mid \Omega_o})$ and $\theta_o(r)$, respectively.

Let $u \in H^2_{\text{loc}}(\Omega)$ be a solution of (1) and put

$$I(r) = \int_{B_r} |Du|^2 + V R |u|^2 \, dx = \int_{B_r} |(\nabla - iA)u|^2 + V R |u|^2 \, dx,$$

(6)

$$H(r) = \int_{\partial B_r} |u|^2 \, dS, \quad N(r) = \frac{rI(r)}{H(r)}.$$

Note that $H = (\frac{1}{r} \nabla - A)^2 + V = \sum_{j=1}^{n} D_j D_j + V$. Our argument is based on the following identity.

**Lemma 2.1.** Suppose Assumption (A) (for A). Then $u \in H^2_{\text{loc}}(\Omega) \cap L^\infty(\Omega)$ satisfies

$$\text{Im}(\int_{B_r} (x \cdot Du) \sum_{j=1}^{n} D_j D_j u \, dx)$$

$$= \frac{r}{2} \int_{\partial B_r} |Du|^2 \, dS - \frac{n - 2}{2} \int_{B_r} |Du|^2 \, dx$$

$$- r \int_{\partial B_r} |n \cdot Du|^2 \, dS + \text{Re}(\sum_{j,k=1}^{n} \int_{B_r} b_{jk}(x) x_j D_k u \, dx)$$

(8)

for every $r > 0$ with $B_r \subset \Omega$. 


Proof. This is a kind of Rellich’s identity. The detailed computation can be seen in [EK]. So we omit it.

We remark that under the assumption $|A|^2, V \in K^\text{loc}_n(\Omega)$ a solution $u \in H^1_\text{loc}(\Omega)$ is locally bounded (see [Ku2]). So we can apply Lemma 2.1 for solutions $u \in H^2_\text{loc}(\Omega)$ of (1). This identity implies

**Lemma 2.2.** Suppose Assumption (A). Then for a.e. $r \in (0, R_o)$

\[
I'(r) = \frac{n-2}{r} I(r) + 2 \int_{\partial B_r} |n \cdot Du|^2 dS - \frac{n-2}{r} \int_{B_r} V^R |u|^2 dx + \int_{\partial B_r} V^R |u|^2 dS - \frac{2}{r} I_m \left( \int_{B_r} (x \cdot \nabla u) V^R u dx \right)
\]

(9)

holds, where $R_o = \max \{r; B_r \subset \Omega\}$.

Proof. Note that

\[
I'(r) = \int_{\partial B_r} |Du|^2 + V^R |u|^2 dS \quad \text{a.e. } r.
\]

Lemma 2.1 and this identity imply (9).

Since $x \cdot Du = x \cdot D^* u = i(x \cdot \nabla u) - (x \cdot A)u$, we have

\[
I_m \left( \int_{B_r} (x \cdot Du) V^R u dx \right) = \text{Re} \left( \int_{B_r} (x \cdot \nabla u) V^R u dx \right) + I_m \left( \int_{B_r} (x \cdot Du) V^R u dx \right).
\]

Put

\[
J(V^R; u; r) = \int_{\partial B_r} V^R |u|^2 dS - \frac{n-2}{r} \int_{B_r} V^R |u|^2 dx - \frac{1}{r} \int_{B_r} (x \cdot \nabla (|u|^2)) V^R dx.
\]

(10)

By integration by parts we have

\[
J(V^R; u; r) \geq - \frac{1}{r} \int_{B_r} (2V^R + x \cdot \nabla V^R)^{-1} |u|^2 dx.
\]

Then from this observation and Lemma 2.2 we obtain the following estimate:

\[
I'(r) \geq \frac{n-2}{r} I(r) + 2 \int_{\partial B_r} |n \cdot Du|^2 dS - \frac{1}{r} \int_{B_r} (2V^R + x \cdot \nabla V^R)^{-1} |u|^2 dx - \frac{2}{r} \int_{B_r} (|x||V^R| + |x||B|)|Du||u| dx
\]

(11)

for a.e. $r \in (0, R_o)$. To control the last two terms we need the following inequality.
Lemma 2.3. Let $W \in K_{loc}^{1}(\Omega)$ and $u \in H_{loc}^{1}(\Omega)$. Then there exists a constant $C(n)$ such that

$$\int_{B_r} |W||u|^2 \, dx \leq C(n)\eta(r; W\chi_{\Omega})(\int_{B_r} |Du|^2 \, dx + \frac{1}{r} \int_{\partial B_r} |u|^2 \, dS)$$

for $B_r \subset \Omega_0, \Omega_0 \subset \Omega$.

Proof. Since $|u| \in H_{loc}^{1}(\Omega)$, we know that

$$\int_{B_r} |W||u|^2 \, dx \leq C(n)\eta(r; W\chi_{\Omega})(\int_{B_r} |\nabla|u|^2 \, dx + \frac{1}{r} \int_{\partial B_r} |u|^2 \, dS)$$

holds ([FGL]). Noting $|\nabla|u|^2 \leq |Du|$ a.e., we obtain the desired inequality. \hfill \Box

Lemma 2.4. There exists a constant $r_* > 0$ such that $H(r) > 0$ for every $r \in (0, r_*)$ unless $u \equiv 0$ in $B_{r_*}$.

Proof. Note that the constant $r_* > 0$ is determined by the condition $\eta^R_r(r) \leq \frac{1}{2C(n)}$ for $r \in (0, r_*)$, where $\eta^R_r(r) = \eta(r; (V^R)^-)$ and $C(n)$ is the constant in Lemma 2.3. Since $V^R \in K_{loc}^{1}(\Omega)$, the constant $r_* > 0$ satisfying this condition exists.

Suppose $H(r_o) = 0$ for some $r_o \in (0, r_*)$. Noting a simple computation yields

$$I(r) = \Re(\int_{\partial B_r} u \rho \overline{u} \, dS) = \Re(\int_{\partial B_r} (n \cdot Du) \overline{u} \, dS),$$

we have $I(r_o) = 0$. Since Lemma 2.3 implies

$$I(r) \geq \int_{B_r} |Du|^2 \, dx - \int_{B_r} (V^R)^- |u|^2 \, dx$$

$$\geq \int_{B_r} |Du|^2 \, dx - C(n)\eta^R_r(r)(\frac{H(r)}{r})^2 + \int_{B_r} |Du|^2 \, dx,$$

the choice of $r_*$ yields

$$0 = I(r_o) \geq \frac{1}{2} \int_{B_{r_o}} |Du|^2 \, dx.$$

Hence we obtain $|Du(x)| = 0$ a.e. $x \in B_{r_o}$. Since $|\nabla|u|| \leq |Du|$ a.e., $|u|$ is constant in $B_{r_o}$. $H(r_o) = 0$ implies $|u| \equiv 0$ in $B_{r_o}$. The conclusion can be obtained by an argument similar to the one in [Ku1, Theorem 1.5]. \hfill \Box

Hence we may assume $H(r) > 0$ for every $r \in (0, r_*)$.

Lemma 2.5. There exists an absolute constant $C_0 > 0$ such that

$$\int_{B_r} |Du|^2 \, dx \leq C_0 I(r)$$

for every $r \in \Gamma = \{r \in (0, r_*); N(r) > 1\}$.

Proof. Since $\frac{H(r)}{r} < I(r)$ for $r \in \Gamma$, (15) implies

$$\int_{B_r} |Du|^2 \, dx \leq \frac{1 + C(n)\eta^R_r(r)}{1 - C(n)\eta^R_r(r)} I(r) \leq 3I(r)$$

for all $r \in \Gamma$. This concludes the desired estimate. \hfill \Box
Using (11) and Lemmas 2.3, 2.5 we obtain, for a.e. $r \in \Gamma$,

\begin{equation}
I'(r) \geq \frac{n-2}{r} I(r) + 2 \int_{\partial B_r} |n \cdot Du|^2 dS - C \frac{\theta(r)}{r} I(r),
\end{equation}

\begin{equation}
\theta(r) \eta(r; (2V^R + x \cdot \nabla V^R)^-) + \eta(r; |x| V^F)^{1/2} + \eta(r; |x||B|)^{1/2}.
\end{equation}

On the other hand, we have the following identity:

\begin{equation}
H'(r) = \frac{n-1}{r} H(r) + 2 \text{Re}\left(\int_{\partial B_r} u_\rho \bar{n} dS\right), \quad u_\rho = \frac{x}{r} \cdot \nabla u.
\end{equation}

Hence (14) and (19) imply

\begin{equation}
H'(r) = \frac{n-1}{r} H(r) + 2 I(r),
\end{equation}

\begin{equation}
\frac{d}{dr} \left(\frac{H(r)}{r^{n-1}}\right) = -2 \frac{I(r)}{H(r)} = 2 \frac{N(r)}{r}.
\end{equation}

Therefore, for a.e. $r \in \Gamma$ we obtain

\begin{equation}
\frac{N'(r)}{N(r)} = \frac{1}{r} + \frac{I'(r)}{I(r)} - \frac{H'(r)}{H(r)} \geq 2 \left(\int_{\partial B_r} |n \cdot Du|^2 dS - I(r) \frac{I(r)}{H(r)}\right) - C \frac{\theta(r)}{r}
\end{equation}

\begin{equation}
\geq -C \frac{\theta(r)}{r}.
\end{equation}

In the last inequality we used Schwarz’s inequality. This differential inequality yields the following growth estimate for $N(r)$.

**Theorem 2.1.** (i) If $\int_0^{r_o} \frac{\theta(r)}{r} dr < +\infty$ for some $r_o > 0$, then

\begin{equation}
N(r) \leq \max(1, N(r_o)) \exp(C \int_0^{r_o} \frac{\theta(r)}{r} dr)
\end{equation}

for every $r \in (0, r_o)$.

(ii) In general, for each $r_1 \in (0, r_o)$ there exist constants $C, L(r_1) > 0$ such that

\begin{equation}
N(r) \leq \frac{L(r_1)}{r^{C\theta(r_1)}}
\end{equation}

for every $r \in (0, r_1/2)$.

This theorem implies Theorems 1.1 and 1.2 by a standard argument. For the details see [Ku1], [GL2].

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