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A UNIQUE CONTINUATION THEOREM FOR THE SCHRÖDINGER EQUATION WITH SINGULAR MAGNETIC FIELD

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ABSTRACT. We show a unique continuation theorem for the Schrödinger equation $(\frac{1}{i}\nabla - \mathbf{A})^2 u + Vu = 0$ with singular coefficients $\mathbf{A}$ and V.

1. MAIN RESULTS

In this paper we show a unique continuation theorem for the Schrödinger operator $H = (\frac{1}{i}\nabla - \mathbf{A})^2 + V$ with singular magnetic field. In fact we shall establish, under some assumptions on $\mathbf{A}$ and V, the following estimate:

$$\int_{B_{2r}(x_o)} |u|^2\, dx \le C \int_{B_r(x_o)} |u|^2\, dx.$$

The latter holds for $r > 0$ and $x_o \in \Omega$ with $B_{2r}(x_o) \subset \Omega$ and for solutions $u \in H^2_{\mathrm{loc}}(\Omega)$ of

$$Hu = (\frac{1}{i}\nabla - \mathbf{A})^2 u + Vu = 0 \quad in \quad \Omega \subset \mathbf{R}^n, \tag{1}$$

where $n \ge 3, \Omega$ is a domain, $\mathbf{A}(x) = (A_j(x))_{j=1}^n$ is real-valued and $V(x) = V^R(x) + iV^I(x)$ is complex-valued. This estimate implies a strong unique continuation theorem (Corollary 1.1). The proof uses a Rellich's type identity and the variational method (originally due to Garofalo and Lin, see e.g., [GL1], [GL2], [Ku1]), which does not need Carleman type estimates. We emphasize that our method requires neither higher integrability nor pointwise estimates for $\mathbf{A}$ compared with previous results [BKRS], [H], [Wo1], [So], [GL1], [GL2], [Ku1].

Throughout this paper we use the notation

$$\nabla = (\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}), \quad V^+(x) = \max(V(x), 0),\ V^-(x) = \max(-V(x), 0),$$

$$\mathbf{B} = (b_{jk})_{j,k=1}^n, \quad b_{jk}(x) = \frac{\partial A_j}{\partial x_k} - \frac{\partial A_k}{\partial x_j},$$

$$D_j = \frac{1}{i}\frac{\partial}{\partial x_j} - A_j, \quad {D_j}^* = -\frac{1}{i}\frac{\partial}{\partial x_j} - A_j,$$

$$H^m_{\mathrm{loc}}(\Omega) = \{u \in L^2_{\mathrm{loc}}(\Omega); \quad D^\alpha u \in L^2_{\mathrm{loc}}(\Omega), |\alpha| \le m\}.$$

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Note that $\overline{D_j u} = D_j^* \overline{u}$ for a complex-valued function u.

To state our main results, first we recall the definition of the Kato class $K_n^{\rm loc}(\Omega)$.

Definition 1.1. We say $f \in L^1_{\rm loc}(\Omega)$ belongs to the Kato class $K_n^{\rm loc}(\Omega)$ if $\lim_{r\to 0} \eta(r; f\chi_{\Omega_o}) = 0$ for every compact subdomain Ω_o of Ω. Here

$$\eta(r; f\chi_{\Omega_o}) = \sup_{x\in\mathbf{R}^n} \int_{\{|x-y|<r\}\cap\Omega_o} \frac{|f(y)|}{|x-y|^{n-2}}\, dy.$$

The assumptions on $\mathbf{A}$ and V in this paper are the following.

Assumption (A): Let $x_o \in \Omega$ be fixed.

1. $\mathbf{A} \in L^4_{\rm loc}(\Omega)$, $\nabla\mathbf{A} \in L^2_{\rm loc}(\Omega)$, $|\mathbf{A}|^2 \in K_n^{\rm loc}(\Omega)$, $(|x-x_o||\mathbf{B}|)^2 \in K_n^{\rm loc}(\Omega)$;
2. $V \in K_n^{\rm loc}(\Omega)$, $(|x-x_o||V^I|)^2 \in K_n^{\rm loc}(\Omega)$, $(2V^R+(x-x_o)\cdot\nabla V^R)^- \in K_n^{\rm loc}(\Omega)$.

Taking an arbitrary compact subdomain $\Omega_o \subset \Omega$ such that $x_o \in \Omega_o$, we will use the notation

$$\begin{aligned}\theta_o(r) &= \eta_o(r; (2V^R + (x-x_o)\cdot\nabla V^R)^-) \\ &\quad + \eta_o(r; ((x-x_o)V^I)^2)^{1/2} + \eta_o(r; ((x-x_o)|\mathbf{B}|)^2)^{1/2},\end{aligned}$$

where $\eta_o(r; f) = \eta(r; f\chi_{\Omega_o})$.

Theorem 1.1. *Suppose Assumption (A) and* $\int_0^{r_o} \frac{\theta_o(r)}{r}\, dr < +\infty$ *for some* $r_o > 0$. *Let* $u \in H^2_{\rm loc}(\Omega)$ *be a solution of* (1). *Then there exist constants* $r_*, C > 0$ *such that*

$$\int_{B_{2r}(x_o)} |u|^2\, dx \le C \int_{B_r(x_o)} |u|^2\, dx \tag{2}$$

for every $0 < r < r_*$.

When we do not assume $\int_0^{r_o} \frac{\theta_o(r)}{r}\, dr < +\infty$, we have

Theorem 1.2. *Suppose Assumption (A) and let* $u \in H^2_{\rm loc}(\Omega)$ *be a solution of* (1). *Then, for each* $r_1 \in (0, r_*)$, *there exist constants* $C > 0$ *and* $L(r_1) > 0$ *such that*

$$\int_{B_{2r}(x_o)} |u|^2\, dx \le C \exp\left(\frac{L(r_1)}{r^{C\theta_o(r_1)}}\right) \int_{B_r(x_o)} |u|^2\, dx \tag{3}$$

for every $r_1/2 > r > 0$.

The constant r_* is determined by the condition that $\eta(r; (V^R)^-) \le \frac{1}{2C(n)}$ hold for all $r \in (0, r_*)$, where $C(n)$ is a constant depending only on n (Lemma 2.3). The constants C and $L(r_1)$ do not depend on r, but depend on u. It is well known that these yield unique continuation theorems (e.g., [GL1], [GL2], [Ku1]).

Corollary 1.1. *Suppose Assumption (A) and the condition* $\int_0^{r_o} \frac{\theta_o(r)}{r}\, dr < +\infty$, $r_o > 0$, *for every* $x_o \in \Omega$. *Then H has SUCP (strong unique continuation property); if* $u \in H^2_{\rm loc}(\Omega)$ *is a solution of* (1) *and satisfies, for some* $x_o \in \Omega$ *and for every* $m > 0$,

$$\int_{B_r(x_o)} |u|^2\, dx = O(r^m) \quad (r \to 0),$$

then $u \equiv 0$ *in* Ω.

Corollary 1.2. *Suppose Assumption (A) for every $x_o \in \Omega$. Then H has a unique continuation property; if $u \in H^2_{\mathrm{loc}}(\Omega)$ is a solution of (1) and satisfies, for some $x_o \in \Omega$ and $A, \alpha > 0$,*

$$\int_{B_r(x_o)} |u|^2\,dx = O(\exp(-\frac{A}{r^\alpha})) \quad (r \to 0),$$

then $u \equiv 0$ in Ω.

In particular, H has WUCP (weak unique continuation property); if u vanishes on a subdomain Ω' of Ω, then $u \equiv 0$ in Ω.

Since $Hu = -\Delta u + 2i\mathbf{A}\cdot\nabla u + i(\mathrm{div}\mathbf{A})u + |\mathbf{A}|^2 u + Vu$, putting $\mathbf{b} = 2i\mathbf{A}, W = i(\mathrm{div}\mathbf{A}) + |\mathbf{A}|^2 + V$, we can rewrite the equation (1) in the following way:

$$Hu = -\Delta u + \mathbf{b}\cdot\nabla u + Wu. \tag{4}$$

Although we can apply results of [BKRS], [Wo1], [Wo2], [So], [H] [GL2], [Ku1] etc. under suitable conditons on $\mathbf{b}$ and W, our theorems cannot be covered by these previous results. First, previous results require a stronger condition on $\mathrm{div}\mathbf{A}$ even for WUCP. For instance, [Wo2] require $W \in L^{n/2}_{\mathrm{loc}}$ for WUCP and hence $\mathrm{div}\mathbf{A} \in L^{n/2}_{\mathrm{loc}}$ etc. On the other hand, our method only requires $\mathrm{div}\mathbf{A} \in L^2_{\mathrm{loc}}$ for $\mathrm{div}\mathbf{A}$ and, instead of that, $(|x-x_o||\mathbf{B}|)^2 \in K^{\mathrm{loc}}_n(\Omega)$ (or $(|x-x_o||\mathbf{B}|)^2 \in F^{\mathrm{loc}}_t$) for the magnetic field $\mathbf{B}$ (see Example 1.1). Secondly, to obtain SUCP [BKRS], [H] require $|\mathbf{A}| \in L^q_{\mathrm{loc}}, q > \frac{3n-2}{2}$ (for related results see also [Wo1]) and [So], [GL2], [Ku1] require a pointwise estimate

$$|\mathbf{A}(x)| \le \frac{f(|x-x_o|)}{|x-x_o|}, \quad \int_0^{r_o} \frac{f(t)}{t}\,dt < +\infty$$

for $\mathbf{A}$, but our method requires neither higher integrability nor pointwise estimates.

Remark 1.1. These theorems also hold even if we replace the class $K^{\mathrm{loc}}_n(\Omega)$ in Assumption (A) by the more general one $Q^{\mathrm{loc}}_t(\Omega) = K^{\mathrm{loc}}_n(\Omega) + F^{\mathrm{loc}}_t(\Omega)$, $1 < t \le n/2$, where $F^{\mathrm{loc}}_t(\Omega)$ is the Fefferman-Phong class. However, in this case we must assume an additional condition

$$\lim_{r\to 0}\sup_{x\in\Omega} \|(V^R)^-\|_{Q_t(B_r(x)\cap\Omega)} \le \epsilon_o$$

for sufficiently small $\epsilon_o > 0$ and take $H^2_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ as a solution class. See [Ku1].

Remark 1.2. A sufficient condition to assure $\int_0^{r_o} \frac{\theta_o(r)}{r}\,dr < +\infty$ is that

$$(2V^R + (x-x_o)\cdot\nabla V^R)^-, ((x-x_o)V^I)^2, ((x-x_o)|\mathbf{B}|)^2 \in K^{\mathrm{loc}}_{n,\delta}(\Omega)(\subset K^{\mathrm{loc}}_n(\Omega))$$

hold for some $\delta > 0$, where we say $f \in K^{\mathrm{loc}}_{n,\delta}(\Omega)$ if f satisfies, for every compact subdomain Ω_o of Ω,

$$\lim_{r\to 0}\sup_{x\in\mathbf{R}^n} \int_{\{|x-y|<r\}\cap\Omega_o} \frac{|f(y)|}{|x-y|^{n-2+\delta}}\,dy = 0.$$

Remark 1.3. By using the approximation argument in [Ku1] we can show unique continuation theorems similar to the theorems above even for H^1_{loc} solutions.

Finally, we must remark that Kalf [Ka] proved WUCP under the assumptions $\mathbf{A} \in H^1_{\mathrm{loc}}(\Omega)$, $(V^R)^2 \in K^{\mathrm{loc}}_n(\Omega)$ $(V^I \equiv 0)$, but his method cannot be applied to SUCP. For WUCP our results complement his result; compare that one needs

$(V^R)^2 \in K_n^{\rm loc}(\Omega)$ in his case and $(2V^R + (x-x_o)\cdot\nabla V^R)^- \in K_n^{\rm loc}(\Omega)$ in our case (see Examples 1.2, 1.3).

Example 1.1. Let $A_i = C\frac{x_i}{|x|}$. Then $\mathbf{B} \equiv 0$ and $\nabla\mathbf{A} \in L^2_{\rm loc}$, $\mathbf{A} \in L^4_{\rm loc}$ when $n \geq 5$. Note that $|\mathbf{A}|^2 = C^2/|x|^2 \notin K_n^{\rm loc}$, but $|\mathbf{A}|^2 = C^2/|x|^2 \in F_t^{\rm loc}$ for $1 < t < n/2$. Hence we can apply our method to show SUCP for this $\mathbf{A}$ (see Remark 1.1).

Example 1.2. Let $x_o = O, B_1 = B_1(O)$ and

$$V(x) = \frac{1}{|x|^l|\log|x||^m}, \quad m, l > 0.$$

Then we have

$$\begin{aligned} V \in K_n^{\rm loc}(B_1) &\Leftrightarrow (m>1, l=2) \text{ or } (m \in \mathbf{R}^1, l<2),\\ V^2 \in K_n^{\rm loc}(B_1) &\Leftrightarrow (m>1/2, l=1) \text{ or } (m \in \mathbf{R}^1, l<1), \quad (5)\\ 2V + x\cdot\nabla V \in K_n^{\rm loc}(B_1) &\Leftrightarrow (m>0, l=2) \text{ or } (m \in \mathbf{R}^1, l<2). \end{aligned}$$

So in this example the condition $V^2 \in K_n^{\rm loc}(\Omega)$ is stronger than $(2V + x\cdot\nabla V)^- \in K_n^{\rm loc}(\Omega)$.

Example 1.3. Let $N \in \mathbf{N}$ and $R = (R_1, \cdots, R_N) \in \mathbf{R}^{3N}$ be fixed and

$$V(x) = \sum_{j=1}^N \frac{1}{|x_j - R_j|}, \quad x = (x_1, \cdots, x_N) \in \mathbf{R}^{3N}, \quad x_j \in \mathbf{R}^3.$$

Then $V^2 \notin K_n^{\rm loc}$, but $2V + (x-x_o)\cdot\nabla V \in K_n^{\rm loc}$ for every $x_o \in \mathbf{R}^{3N}$.

2. Proof of the theorems

We may assume x_o is the origin O and write $B_r = B_r(O)$. For the sake of simplicity, we also use Ω, $\eta(r;f)$ and $\theta(r)$ instead of Ω_o, $\eta(r;f\chi_{\Omega_o})$ and $\theta_o(r)$, respectively.

Let $u \in H^2_{\rm loc}(\Omega)$ be a solution of (1) and put

$$I(r) = \int_{B_r} |Du|^2 + V^R|u|^2\,dx = \int_{B_r} |(\nabla - i\mathbf{A})u|^2 + V^R|u|^2\,dx, \quad (6)$$

$$H(r) = \int_{\partial B_r} |u|^2\,dS, \quad N(r) = \frac{rI(r)}{H(r)}. \quad (7)$$

Note that $H = (\frac{1}{i}\nabla - \mathbf{A})^2 + V = \sum_{j=1}^n D_jD_j + V$. Our argument is based on the following identity.

Lemma 2.1. *Suppose Assumption (A) (for $\mathbf{A}$). Then $u \in H^2_{\rm loc}(\Omega)\cap L^\infty_{\rm loc}(\Omega)$ satisfies*

$$\begin{aligned} &Im(\int_{B_r}(x\cdot\overline{Du})\sum_{j=1}^n D_jD_ju\,dx)\\ &= \frac{r}{2}\int_{\partial B_r}|Du|^2\,dS - (\frac{n-2}{2})\int_{B_r}|Du|^2\,dx\\ &- r\int_{\partial B_r}|n\cdot Du|^2\,dS + Re(\sum_{j,k=1}^n\int_{B_r} b_{jk}(x)x_jD_ku\overline{u}\,dx) \quad (8)\end{aligned}$$

for every $r > 0$ with $B_r \subset \Omega$.

Proof. This is a kind of Rellich's identity. The detailed computation can be seen in [EK]. So we omit it. □

We remark that under the assumption $|\mathbf{A}|^2, V \in K_n^{\mathrm{loc}}(\Omega)$ a solution $u \in H^1_{\mathrm{loc}}(\Omega)$ is locally bounded (see [Ku2]). So we can apply Lemma 2.1 for solutions $u \in H^2_{\mathrm{loc}}(\Omega)$ of (1). This identity implies

Lemma 2.2. *Suppose Assumption (A). Then for a.e.* $r \in (0, R_o)$

$$\begin{aligned} I'(r) &= \frac{n-2}{r} I(r) + 2\int_{\partial B_r} |n \cdot Du|^2\, dS \\ &- \frac{n-2}{r}\int_{B_r} V^R|u|^2\, dx + \int_{\partial B_r} V^R |u|^2\, dS \\ &- \frac{2}{r} Im(\int_{B_r} (x\cdot \overline{Du}) V u\, dx) \\ &- \frac{2}{r} Re(\sum_{j,k=1}^{n} \int_{B_r} b_{jk}(x) x_j (D_k u)\overline{u}\, dx) \end{aligned} \tag{9}$$

holds, where $R_o = \max\{r; B_r \subset \Omega\}$.

Proof. Note that

$$I'(r) = \int_{\partial B_r} |Du|^2 + V^R|u|^2\, dS \quad \text{a.e. } r.$$

Lemma 2.1 and this identity imply (9). □

Since $x\cdot\overline{Du} = x\cdot D^*\overline{u} = i(x\cdot\nabla\overline{u}) - (x\cdot\mathbf{A})\overline{u}$, we have

$$Im(\int_{B_r}(x\cdot\overline{Du})Vu\,dx) = Re(\int_{B_r}(x\cdot\nabla\overline{u})V^R u\, dx) + Im(\int_{B_r}(x\cdot\overline{Du})V^I u\,dx).$$

Put

$$\begin{aligned} J(V^R;u;r) &= \int_{\partial B_r} V^R|u|^2\, dS - \frac{n-2}{r}\int_{B_r} V^R|u|^2\, dx \\ &- \frac{1}{r}\int_{B_r} (x\cdot\nabla(|u|^2))V^R\, dx. \end{aligned} \tag{10}$$

By integration by parts we have

$$J(V^R;u;r) \geq -\frac{1}{r}\int_{B_r} (2V^R + x\cdot\nabla V^R)^-|u|^2\, dx.$$

Then from this observation and Lemma 2.2 we obtain the following estimate:

$$\begin{aligned} I'(r) \geq &\frac{n-2}{r} I(r) + 2\int_{\partial B_r} |n\cdot Du|^2\, dS \\ &- \frac{1}{r}\int_{B_r}(2V^R + x\cdot\nabla V^R)^-|u|^2\, dx \\ &- \frac{2}{r}\int_{B_r} (|x||V^I| + |x||\mathbf{B}|)|Du||u|\, dx \end{aligned} \tag{11}$$

for a.e. $r \in (0, R_o)$. To control the last two terms we need the following inequality.

Lemma 2.3. *Let $W \in K_n^{\mathrm{loc}}(\Omega)$ and $u \in H^1_{\mathrm{loc}}(\Omega)$. Then there exists a constant $C(n)$ such that*

$$\int_{B_r} |W||u|^2\,dx \le C(n)\eta(r; W\chi_{\Omega_o})\Big(\int_{B_r} |Du|^2\,dx + \frac{1}{r}\int_{\partial B_r} |u|^2\,dS\Big) \tag{12}$$

for $B_r \subset \Omega_o, \overline{\Omega_o} \subset \Omega$.

Proof. Since $|u| \in H^1_{\mathrm{loc}}(\Omega)$, we know that

$$\int_{B_r} |W||u|^2\,dx \le C(n)\eta(r; W\chi_{\Omega_o})\Big(\int_{B_r} |\nabla|u||^2\,dx + \frac{1}{r}\int_{\partial B_r} |u|^2\,dS\Big) \tag{13}$$

holds([FGL]). Noting $|\nabla|u|| \le |Du|$ a.e., we obtain the desired inequality. □

Lemma 2.4. *There exists a constant $r_* > 0$ such that $H(r) > 0$ for every $r \in (0, r_*)$ unless $u \equiv 0$ in B_{r_*}.*

Proof. Note that the constant $r_* > 0$ is determined by the condition $\eta_-^R(r) \le \frac{1}{2C(n)}$ for $r \in (0, r_*)$, where $\eta_-^R(r) = \eta(r; (V^R)^-)$ and $C(n)$ is the constant in Lemma 2.3. Since $V^R \in K_n^{\mathrm{loc}}(\Omega)$, the constant $r_* > 0$ satisfying this condition exists.

Suppose $H(r_o) = 0$ for some $r_o \in (0, r_*)$. Noting a simple conputation yields

$$I(r) = Re\Big(\int_{\partial B_r} u_\rho \overline{u}\,dS\Big) = Re\Big(\int_{\partial B_r} (n \cdot Du)\overline{u}\,dS\Big), \tag{14}$$

we have $I(r_o) = 0$. Since Lemma 2.3 implies

$$\begin{aligned} I(r) &\ge \int_{B_r} |Du|^2\,dx - \int_{B_r} (V^R)^-|u|^2\,dx \\ &\ge \int_{B_r} |Du|^2\,dx - C(n)\eta_-^R(r)\Big(\frac{H(r)}{r} + \int_{B_r} |Du|^2\,dx\Big), \end{aligned} \tag{15}$$

the choice of r_* yields

$$0 = I(r_o) \ge \frac{1}{2}\int_{B_{r_o}} |Du|^2\,dx. \tag{16}$$

Hence we obtain $|Du(x)| = 0$ a.e. $x \in B_{r_o}$. Since $|\nabla|u|| \le |Du|$ a.e., $|u|$ is constant in B_{r_o}. $H(r_o) = 0$ implies $|u| \equiv 0$ in B_{r_o}. The conclusion can be obtained by an argument similar to the one in [Ku1, Theorem 1.5]. □

Hence we may assume $H(r) > 0$ for every $r \in (0, r_*)$.

Lemma 2.5. *There exists an absolute constant $C_0 > 0$ such that*

$$\int_{B_r} |Du|^2\,dx \le C_0 I(r) \tag{17}$$

for every $r \in \Gamma = \{r \in (0, r_*); N(r) > 1\}$.

Proof. Since $\frac{H(r)}{r} < I(r)$ for $r \in \Gamma$, (15) implies

$$\int_{B_r} |Du|^2\,dx \le \frac{1 + C(n)\eta_-^R(r)}{1 - C(n)\eta_-^R(r)} I(r) \le 3I(r)$$

for all $r \in \Gamma$. This concludes the desired estimate. □

Using (11) and Lemmas 2.3, 2.5 we obtain, for a.e. $r \in \Gamma$,

$$I'(r) \geq \frac{n-2}{r} I(r) + 2\int_{\partial B_r} |n \cdot Du|^2 \, dS - C\frac{\theta(r)}{r} I(r), \tag{18}$$

$$\theta(r)\eta(r; (2V^R + x \cdot \nabla V^R)^-) + \eta(r; (|x|V^I)^2)^{1/2} + \eta(r; (|x||\mathbf{B}|)^2)^{1/2}.$$

On the other hand, we have the following identity:

$$H'(r) = \frac{n-1}{r} H(r) + 2Re(\int_{\partial B_r} u_\rho \overline{u} \, dS), \quad u_\rho = \frac{x}{r} \cdot \nabla u. \tag{19}$$

Hence (14) and (19) imply

$$\begin{aligned} H'(r) &= \frac{n-1}{r} H(r) + 2I(r), \\ \frac{d}{dr}(\log \frac{H(r)}{r^{n-1}}) &= 2\frac{I(r)}{H(r)} = 2\frac{N(r)}{r}. \end{aligned} \tag{20}$$

Therefore, for a.e. $r \in \Gamma$ we obtain

$$\begin{aligned} \frac{N'(r)}{N(r)} &= \frac{1}{r} + \frac{I'(r)}{I(r)} - \frac{H'(r)}{H(r)} \\ &\geq 2\left(\frac{\int_{\partial B_r} |n \cdot Du|^2 \, dS}{I(r)} - \frac{I(r)}{H(r)}\right) - C\frac{\theta(r)}{r} \\ &\geq -C\frac{\theta(r)}{r}. \end{aligned} \tag{21}$$

In the last inequality we used Schwarz's inequality. This differential inequality yields the following growth estimate for $N(r)$.

Theorem 2.1. *(i) If* $\int_0^{r_o} \frac{\theta(r)}{r} \, dr < +\infty$ *for some* $r_o > 0$*, then*

$$N(r) \leq \max(1, N(r_*)) \exp(C \int_0^{r_*} \frac{\theta(r)}{r} \, dr) \tag{22}$$

for every $r \in (0, r_*)$.
(ii)
In general, for each $r_1 \in (0, r_*)$ *there exist constants* $C, L(r_1) > 0$ *such that*

$$N(r) \leq \frac{L(r_1)}{r^{C\theta(r_1)}} \tag{23}$$

for every $r \in (0, r_1/2)$.

This theorem implies Theorems 1.1 and 1.2 by a standard argument. For the details see [Ku1], [GL2].

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