NOT ALL JULIA SETS ARE QUASI-SELF-SIMILAR

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Abstract. We show that there exist rational functions, whose Julia set fails to be quasi-self-similar.

1.

One of the conspicuous features of the Julia sets of rational functions is that small parts of them look very much like some large parts. Sullivan has introduced a proper concept to describe the situation: the quasi-self-similarity. He also established the quasi-self-similarity of the Julia sets of all hyperbolic rational functions [1, Theorem 8.6], [3, Theorem 7], [8, p. 742]. One of the open problems listed at the end of [3] asks, whether the same is true of all rational functions of degree \( \geq 2 \). The purpose of the present note is to show that this is not the case.

2.

Let \( c \in (0, 1] \). A set \( E \) in the euclidean \( n \)-space \( \mathbb{R}^n \) is \( c \)-porous if each closed ball \( \overline{B}^n(x,r) \subset \mathbb{R}^n \) contains a point \( z \) such that the open ball \( B^n(z,cr) \) does not meet \( E \); \( E \) is porous if it is \( c \)-porous for some \( c \) (see e.g. [10]). For instance, Cantor sets with constant ratio in \( \mathbb{R}^n \) are porous in \( \mathbb{R}^n \). Given \( k > 0 \), we let \( \phi_k \) stand for the similarity map \( x \mapsto kx \), \( x \in \mathbb{R}^n \). A nonempty set \( E \subset \mathbb{R}^n \) is called \( K \)-quasi-self-similar if there is an \( r_0 > 0 \) such that, given any closed ball \( \overline{B}^n(x,r) \) with \( t = r_0/r > 1 \), there exists a \( K \)-quasi-isometry \( f : \phi_t(\overline{B}^n(x,r) \cap E) \to E \), i.e., \( f \) satisfies

\[
1 \leq |y - z| \leq |f(y) - f(z)| \leq K|y - z| \quad \text{for all } y, z \in \phi_t(\overline{B}^n(x,r) \cap E).
\]

Quasi-self-similarity means \( K \)-quasi-self-similarity for some \( K \geq 1 \). See [1, p. 121], [3, p. 65], [4, p. 183]. The constant \( r_0 \) is called a standard size of \( E \) [4, p. 183]. We are going to show that quasi-self-similarity implies porosity under some mild restrictions.

Lemma. Let \( E \subset \mathbb{R}^n \) be a compact, nowhere dense, quasi-self-similar set. Then \( E \) is porous in \( \mathbb{R}^n \).
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and let $J(f_c)$ denote the Julia set of $f_c$. Note that $J(f_c)$ is a compact nowhere dense subset of $\mathbb{C}$, because $\infty$ belongs to the Fatou set of $f_c$. Shishikura [5], [6] has recently shown that there are values of $c$ for which $\text{dim}_H(J(f_c)) = 2$. More precisely, there is a residual subset $F$ of the boundary of the Mandelbrot set such that if $c \in F$, then $\text{dim}_H(J(f_c)) = 2$. Hence we have

**Corollary 2.** There are values $c \in \mathbb{C}$ such that $J(f_c)$ fails to be quasi-self-similar.

**Remark 1.** As mentioned above, the Julia set of any hyperbolic rational function is quasi-self-similar (see [2, pp. 89–93] for basic properties of hyperbolic rational maps). It follows from Corollary 1 that $\text{dim}_H(J(f)) < 2$ for such functions. Sullivan [8, Theorem 4] has deduced this result relying on properties of conformal measures defined on Julia sets.

**Remark 2.** Of course, the same question can be proposed in the context of finitely generated Kleinian groups; that is, is the limit set of any finitely generated Kleinian group of $\mathbb{R}^2$ quasi-self-similar? The answer is again in the negative. This follows, in view of Corollary 1, from a result of Sullivan [7], according to which there are finitely generated Kleinian groups of $\mathbb{R}^2$ whose limit set has Hausdorff dimension two. Note, however, that the Hausdorff dimension of a geometrically finite Kleinian group of $\mathbb{R}^n$ is always less than $n$ [9, Theorem D].

**REFERENCES**


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