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FIXED POINT THEORY FOR COMPACT UPPER SEMI–CONTINUOUS OR LOWER SEMI–CONTINUOUS SET VALUED MAPS

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ABSTRACT. Fixed point theory is presented for compact u.s.c. and l.s.c. set valued maps.

1. INTRODUCTION

This paper presents new fixed point theory for compact upper semi–continuous (u.s.c.) and lower semi–continuous (l.s.c.) set valued mappings. Our results generalize some well known fixed point results for convex and nonconvex maps. Our paper is divided into two main sections. In section 2 we use a recent Leray–Schauder type alternative due to Ben–El–Mechaiekh and Idzik [3] for u.s.c., compact approachable maps with nonempty, compact values to establish a new fixed point result for such maps. Our theory generalizes some fixed point results in Ben–El–Mechaiekh and Deguire [2] and Granas [9]. In section 3 we establish via Michael's selection theorem [11] a nonlinear alternative of Leray–Schauder type for l.s.c., compact multifunctions with closed, convex values. Also a new fixed point result is given. This generalizes some results in Himmelberg, Porter and Van Vleck [10] and Tarafdar and Výborný [17]. A brief discussion of nonconvex maps is also given.

We now gather together some definitions and known facts. Let X and Y be topological spaces. A multifunction $f : X \to Y$ is a point to set function such that for each $x \in X$, $f(x)$ is a nonempty subset of Y. The function f is u.s.c. if the set $f^{-1}(B) = \{x \in X : f(x) \cap B \neq \emptyset\}$ is closed for any closed set B in Y (equivalently, $f : X \to Y$ is u.s.c. if for any net $\{x_\alpha\}$ in X and any closed set B in Y with $x_\alpha \to x_0 \in X$ and $f(x_\alpha) \cap B \neq \emptyset$ for all α we have $f(x_0) \cap B \neq \emptyset$). The function $f : X \to Y$ is l.s.c. if the set $f^{-1}(A)$ is open for any open set A in Y. $f : X \to Y$ is compact if $f(X)$ is relatively compact in Y. Finally the function f is said to be point–compact if for each $x \in X$, $f(x)$ is a compact subset of Y. Now we state a result [16, proposition 1] which will be used frequently in this paper.

Theorem 1.1. *Let X and Y be topological spaces and $f : X \to Y$ be a u.s.c., point–compact multifunction. Suppose $\{x_\alpha\}$ is a net in X such that $x_\alpha \to x_0$. If $y_\alpha \in f(x_\alpha)$ for each α, then there is a $y_0 \in f(x_0)$ and a subnet $\{y_\beta\}$ of the net $\{y_\alpha\}$ such that $y_\beta \to y_0$.*

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Theorem 1.1 immediately yields the following well known result.

Theorem 1.2. *Let X and Y be topological spaces and $f : X \to Y$ be a u.s.c., point–compact multifunction. Suppose $\{x_\alpha\}$ is a net in X and $y_\alpha \in f(x_\alpha)$ for each α. If $x_\alpha \to x_0$ and $y_\alpha \to y_0$, then $y_0 \in f(x_0)$.*

2. The upper semi–continuous case

We begin by describing the class of approachable set valued maps. Let X and Y be subsets of Hausdorff topological vector spaces E_1 and E_2 respectively, and let $F : X \to Y$ be a multifunction. Given two open neighborhoods U and V of the origins in E_1 and E_2 respectively, a (U, V)–continuous selection of F is a continuous function $s : X \to Y$ satisfying

$$s(x) \in (F[(x + U) \cap X] + V) \cap Y \quad \text{for every} \quad x \in X.$$

F is said to be approachable if it has a (U, V)–continuous selection for any open neighborhoods U and V of the origins in E_1 and E_2 respectively. Let

$$\mathcal{A}_0(X, Y) = \{\, F : X \to Y : F \text{ is approachable } \}$$

and

$$\mathcal{A}(X, Y) = \{\, F \in \mathcal{A}_0(X, Y) : F \text{ is u.s.c. and compact valued } \}.$$

Remark. For examples of approachable maps we refer the reader to [2, 3].

We now gather together three theorems from the literature which will be used in this section.

Theorem 2.1 ([2], pg.486, proposition 3.3)**.** *Let X, Y and Z be subsets of the Hausdorff topological vector spaces E_1, E_2 and E_3 respectively. Suppose $F : X \to Y$ is a multivalued map with $F \in \mathcal{A}(X, Y)$ and $r : Z \to X$ is a continuous map (single valued). If Z is compact then $Fr \in \mathcal{A}(Z, Y)$.*

Theorem 2.2 ([2], pg.496, corollary 7.3)**.** *Let X be a convex subset of a locally convex Hausdorff linear topological space E with $F \in \mathcal{A}(X, X)$ a compact map. Then F has a fixed point (i.e. there exists $x \in X$ with $x \in F(x)$).*

Theorem 2.3 ([3], pg.108, Theorem)**.** *Let X be a closed subset of a locally convex Hausdorff linear topological space E with $0 \in int\, X$. In addition assume $F : X \to E$ is a u.s.c., approachable, compact map with nonempty, closed values. Then either*

(A1) *F has a fixed point in X; or*
(A2) *there exists $(\lambda, x) \in (0, 1) \times \partial X$ such that $x \in \lambda\, F(x)$.*

We now state and prove the main result of this section.

Theorem 2.4. *Let C be a complete convex subset of a metrizable (metric d) locally convex linear topological space E with Q a closed, convex, proper subset of C and $0 \in Q$. Assume $F : Q \to C$ is a u.s.c., approachable, compact map with nonempty, compact values. In addition suppose*

$$(2.1) \qquad \begin{cases} \textit{if } \{(x_j, \lambda_j)\}_{j=1}^{\infty} \textit{ is a sequence in } \partial Q \times [0, 1] \textit{ converging to} \\ (x, \lambda) \textit{ with } x \in \lambda\, F(x) \textit{ and } 0 \le \lambda < 1, \\ \textit{then there exists } j_0 \in \{1, 2, ...\} \textit{ with} \\ \{\lambda_j\, F(x_j)\} \subseteq Q \textit{ for each } j \ge j_0. \end{cases}$$

Finally assume <u>*either*</u>

$$\text{(2.2a)} \qquad \begin{cases} C \ \text{ is compact and } U_i = \{x \in E : d(x,Q) < \frac{1}{i}\} \subseteq C \\ \text{for } i \text{ sufficiently large,} \end{cases}$$

<u>*or*</u>

$$\text{(2.2b)} \qquad Q \subseteq \overline{co}\,(\overline{F(Q)}) \ \text{ and } U_i \subseteq \overline{co}\,(\overline{F(Q)}) \ \text{ for } i \text{ sufficiently large,}$$

<u>*or*</u>

$$\text{(2.2c)} \qquad \begin{cases} Fr \in \mathcal{A}_0(C,C) \ \text{ and } U_i \subseteq C \ \text{ for } i \text{ sufficiently large; here} \\ r : E \to Q \ \text{ is a continuous retraction such that} \\ r(z) \in \partial Q \ \text{ for } z \in E/Q \end{cases}$$

is satisfied. Then F *has a fixed point in* Q.

Remarks. (i) The existence of a continuous retraction $r : E \to Q$ follows immediately from Dugundji's extension theorem [7, 18].

(ii) If $0 \in int\, Q$ we may take

$$r(x) = \frac{x}{\max\{1, \mu(x)\}}\,, \ x \in E.$$

Here μ is the Minkowski functional [15] on Q, i.e. $\mu(x) = \inf\{\alpha > 0 : x \in \alpha\, Q\}$. If $int\, Q = \emptyset$ then $\partial Q = Q$. As a result we may (and do) choose the retraction r above so that $r(z) \in \partial Q$ if $z \in E/Q$.

(iii) Notice (2.2a) and (2.2b) are only of interest if the space is normable.

(iv) If F is convex valued then of course Fr is convex valued, so Fr is approachable [3], i.e. (2.2c) holds.

(v) Theorem 2.4 was proved by Furi and Pera [8], by a different method, when F is single valued, compact and continuous. It was extended by O'Regan [13, 14] for the case when F is single valued, condensing (P–concentrative) and continuous.

Proof. Let $Z = C$ if (2.2a) or (2.2c) holds, and let $Z = \overline{co}\,(\overline{F(Q)})$ if (2.2b) is satisfied.

Remark. Note $\overline{co}\,(\overline{F(Q)})$ is compact. To see this let P be a defining system of seminorms. Fix $p \in P$. Since $\overline{F(Q)}$ is compact then $\overline{co}\,(\overline{F(Q)})$ is precompact(totally bounded) [6] in the seminormed space (E,p) for each $p \in P$. Thus $\overline{co}\,(\overline{F(Q)})$ is precompact in E. Now since $\overline{co}\,(\overline{F(Q)})$ is a closed precompact subset of C and C is complete, then $\overline{co}\,(\overline{F(Q)})$ is compact [5].

Now let

$$B = \{\, x \in Z : \, x \in Fr(x) \,\}\,.$$

We first show $B \neq \emptyset$. Notice $Fr : Z \to Z$ is a u.s.c. (since F is u.s.c. and r is continuous), approachable (theorem 2.1 if (2.2a) or (2.2b) occurs and by assumption if (2.2c) occurs), compact map with nonempty, compact values. Theorem 2.2 implies that Fr has a fixed point, so $B \neq \emptyset$. We next claim that B is closed. Let (x_α) be a net in B with $x_\alpha \to x_0 \in Z$. Theorem 1.2 implies $x_0 \in Fr(x_0)$, so $x_0 \in B$. Thus B is closed, and in fact B is compact since $B \subseteq Fr(B) \subseteq F(Q)$.

It remains to show $B \cap Q \neq \emptyset$. To do this we argue by contradiction. Suppose $B \cap Q = \emptyset$. Then since B is compact and Q is closed there exists $\delta > 0$ with $dist(B, Q) > \delta$. Choose $m \in \{1, 2, ...\}$ such that $1 < \delta m$ and $U_i \subseteq Z$ for $i \in \{m, m+1, ...\}$; here $U_i = \left\{ x \in E : \, d(x, Q) < \frac{1}{i} \right\}$. Fix $i \in \{m, m+1, ...\}$. Since $dist(B, Q) > \delta$, then $B \cap \overline{U_i} = \emptyset$. Now $0 \in U_i$ and $Fr : \overline{U_i} \to C$ is a u.s.c., approachable, compact map with nonempty, compact values. Theorem 2.3 implies (since $B \cap \overline{U_i} = \emptyset$) that there exists $(y_i, \lambda_i) \in \partial U_i \times (0, 1)$ with $y_i \in \lambda_i \, Fr(y_i)$. Notice in particular since $y_i \in \partial U_i$ that

$$\{\lambda_i \, Fr(y_i)\} \not\subseteq Q \quad \text{for each} \quad i \in \{m, m+1, ...\}. \tag{2.3}$$

Now consider

$$D = \{ \, x \in E : \, x \in \lambda \, Fr(x) \quad \text{for some} \quad \lambda \in [0, 1] \, \}.$$

We first show D is closed. To see this let (x_α) be a net in D (i.e. for each α, $x_\alpha \in \lambda_\alpha \, Fr(x_\alpha)$ for some $\lambda_\alpha \in [0, 1]$) with $x_\alpha \to x_0 \in E$. Without loss of generality assume $\lambda_\alpha \to \lambda_0 \in [0, 1]$. We claim that

$$N : E \times [0, 1] \to C \quad \text{given by} \quad N(x, \lambda) = \lambda \, Fr(x)$$

is u.s.c. with nonempty, compact values. If the claim is true then theorem 1.2 implies $x_0 \in N(x_0, \lambda_0)$, i.e. $x_0 \in \lambda_0 \, Fr(x_0)$, so D is closed. To see that N is u.s.c. let Ω be a closed subset of C, (y_α, t_α) a net in $E \times [0, 1]$, $(y_\alpha, t_\alpha) \to (y_0, t_0)$ and $t_\alpha \, Fr(y_\alpha) \cap \Omega \neq \emptyset$. We must show $t_0 \, Fr(y_0) \cap \Omega \neq \emptyset$. Suppose $w_\alpha \in Fr(y_\alpha)$ with $t_\alpha \, w_\alpha \in \Omega$. Since Fr is u.s.c., there exist (theorem 1.1) $w_0 \in Fr(y_0)$ and a subnet (w_β) of (w_α) with $w_\beta \to w_0$. Since Ω is closed we have $t_0 \, w_0 \in \Omega$. Consequently $t_0 \, Fr(y_0) \cap \Omega \neq \emptyset$, so $N : E \times [0, 1] \to C$ is u.s.c. Thus D is closed and also compact since $D \subseteq co\,(F(Q) \cup \{0\})$. This together with $d(y_j, Q) = \frac{1}{j}$, $|\lambda_j| \leq 1$ (for $j \in \{m, m+1, ...\}$), implies that we may assume without loss of generality that $\lambda_j \to \lambda^\star$ and $y_j \to y^\star \in \partial Q$. Also, since $y_j \in \lambda_j \, Fr(y_j)$, we have from theorem 1.2 that $y^\star \in \lambda^\star \, Fr(y^\star)$ (note from above that $N : \overline{U_m} \times [0, 1] \to C$ given by $N(u, \lambda) = \lambda \, Fr(u)$ is u.s.c. with nonempty, compact values). If $\lambda^\star = 1$, then $y^\star \in Fr(y^\star)$, which contradicts $B \cap Q = \emptyset$. Hence we may assume $0 \leq \lambda^\star < 1$. But in this case, (2.1) with $x_j = r(y_j) \in \partial Q$, $x = y^\star = r(y^\star)$ implies that there exists $j_0 \in \{1, 2, ...\}$ with $\{\lambda_j \, Fr(y_j)\} \subseteq Q$ for each $j \geq j_0$. This contradicts (2.3). Thus $B \cap Q \neq \emptyset$, so there exists $x \in Q$ with $x \in Fr(x)$, i.e. $x \in F(x)$. $\square$

Let X and Y be subsets of a locally convex Hausdorff linear topological space E. More generally, let

$$\mathcal{B}_0(X, Y) = \{F : X \to Y : \, F \ \text{ satisfies property } \ B\}$$

and let $\mathcal{B}(X, Y)$ be the set of maps $F : X \to Y$ which are u.s.c., compact with nonempty, compact values and satisfy a property B (which we will not specify). Suppose the following properties are satisfied:

$$\text{if } X \text{ is convex and } F \in \mathcal{B}(X, Y) \text{ then } F \text{ has a fixed point in } X \tag{2.4}$$

and

$$\begin{cases} \text{if } X \text{ is closed and convex, } 0 \in int\, X \text{ and } F \in \mathcal{B}(X, E) \\ \text{ then either} \\ (A1).\, F \text{ has a fixed point in } X; \text{ or} \\ (A2). \text{ there exists } (\lambda, x) \in (0, 1) \times \partial X \text{ such that } x \in \lambda \, F(x). \end{cases} \tag{2.5}$$

With the above properties, essentially the same reasoning as in theorem 2.4 establishes the following result.

Theorem 2.5. *Let C be a complete convex subset of a metrizable (metric d) locally convex linear topological space E with Q a closed, convex, proper subset of C, $0 \in Q$ and $U_i = \{x \in E : d(x,Q) < \frac{1}{i}\} \subseteq C$ for i sufficiently large. Assume $F \in \mathcal{B}(Q,C)$, (2.1) holds and also that*

$$\begin{cases} Fr \in \mathcal{B}_0(C,C); \text{ here } r : E \to Q \text{ is a continuous retraction} \\ \text{such that } r(z) \in \partial Q \text{ for } z \in E/Q \end{cases} \tag{2.6}$$

is satisfied. Then F has a fixed point in Q.

Remark. For examples of classes of maps $\mathcal{B}_0(X,Y)$ we refer the reader to [9]. For example, property B could be that the values of the maps are R_δ sets.

3. The lower semi–continuous case

We begin by establishing a nonlinear alternative of Leray–Schauder type for l.s.c. set valued maps.

Theorem 3.1. *Let Q be either (i) a closed convex subset of a Fréchet space (complete metrizable locally convex linear topological space) E, or (ii) a complete, convex, compact, metrizable subset of a locally convex Hausdorff linear topological space E. Assume U is a relatively open subset of Q, $0 \in U$ and U is convex. Suppose $F : \overline{U} \to Q$ is a l.s.c., compact map with closed, convex values. Then either*

(A1) *F has a fixed point in $\overline{U}$; or*
(A2) *there exist $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.*

Remark. $\overline{U}$ and ∂U denote the closure and boundary of U in Q respectively.

Proof. Define the retraction $r : Q \to \overline{U}$ by

$$r(x) = \frac{x}{\max\{1, \mu(x)\}}\,,\ x \in Q,$$

where μ is the Minkowski functional [15] on $\overline{U}$, i.e. $\mu(x) = \inf\{\alpha > 0 : x \in \alpha\overline{U}\}$. Consider the mapping $Fr : Q \to Q$. Notice Fr is an l.s.c., compact map with closed, convex values. Michael's selection theorem [11, theorem 1.2] implies that there exists a continuous function $f : Q \to Q$ such that $f(x) \in Fr(x)$ for all $x \in Q$. Next Tychonoff's theorem [7] implies that f has a fixed point, which is also of course a fixed point of Fr, i.e. there exists $y \in Q$ with $y \in Fr(y)$. Thus if $z = r(y) \in \overline{U}$ we have $z \in rF(z)$, i.e. $z = r(w)$ for some $w \in F(z)$, i.e. $z \in \overline{U}$ is a fixed point of rF. Now either $w \in \overline{U}$ or not. If $w \in \overline{U}$ then $r(w) = w$, so $(A1)$ occurs. If $w \notin \overline{U}$ then $z = r(w) = \frac{w}{\mu(w)}$, so $z \in \lambda F(z)$ where $\lambda = \frac{1}{\mu(w)} < 1$, i.e. $(A2)$ occurs. □

Remark. In theorem 3.1 conditions were put on Q, E and F so that Michael's selection theorem could be applied. Of course results could be established if we used other selection theorems for l.s.c. set valued maps (convex and nonconvex); see [1, 4, 12] for example. This remark also applies to the next theorem.

We now prove a new fixed point result for l.s.c., compact set valued maps.

Theorem 3.2. *Let C be either (i) a closed convex subset of a Fréchet space (metric d) E or (ii) a complete, convex, compact subset of a metrizable (metric d) locally convex linear topological space E. Assume Q is a closed, convex, proper subset of C with $0 \in Q$ and $U_i = \{x \in E : d(x,Q) < \frac{1}{i}\} \subseteq C$ for i sufficiently large. Suppose $F : Q \to C$ is an l.s.c., compact map with closed, convex values, and that* (2.1) *holds. Then F has a fixed point.*

Proof. Let $r : E \to Q$ be a continuous retraction with $r(z) \in \partial Q$ for $z \in E/Q$. By Michael's selection theorem [11] (since $Fr : C \to C$ is an l.s.c., compact map with closed, convex values) there exists a continuous function $f : C \to C$ such that $f(x) \in Fr(x)$ for all $x \in C$. Consider

$$B = \{\, x \in C : \, x = f(x)\,\}\,.$$

Now $B \neq \emptyset$ by Tychonoff's theorem. Also B is clearly closed and compact, since $B \subseteq f(B) \subseteq F(Q)$. It remains to show $B \cap Q \neq \emptyset$. Suppose $B \cap Q = \emptyset$. Then there exists $\delta > 0$ with $dist(B,Q) > \delta$. Choose $m \in \{1, 2, ...\}$ such that $1 < \delta m$ and $U_i \subseteq C$ for $i \in \{m, m+1, ...\}$. Fix $i \in \{m, m+1, ...\}$. Since $dist(B,Q) > \delta$, then $B \cap \overline{U_i} = \emptyset$. Now $f : \overline{U_i} \to C$ is a continuous, compact map, so by the nonlinear alternative for single valued maps [7] there exists (since $B \cap \overline{U_i} = \emptyset$) $(y_i, \lambda_i) \in \partial U_i \times (0,1)$ with $y_i = \lambda_i\, f(y_i)$. Now since $y_i \in \partial U_i$ we have $\lambda_i\, f(y_i) \notin Q$ for $i \in \{m, m+1, ...\}$, and so

$$\{\lambda_i\, Fr(y_i)\} \nsubseteq Q \quad \text{for each} \quad i \in \{m, m+1, ...\}. \tag{3.1}$$

Let

$$D = \{\, x \in C : \, x = \lambda\, f(x) \ \text{ for some } \ \lambda \in [0,1]\,\}.$$

Since D is compact and $d(y_j, Q) = \frac{1}{j}$, $|\lambda_j| \leq 1$ (for $j \in \{m, m+1, ...\}$), we may assume without loss of generality that $\lambda_j \to \lambda^\star$ and $y_j \to y^\star \in \partial Q$. Also, since $y_j = \lambda_j\, f(y_j)$ we have $y^\star = \lambda^\star\, f(y^\star)$. Now $\lambda^\star \neq 1$ since $B \cap Q = \emptyset$. Thus $0 \leq \lambda^\star < 1$ and $y^\star \in \lambda^\star Fr(y^\star)$. But in this case (2.1) with $x_j = r(y_j)$, $x = y^\star = r(y^\star)$ implies that there exists $j_0 \in \{1, 2, ...\}$ with $\{\lambda_j\, Fr(y_j)\} \subseteq Q$ for each $j \geq j_0$. This contradicts (3.1), so $B \cap Q \neq \emptyset$. $\square$

Remark. One could use the ideas in this section together with ideas in [14] to obtain new results of Leray–Schauder and Furi–Pera type for l.s.c., condensing (P–concentrative) maps.

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