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SERRE DUALITY FOR NONCOMMUTATIVE PROJECTIVE SCHEMES

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ABSTRACT. We prove the Serre duality theorem for the noncommutative projective scheme proj A when A is a graded noetherian PI ring or a graded noetherian AS-Gorenstein ring.

0. INTRODUCTION

Let k be a field and let $A = \bigoplus_{i\geq 0} A_i$ be an $\mathbb{N}$-graded right noetherian k-algebra. In this paper we always assume that A is *locally finite* in the sense that each A_i is finite dimensional over k. If $A_0 = k$, then A is called connected graded. We denote by Gr A the category of graded right A-modules and by gr A the subcategory consisting of noetherian right A-modules. The augmentation ideal $A_{\geq 1} = \bigoplus_{i\geq 1} A_i$ is denoted by $\mathfrak{m}$. Let $M = \bigoplus_i M_i$ be a graded right A-module and x a homogeneous element of M. We say x is $\mathfrak{m}$-torsion if $x\mathfrak{m}^n = 0$ for some n. All $\mathfrak{m}$-torsion elements form a submodule of M, which is denoted by $\tau(M)$. The module M is call $\mathfrak{m}$-*torsion* (respectively $\mathfrak{m}$-*torsion-free*) if $\tau(M) = M$ (respectively $\tau(M) = \{0\}$). Let Tor A denote the subcategory of Gr A of all $\mathfrak{m}$-torsion modules and tor A denote the intersection of Tor A and gr A. The (degree) shift of M, denoted by $s(M)$, is defined by $s(M)_i = M_{i+1}$, and we use $M(n)$ for the n-th power of a shift $s^n(M)$. Given an $\mathbb{N}$-graded right noetherian ring A, the noncommutative projective scheme of A is defined as Proj $A := (\text{QGr } A, \mathcal{A}, s)$ where QGr A is the quotient category Gr A/Tor A, $\mathcal{A}$ is the image of A_A in QGr A and s is the degree shift. Sometime it is easier to work on noetherian objects, and the triple proj $A := (\text{qgr } A, \mathcal{A}, s)$ is also called the projective scheme of A, where qgr A = gr A/tor A. (See [AZ] for more details.) The canonical functor from Gr A to QGr A (and from gr A to qgr A) is denoted by π. If $M \in$ Gr A, we will use the corresponding calligraphic letter $\mathcal{M}$ for $\pi(M)$ if no confusion occurs. For example, $\mathcal{A} = \pi(A_A)$.

Let $X =$ proj A. The global section is $\mathrm{H}^0(X, \mathcal{N}) = \mathrm{Hom}_{\mathrm{QGr}\ A}(\mathcal{A}, \mathcal{N})$ and the cohomology is $\mathrm{H}^i(X, \mathcal{N}) = \mathrm{Ext}^i_{\mathrm{QGr}\ A}(\mathcal{A}, \mathcal{N})$ for all $i > 0$. In the commutative case,

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the Serre duality theorem [H2, III.7.6] says there is an object ω^0 in qgr A such that

$$\theta^0 : \operatorname{Hom}_{\text{qgr } A}(\mathcal{M}, \omega^0) \longrightarrow \mathrm{H}^d(X, \mathcal{M})^* \tag{0-1}$$

is a natural isomorphism for all $\mathcal{M} \in$ qgr A. Here * is the vector space dual and d is the cohomological dimension of X defined by

$$cd(X) = \max\{i \mid \mathrm{H}^i(X, \mathcal{M}) \neq 0 \text{ for some } \mathcal{M} \in \text{QGr } A\}.$$

It is easy to see that such ω^0 is unique, and it is called a *dualizing sheaf* on $X =$ proj A. In this paper we will prove a version of the Serre Duality Theorem for noncommutative projective schemes proj A. The first idea is to study $\mathrm{H}^d(X, -)^*$ as a functor from qgr A to Mod k. In general Mod A is the category of (ungraded) right A-modules. Using a graded version of Watts' theorem we will prove (0-1) for noncommutative rings under some hypotheses. These hypotheses can be checked for a class of rings including noetherian PI rings. The second idea is to use the balanced dualizing complex which was introduced and studied in [Ye]. The existence of balanced dualizing complex implies (0-1) holds for some ω^0 constructed from the dualizing complex. Since noetherian AS-Gorenstein rings admit balanced dualizing complexes, (0-1) holds for such rings.

In general we should not expect a dualizing sheaf ω^0 to exist in qgr A, and we can ask if (0-1) holds for some ω^0 in QGr A. This was answered affirmatively by P. Jørgensen recently in [Jø]. However when one uses the duality, there will be many advantages if ω^0 is in qgr A. Hence it is still important to work out for what other classes of graded rings the dualizing sheaf ω^0 is in qgr A.

1. Watts' Theorem

Let $(\mathcal{C}, \mathcal{A}, s)$ be a triple consisting of a k-linear category $\mathcal{C}$, an object $\mathcal{A}$ in $\mathcal{C}$ and an automorphism s of $\mathcal{C}$. Both (gr A, A_A, s) and (qgr $A, \mathcal{A}, s$) are examples of such triples. All functors in this paper will preserve the k-linear structure. As the degree shift, $s^n\mathcal{M}$ will be denoted by $\mathcal{M}(n)$ for all $n \in \mathbb{Z}$. Let Γ denote the representing functor $\bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{C}}(s^{-i}\mathcal{A}, -)$. Note that in [AZ, Sec. 4], Γ is the functor $\bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{C}}(\mathcal{A}, s^i(-))$. But it is easy to see that these two Γs are naturally isomorphic, and hence we will not distinguish them. For $\mathcal{M}$ in $\mathcal{C}$, $\Gamma(\mathcal{M})$ is a $\mathbb{Z}$-graded k-module with degree i part being $\operatorname{Hom}_{\mathcal{C}}(\mathcal{A}(-i), \mathcal{M})$. By composition of morphisms, $\Gamma(\mathcal{A})$ is a $\mathbb{Z}$-graded k-algebra and $\Gamma(\mathcal{M})$ is a $\mathbb{Z}$-graded right $\Gamma(\mathcal{A})$-module with multiplication $mr = ms^{-i}(r)$ for all $m \in \operatorname{Hom}_{\mathcal{C}}(\mathcal{A}(-i), \mathcal{M})$ and $r \in \operatorname{Hom}_{\mathcal{C}}(\mathcal{A}(-j), \mathcal{A})$.

If F is a covariant functor from $\mathcal{C}$ to Mod k, then $\underline{F}$ denotes the functor $\bigoplus_{i\in\mathbb{Z}} F(s^i(-))$. If F is a contravariant functor from $\mathcal{C}$ to Mod k, then $\underline{F}$ denotes $\bigoplus_{i\in\mathbb{Z}} F(s^{-i}(-))$. In the case of a bi-functor $\operatorname{Ext}^q_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$, we have

$$\underline{\operatorname{Ext}}^q_{\mathcal{C}}(\mathcal{M}, \mathcal{N}) \cong \bigoplus_{i\in\mathbb{Z}} \operatorname{Ext}^q_{\mathcal{C}}(\mathcal{M}, \mathcal{N}(i)) \cong \bigoplus_{i\in\mathbb{Z}} \operatorname{Ext}^q_{\mathcal{C}}(\mathcal{M}(-i), \mathcal{N}).$$

Now let $F : \mathcal{C} \longrightarrow$ Mod k be a contravariant functor. We use $\underline{F}(\mathcal{A})$ for the $\mathbb{Z}$-graded k-module $\bigoplus_{i\in\mathbb{Z}} F(\mathcal{A}(-i))$. For every $x \in F(\mathcal{A}(-i))$ and $r \in \Gamma(\mathcal{A})_j = \operatorname{Hom}_{\mathcal{C}}(\mathcal{A}(-j), \mathcal{A})$, we define a right $\Gamma(\mathcal{A})$-action by $xr = F(s^{-i}(r))(x)$. Clearly

$xr \in F(\mathcal{A}(-i-j))$ and, for every $w \in \Gamma(\mathcal{A})_l$,

$$\begin{aligned} x(rw) &= F(s^{-i}(rw))(x) = F(s^{-i}(rs^{-j}(w)))(x) \\ &= F(s^{-i}(r)s^{-i-j}(w))(x) = F(s^{-i-j}(w))F(s^{-i}(r))(x) \\ &= (xr)w. \end{aligned}$$

Hence $\underline{F}(\mathcal{A})$ has a natural graded right $\Gamma(\mathcal{A})$-module structure.

Proposition 1.1. *Let $(\mathcal{C}, \mathcal{A}, s)$ be a triple as above, and let F be a contravariant functor from $\mathcal{C}$ to* Mod k. *Then there is a natural transformation $\sigma : F \longrightarrow \mathrm{Hom}_{\mathrm{Gr}\ A}(\Gamma(-), \underline{F}(\mathcal{A}))$ such that $\sigma_{\mathcal{A}(-i)}$ are isomorphisms for all $i \in \mathbb{Z}$, where $A = \Gamma(\mathcal{A})$.*

Proof. Let $\mathcal{M}$ be an object in $\mathcal{C}$. For every $x \in F(\mathcal{M})$ and $m_j \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{A}(-j), \mathcal{M})$, we define $\sigma_{\mathcal{M}} : F(\mathcal{M}) \longrightarrow \mathrm{Hom}_{\mathrm{Gr}\ A}(\Gamma(\mathcal{M}), \underline{F}(\mathcal{A}))$ by

$$x \longmapsto \{\sigma_{\mathcal{M}}(x) : m_j \longmapsto F(m_j)(x)\}.$$

We claim that $\sigma_{\mathcal{M}}(x)$ is an A-homomorphism. For every $r \in A_i = \mathrm{Hom}_{\mathcal{C}}(\mathcal{A}(-i), \mathcal{A})$, $\sigma_{\mathcal{M}}(x)$ maps $m_j r$ to

$$\begin{aligned} F(m_j r)(x) &= F(m_j s^{-j}(r))(x) = F(s^{-j}(r))F(m_j)(x) \\ &= (F(m_j)(x))r = \sigma_{\mathcal{M}}(x)(m_j)r. \end{aligned}$$

Hence $\sigma_{\mathcal{M}}(x)$ is an A-homomorphism and consequently $\sigma_{\mathcal{M}}$ is well defined.

For every morphism $f : \mathcal{M} \longrightarrow \mathcal{N}$ in $\mathcal{C}$, consider the following diagram:

$$\begin{array}{ccc} F(\mathcal{N}) & \xrightarrow{\sigma_{\mathcal{N}}} & \mathrm{Hom}_{\mathrm{Gr}\ A}(\Gamma(\mathcal{N}), \underline{F}(\mathcal{A})) \\ {\scriptstyle F(f)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Hom}_{\mathrm{Gr}\ A}(\Gamma(f), \underline{F}(\mathcal{A}))} \\ F(\mathcal{M}) & \xrightarrow{\sigma_{\mathcal{M}}} & \mathrm{Hom}_{\mathrm{Gr}\ A}(\Gamma(\mathcal{M}), \underline{F}(\mathcal{A})). \end{array}$$

For $y \in F(\mathcal{N})$ and $m \in \Gamma(\mathcal{M})_j$,

$$\mathrm{Hom}_{\mathrm{Gr}\ A}(\Gamma(f), \underline{F}(\mathcal{A}))\sigma_{\mathcal{N}}(y) :\ m \longmapsto \sigma_{\mathcal{N}}(y)(fm) = F(fm)(y)$$

and

$$\sigma_{\mathcal{M}} F(f)(y) :\ m \longmapsto F(m)(F(f)(y)) = F(fm)(y).$$

Hence σ is a natural transformation.

If $\mathcal{M} = \mathcal{A}(-i)$, $\Gamma(\mathcal{A}(-i)) \cong \Gamma(\mathcal{A})(-i)$. For every $x \in F(\mathcal{A}(-i))$, $\sigma_{\mathcal{A}(-i)}(x)$ is an A-homomorphism sending 1 to x. Hence $\sigma_{\mathcal{A}(-i)}$ is an isomorphism for all i. $\square$

The next lemma can be proved easily by using the Five Lemma [Ro, 3.32] as in the proof of Watts' original theorem [Ro, 3.33].

Lemma 1.2. *Let F and H be two contravariant left exact functors from $\mathcal{C}$ to another category $\mathcal{D}$. Suppose $\{\mathcal{G}_i\}_I$ is a set of generators such that every object $\mathcal{M} \in \mathcal{C}$ is finitely presented by $\{\mathcal{G}_i\}_I$. If $\sigma : F \longrightarrow H$ is a natural transformation and $\sigma_{\mathcal{G}_i}$ is an isomorphism for all $i \in I$, then σ is a natural isomorphism from F to H.*

Now we apply Proposition 1.1 and Lemma 1.2 to (gr A, A_A, s).

Theorem 1.3 (Watts' Theorem for gr A)**.** *Let A be a graded right noetherian algebra. Let F be a contravariant left exact functor from* gr A *to* Mod k. *Then $F \cong \mathrm{Hom}_{\mathrm{Gr}\ A}(-, B)$ for a $\mathbb{Z}$-graded (not necessarily noetherian) right A-module $B = \underline{F}(A_A)$.*

Proof. First, the functor Γ is naturally isomorphic to the identity functor and the algebra A is naturally isomorphic to $\Gamma(A_A)$. Hence, by Proposition 1.1, $\sigma : F \longrightarrow \mathrm{Hom}_{\mathrm{Gr}\ A}(-, B)$ is a natural transformation. Note that $\mathrm{Hom}_{\mathrm{Gr}\ A}(-, B)$ is a left exact functor. Since A is right noetherian, $\{A(i) \mid i \in \mathbb{Z}\}$ is a set of generators and every object in gr A is finitely presented. By Proposition 1.1, the hypotheses of Lemma 1.2 hold. Therefore $F \cong \mathrm{Hom}_{\mathrm{Gr}\ A}(-, B)$. □

Similarly a version of Watts' theorem holds for the triple (qgr $A, \mathcal{A}, s$). By [AZ, 4.5], if A satisfies the condition χ_1, then A can be recovered from the triple (qgr $A, \mathcal{A}, s$) up to $\mathfrak{m}$-torsion. Recall that A satisfies χ_i if $\underline{\mathrm{Ext}}^i_{\mathrm{gr}\ A}(A/\mathfrak{m}, M)$ are finite dimensional over k for all graded noetherian right A-modules M. If χ_i holds for all i, then we say A satisfies χ.

Theorem 1.4 (Watts' Theorem for qgr A)**.** *Let A be an $\mathbb{N}$-graded right noetherian algebra satisfying χ_1. Let F be a contravariant left exact functor from* qgr A *to* Mod k*. If $B = \underline{F}(\mathcal{A})_{\geq 0}$ is a noetherian right A-module, then $F \cong \mathrm{Hom}_{\mathrm{qgr}\ A}(-, \mathcal{B})$ where $\mathcal{B} = \pi(B)$.*

Proof. Since A satisfies χ_1, by [AZ, 4.6(2)] we may assume that $A = \Gamma(\mathcal{A})_{\geq 0}$. By Proposition 1.1, $\sigma : F \longrightarrow \mathrm{Hom}_{\mathrm{Gr}\ \Gamma(\mathcal{A})}(\Gamma(-), \underline{F}(\mathcal{A}))$ is a natural transformation. Clearly the following is also a natural transformation:

$$\mathrm{Hom}_{\mathrm{Gr}\ \Gamma(\mathcal{A})}(\Gamma(-), \underline{F}(\mathcal{A})) \to \mathrm{Hom}_{\mathrm{Gr}\ A}(\Gamma(-)_{\geq 0}, \underline{F}(\mathcal{A})_{\geq 0}) \to \mathrm{Hom}_{\mathrm{QGr}\ A}(\pi\Gamma(-), \mathcal{B}).$$

Since $\pi\Gamma \cong Id_{\mathrm{qgr}\ A}$ [AZ, 4.5] and B is noetherian, we have a natural transformation

$$\eta : \mathrm{Hom}_{\mathrm{Gr}\ \Gamma(\mathcal{A})}(\Gamma(-), \underline{F}(\mathcal{A})) \longrightarrow \mathrm{Hom}_{\mathrm{QGr}\ A}(-, \mathcal{B}) = \mathrm{Hom}_{\mathrm{qgr}\ A}(-, \mathcal{B}).$$

Since A satisfies χ_1, by [AZ, 3.13], for $i \gg 0$,

$$\mathrm{Hom}_{\mathrm{qgr}\ A}(\mathcal{A}(-i), \mathcal{B}) \cong B_i = F(\mathcal{A}(-i)) \cong \mathrm{Hom}_{\mathrm{Gr}\ \Gamma(\mathcal{A})}(\Gamma(\mathcal{A}(-i)), \underline{F}(\mathcal{A})).$$

Hence $\eta_{\mathcal{A}(-i)}$ is an isomorphism for $i \gg 0$. Thus $\eta\sigma$ is a natural transformation from F to $\mathrm{Hom}_{\mathrm{qgr}\ A}(-, \mathcal{B})$ such that $(\eta\sigma)_{\mathcal{A}(-i)}$ is an isomorphism for $i \gg 0$.

For every p, $\{\mathcal{A}(-i) | i \geq p\}$ is a set of generators for the noetherian category qgr A. Then the hypotheses of Lemma 1.2 hold and therefore $F \cong \mathrm{Hom}_{\mathrm{qgr}\ A}(-, \mathcal{B})$. □

2. Duality theorems

Let A be a graded right noetherian algebra and let M and N be graded right noetherian A-modules. Suppose the projective dimension of M, denoted by $pd(M)$, is $d < \infty$. Then $\mathrm{Ext}^i_{\mathrm{gr}\ A}(M, -) = 0$ for all $i \geq d + 1$. Hence a short exact sequence

$$0 \longrightarrow K \longrightarrow L \longrightarrow N \longrightarrow 0$$

yields a long exact sequence in Mod k

$$\begin{array}{l} 0 \longrightarrow \mathrm{Hom}_{\mathrm{gr}\ A}(M, K) \longrightarrow \mathrm{Hom}_{\mathrm{gr}\ A}(M, L) \longrightarrow \mathrm{Hom}_{\mathrm{gr}\ A}(M, N) \longrightarrow \\ \quad \mathrm{Ext}^1_{\mathrm{gr}\ A}(M, K) \longrightarrow \mathrm{Ext}^1_{\mathrm{gr}\ A}(M, L) \longrightarrow \mathrm{Ext}^1_{\mathrm{gr}\ A}(M, N) \longrightarrow \\ \qquad\qquad \cdots \qquad \cdots \\ \quad \mathrm{Ext}^d_{\mathrm{gr}\ A}(M, K) \longrightarrow \mathrm{Ext}^d_{\mathrm{gr}\ A}(M, L) \longrightarrow \mathrm{Ext}^d_{\mathrm{gr}\ A}(M, N) \longrightarrow 0. \end{array}$$

Applying the contravariant exact functor $V \longmapsto V^* = \mathrm{Hom}_k(V,k)$, we obtain

$$\begin{array}{l} 0 \longleftarrow \mathrm{Hom}_{\mathrm{gr}\ A}(M,K)^* \longleftarrow \mathrm{Hom}_{\mathrm{gr}\ A}(M,L)^* \longleftarrow \mathrm{Hom}_{\mathrm{gr}\ A}(M,N)^* \longleftarrow \\ \qquad \mathrm{Ext}^1_{\mathrm{gr}\ A}(M,K)^* \longleftarrow \mathrm{Ext}^1_{\mathrm{gr}\ A}(M,L)^* \longleftarrow \mathrm{Ext}^1_{\mathrm{gr}\ A}(M,N)^* \longleftarrow \\ \qquad \qquad \cdots \qquad \cdots \\ \qquad \mathrm{Ext}^d_{\mathrm{gr}\ A}(M,K)^* \longleftarrow \mathrm{Ext}^d_{\mathrm{gr}\ A}(M,L)^* \longleftarrow \mathrm{Ext}^d_{\mathrm{gr}\ A}(M,N)^* \longleftarrow 0. \end{array}$$

Thus $\mathrm{Ext}^d_{\mathrm{gr}\ A}(M,-)^*$ is a contravariant left exact functor from gr A to Mod k. By Theorem 1.3, $\mathrm{Ext}^d_{\mathrm{gr}\ A}(M,-)^* \cong \mathrm{Hom}_{\mathrm{Gr}\ A}(-,B)$ for some graded right A-module B. By the long exact sequence above, $\{\mathrm{Ext}^{d-i}_{\mathrm{gr}\ A}(M,-)^* \mid i \geq 0\}$ is a δ-functor in the sense of [H2, p.205]. Since $\{\mathrm{Ext}^i_{\mathrm{gr}\ A}(-,B) \mid i \geq 0\}$ is a universal δ-functor [H2, III.1.4], there is a natural transformation

$$\theta^i : \mathrm{Ext}^i_{\mathrm{Gr}\ A}(-,B) \longrightarrow \mathrm{Ext}^{d-i}_{\mathrm{gr}\ A}(M,-)^*$$

with θ^0 being the inverse of the natural isomorphism given in Theorem 1.3 for the functor $F = \mathrm{Ext}^d_{\mathrm{gr}\ A}(M,-)^*$. Hence we have proved part (a) of the following theorem.

Theorem 2.1. *Let M be a noetherian graded right A-module with $pd(M) = d$.*

(a) For each $i \geq 0$, there is a natural transformation

$$\theta^i : \mathrm{Ext}^i_{\mathrm{Gr}\ A}(-,B) \longrightarrow \mathrm{Ext}^{d-i}_{\mathrm{gr}\ A}(M,-)^*$$

where B is a graded right A-module and θ_0 is the inverse of the natural isomorphism given in Theorem 1.3.

(b) These θ^i are isomorphisms for all i if and only if $\mathrm{Ext}^{d-i}_{\mathrm{gr}\ A}(M, A(n)) = 0$ for all $i \geq 1$ and all $n \in \mathbb{Z}$.

Proof. (b) If θ^i is an isomorphism and $i \geq 1$, then

$$\mathrm{Ext}^{d-i}_{\mathrm{gr}\ A}(M, A(n)) \cong \mathrm{Ext}^i_{\mathrm{Gr}\ A}(A(n), B)^* = 0.$$

Conversely suppose that $\mathrm{Ext}^{d-i}_{\mathrm{gr}\ A}(M, A(n)) = 0$ for all $n \in \mathbb{Z}$ and all $i \geq 1$. Since $\{A(n) \mid n \in \mathbb{Z}\}$ is a set of generators for gr A, by [H2, p.206], $\mathrm{Ext}^{d-i}_{\mathrm{gr}\ A}(M,-)^*$ is coeffaceable for all $i \geq 1$. By [H2, III 1.3A], $\{\mathrm{Ext}^{d-i}_{\mathrm{gr}\ A}(M,-)^* \mid i \geq 0\}$ is universal. The derived functor $\{\mathrm{Ext}^i_{\mathrm{Gr}\ A}(-,B) \mid i \geq 0\}$ is always a universal δ-functor [H2, III.1.4]. Two universal δ-functors are naturally isomorphic if θ^0 is a natural isomorphism. □

From now on we consider the triple (qgr $A, \mathcal{A}, s$).

Theorem 2.2. *Let A be a graded right noetherian ring satisfying χ_1. Let $\mathcal{M}$ be in* qgr A *with* $\max\{\, i \mid \mathrm{Ext}^i_{\mathrm{qgr}\ A}(\mathcal{M},\mathcal{N}) \neq 0$ *for some* $\mathcal{N}\} = d < \infty$.

(a) If $B = \bigoplus_{n\geq 0} \mathrm{Ext}^d_{\mathrm{qgr}\ A}(\mathcal{M}, \mathcal{A}(-n))^*$ is a noetherian graded right A-module, then there is a natural transformation

$$\theta^i : \mathrm{Ext}^i_{\mathrm{qgr}\ A}(-,\mathcal{B}) \longrightarrow \mathrm{Ext}^{d-i}_{\mathrm{qgr}\ A}(\mathcal{M},-)^*$$

for all i, with θ^0 being the inverse of the natural isomorphism given in Theorem 1.4.

(b) Suppose A satisfies χ. Then θ^i is a natural isomorphism for all i if and only if $\mathrm{Ext}^{d-i}_{\mathrm{qgr}\ A}(\mathcal{M}, \mathcal{A}(-n)) = 0$ for all $i \geq 1$ and all $n \gg 0$.

Proof. The proof of part (a) is very similar to the proof of Theorem 2.1(a), namely using the argument before Theorem 2.1. The only difference is that in this case we use Theorem 1.4, instead of Theorem 1.3, for the functor $F = \mathrm{Ext}^d_{\mathrm{qgr}\ A}(\mathcal{M}, -)^*$. For part (b) we need the following modification.

Suppose A satisfies χ. If θ^i is an isomorphism and $i \geq 1$, then, by Serre's finiteness theorem [AZ, 7.5],

$$\mathrm{Ext}^{d-i}_{\mathrm{qgr}\ A}(\mathcal{M}, \mathcal{A}(-n))^* \cong \mathrm{Ext}^i_{\mathrm{qgr}\ A}(\mathcal{A}(-n), \mathcal{B}) \cong \mathrm{H}^i(X, \mathcal{B}(n)) = 0$$

for all $n \gg 0$. Conversely if $\mathrm{Ext}^{d-i}_{\mathrm{qgr}\ A}(\mathcal{M}, \mathcal{A}(-n))^* = 0$ for all $i \geq 1$ and all $n \gg 0$, then $\{\mathrm{Ext}^{d-i}_{\mathrm{qgr}\ A}(\mathcal{M}, -)^* | i \geq 0\}$ is a universal δ-functor because $\{\mathcal{A}(-n) | n \geq p\}$ is a set of generators for qgr A for every p. Two universal δ-functors are naturally isomorphic, and hence each θ^i is a natural isomorphism. □

Let $M = \bigoplus_i M_i$ be a $\mathbb{Z}$-graded module. We say M is *left bounded* (respectively *right bounded*) if $M_i = 0$ for all $i \ll 0$ (respectively $i \gg 0$). If A satisfies χ, then by [AZ, 7.5], each $\mathrm{H}^q(X, \mathcal{A}(i))$ is finite dimensional over k and $\underline{\mathrm{H}}^d(X, \mathcal{A}) := \bigoplus_i \mathrm{H}^q(X, \mathcal{A}(i))$ is right bounded for all $q \geq 1$. For a graded module $M = \bigoplus_i M_i$, let M^* denote $\bigoplus_i M^*_{-i}$. Hence $\underline{\mathrm{H}}^q(X, \mathcal{A})^*$ is left bounded for all $q \geq 1$. Now we prove the Serre duality theorem.

Theorem 2.3 (The Serre Duality for proj A)**.** *Let A be a graded right noetherian ring satisfying χ_1, and assume that $cd($ proj $A) = d < \infty$.*

1. Suppose $\underline{\mathrm{H}}^d(X, \mathcal{A})^*_{\geq 0}$ is a noetherian right A-module. Denote $\pi(\underline{\mathrm{H}}^d(X, \mathcal{A})^*)$ by ω^0.

(a) For each i, there is a natural transformation

$$\theta^i : \mathrm{Ext}^i_{\mathrm{qgr}\ A}(-, \omega^0) \longrightarrow \mathrm{H}^{d-i}(X, -)^*$$

where θ^0 is a natural isomorphism.

(b) Suppose A satisfies χ. Then θ^i are natural isomorphisms for all i if and only if $\underline{\mathrm{H}}^{d-i}(X, \mathcal{A})$ is finite dimensional for $d > i \geq 1$ and $\underline{\mathrm{H}}^0(X, \mathcal{A})$ is left bounded.

2. Conversely, if there is a natural isomorphism

$$\theta^0 : \mathrm{Hom}_{\mathrm{qgr}\ A}(-, \omega^0) \longrightarrow \mathrm{H}^d(X, -)^*$$

for an object ω^0 in qgr *A, then $\underline{\mathrm{H}}^d(X, \mathcal{A})^*_{\geq 0}$ is a noetherian right A-module and $\omega^0 \cong \pi(\underline{\mathrm{H}}^d(X, \mathcal{A})^*)$.*

Proof. Part 1 is an immediate consequence of Theorem 2.2. It remains to prove part 2. Since A satisfies χ_1 and ω^0 is an object in qgr A, by [AZ, 4.5], $\Gamma(\omega^0)_{\geq 0}$ is a noetherian right A-module and $\omega^0 \cong \pi(\Gamma(\omega^0)_{\geq 0})$. Then part 2 follows from the isomorphism

$$\Gamma(\omega^0) \cong \bigoplus_i \mathrm{Hom}(\mathcal{A}(-i), \omega^0) \cong \bigoplus_i \mathrm{H}^d(X, \mathcal{A}(-i))^* = \underline{\mathrm{H}}^d(X, \mathcal{A})^*.$$

□

Following the Serre duality theorem in the commutative case [H2, III.7.6] and Theorem 2.3, it is reasonable to make the following definition.

Definition 2.4. (a) Let $X =$ proj A be a projective scheme with $cd(X) = d$. An object ω^0 in qgr A is called a *dualizing sheaf* of X if there is a natural isomorphism

$$\theta^0 : \mathrm{Hom}_{\mathrm{qgr}\ A}(-, \omega^0) \longrightarrow \mathrm{H}^d(X, -)^*.$$

(b) Suppose X has a dualizing sheaf ω^0. We say X is *classical Cohen-Macaulay* if

$$\theta^i : \mathrm{Ext}^i_{\mathrm{qgr}\ A}(-, \omega^0) \longrightarrow \mathrm{H}^{d-i}(X, -)^*$$

are natural isomorphisms for all i.

It is easy to see that a dualizing sheaf is unique, up to isomorphism, if it exists. By the definition, existence of a dualizing sheaf is independent of the choice of the shift operator. If proj A has a dualizing sheaf ω^0, then, for each i, there is a natural transformation $\theta^i : \mathrm{Ext}^i_{\mathrm{qgr}\ A}(-, \omega^0) \longrightarrow \mathrm{H}^{d-i}(X, -)^*$. Under some hypotheses Theorem 2.3 gives a sufficient and necessary condition for a dualizing sheaf to exist. Namely, to show the existence of a dualizing sheaf is equivalent to showing that $\underline{\mathrm{H}}^d(X, \mathcal{A})^*_{\geq 0}$ is a noetherian right A-module. By [AZ, 7.9], if A is noetherian and satisfies χ, then $\underline{\mathrm{H}}^d(X, \mathcal{A})^*$ is a noetherian left A-module. If A is commutative, then left and right module structures are the same. Hence Theorem 2.3 re-proves the Serre duality theorem in the commutative case.

3. PI rings

It is unknown if $\underline{\mathrm{H}}^d(X, \mathcal{A})^*$ is a noetherian right A-module for all noetherian locally finite algebras satisfying χ. This problem is solved for the following case.

Let A be a noetherian graded algebra such that each $\mathfrak{m}$-torsion-free prime factor of A has a homogeneous normal element of positive degree. By [AZ, 8.12(2)], A satisfies χ and $cd($ proj $A) \leq \mathrm{Kdim}(A_A) - 1 < \infty$ where Kdim is Krull (Rentschler-Gabriel) dimension. Familiar examples of such rings are noetherian PI rings and quantum matrix algebras. A bimodule is called noetherian if it is both left and right noetherian.

Theorem 3.1. *Let A be a graded left and right noetherian algebra such that each $\mathfrak{m}$-torsion-free prime factor of A has a normal element of positive degree. Let M be a graded noetherian A-bimodule. Then $\underline{\mathrm{H}}^i(X, \mathcal{M})^*$ are noetherian A-bimodules for all $i > 0$.*

Proof. By [AZ, 7.9], $\underline{\mathrm{H}}^i(X, \mathcal{M})^*$ are left noetherian for $i > 0$. We need to show that these are right noetherian. By [AZ, 7.2.(2)], it suffices to show that $\underline{\mathrm{H}}^i_{\mathfrak{m}}(M)^*$ are right noetherian for all i, where $\underline{\mathrm{H}}^i_{\mathfrak{m}}(M)$ is defined by

$$\underline{\mathrm{H}}^i_{\mathfrak{m}}(M) = \lim_{n\to\infty} \underline{\mathrm{Ext}}^i_{\mathrm{gr}\ A}(A/\mathfrak{m}^n, M) \cong \lim_{n\to\infty} \underline{\mathrm{Ext}}^i_{\mathrm{gr}\ A}(A/A_{\geq n}, M). \tag{3-1}$$

Since $\lim_{n\to\infty}$ is exact, $\{ \underline{\mathrm{H}}^i_{\mathfrak{m}}(M)^* \mid i \in \mathbb{Z}\}$ is a δ-functor in the sense of [H2, p. 205]. By [AZ, 7.2 (2) and 3.6(3)], we have the following statements:

(i) $\underline{\mathrm{H}}^i_{\mathfrak{m}}(-)^* = 0$ for all $i < 0$ and $i > cd($ proj $A) + 1$.
(ii) $\underline{\mathrm{H}}^0_{\mathfrak{m}}(N)^* = \tau(N)^*$ and hence it is always finite dimensional.
(iii) If N is finite dimensional, then $\underline{\mathrm{H}}^0_{\mathfrak{m}}(N)^* = N^*$ and $\underline{\mathrm{H}}^i_{\mathfrak{m}}(N)^* = 0$ for all $i \geq 1$.
(iv) $\underline{\mathrm{H}}^i_{\mathfrak{m}}(N)^*$ is left bounded for all i and all noetherian A-modules N.

Next we use induction on $\mathrm{Kdim}(M_A)$ to prove that $\underline{\mathrm{H}}^i_{\mathfrak{m}}(M)^*$ are right noetherian for all i. If $\mathrm{Kdim}(M_A) = 0$, i.e., M is finite dimensional, then the statement is obvious. Now let M be a bimodule with $\mathrm{Kdim}(M_A) = n > 0$ and suppose the statement holds for all noetherian bimodules N with $\mathrm{Kdim}(N_A) < n$. By using a long exact sequence, we may assume that M is a critical bimodule and has prime annihilators on both sides. Let $P = ann({}_AM)$ and $Q = ann(M_A)$. Since M is not $\mathfrak{m}$-torsion, both A/P and A/Q are not $\mathfrak{m}$-torsion. Let x be a regular normal element of positive degree in A/P and let σ be the automorphism of A/P determined by $xa = \sigma(a)x$ for all $a \in A/P$. We may think M as an $(A/P, A)$-bimodule. The twisted module ${}^\sigma M$ is defined by $a * m = \sigma(a)m$ for all $a \in A/P$ and $m \in M$. Hence ${}^\sigma M$ is an $(A/P, A)$-bimodule. The left multiplication by x defines a homomorphism from M to ${}^\sigma M(d)$, where $d = \deg(x)$. Since M is critical, we obtain a short exact sequence of bimodules

$$0 \longrightarrow M \longrightarrow {}^\sigma M(d) \longrightarrow N \longrightarrow 0 \tag{3-2}$$

where $N = {}^\sigma M(d)/x\, {}^\sigma M(d)$. Applying $\underline{\mathrm{H}}^{\cdot}_{\mathfrak{m}}(-)^*$ to (3-2), we get

$$\cdots \longrightarrow \underline{\mathrm{H}}^i_{\mathfrak{m}}(N)^* \longrightarrow \underline{\mathrm{H}}^i_{\mathfrak{m}}({}^\sigma M(d))^* \longrightarrow \underline{\mathrm{H}}^i_{\mathfrak{m}}(M)^* \longrightarrow \underline{\mathrm{H}}^{i-1}_{\mathfrak{m}}(N)^* \longrightarrow \cdots. \tag{3-3}$$

Note that the right A-module structure of $\underline{\mathrm{H}}^i_{\mathfrak{m}}(M)^*$ comes from the left A-module structure of M. Then the map $\underline{\mathrm{H}}^i_{\mathfrak{m}}({}^\sigma M(d))^* \longrightarrow \underline{\mathrm{H}}^i_{\mathfrak{m}}(M)^*$ is the right multiplication by x. Hence the cokernel of this right multiplication is $\underline{\mathrm{H}}^i_{\mathfrak{m}}(M)^*/\underline{\mathrm{H}}^i_{\mathfrak{m}}(M)^*x$, which is a submodule of $\underline{\mathrm{H}}^{i-1}_{\mathfrak{m}}(N)^*$ by (3-3). Since M is critical, $\mathrm{Kdim}(N_A) < \mathrm{Kdim}(M_A)$. By the induction hypothesis, $\underline{\mathrm{H}}^{i-1}_{\mathfrak{m}}(N)^*$ is right noetherian and hence $\underline{\mathrm{H}}^i_{\mathfrak{m}}(M)^*/\underline{\mathrm{H}}^i_{\mathfrak{m}}(M)^*x$ is right noetherian. Suppose $\underline{\mathrm{H}}^i_{\mathfrak{m}}(M)^*/\underline{\mathrm{H}}^i_{\mathfrak{m}}(M)^*x = \sum_{i=1}^n \overline{m_i}A$ for some $m_i \in \underline{\mathrm{H}}^i_{\mathfrak{m}}(M)^*$. Since $\underline{\mathrm{H}}^i_{\mathfrak{m}}(M)^*$ is left bounded, by induction on the degree of elements we obtain that $\underline{\mathrm{H}}^i_{\mathfrak{m}}(M)^* = \sum_{i=1}^n m_iA$, which is right noetherian. □

The following corollary is an immediate consequence of Theorems 2.3 and 3.1.

Corollary 3.2. *Let A be as in Theorem 3.1. Then* proj A *has a dualizing sheaf.*

4. Balanced dualizing complexes

Let $\mathcal{C}$ be an abelian category with enough injectives. Let $\boldsymbol{D}(\mathcal{C})$ denote the derived category of $\mathcal{C}$ and $\boldsymbol{D}^+(\mathcal{C})$ the subcategory of $\boldsymbol{D}(\mathcal{C})$ consisting of bounded below complexes. We can define $\mathrm{RHom}_{\mathcal{C}}(M^{\cdot}, N^{\cdot})$ for $N^{\cdot} \in \boldsymbol{D}^+(\mathcal{C})$ and $M^{\cdot} \in \boldsymbol{D}(\mathcal{C})$, by replacing $N^{\cdot}$ with a quasi-isomorphic complex of injectives. Write $\boldsymbol{D}^+_{\mathrm{f}}(\mathcal{C})$ (respectively $\boldsymbol{D}^{\mathrm{b}}_{\mathrm{f}}(\mathcal{C})$) for the subcategory of bounded below (respectively bounded) complexes whose cohomologies are noetherian in $\mathcal{C}$. In this section $\mathcal{C}$ will be either QGr A or Gr A for a graded left and right noetherian ring. Regarding shifts, we denote by $M^{\cdot}[n]$ the shift by n of the complex $M^{\cdot}$, and by $M(n)$ the shift by degree n as defined before if M is in either QGr A or Gr A. See [Ye] and [H1] for details on derived categories.

Lemma 4.1. *Let $M^{\cdot} \in \boldsymbol{D}^{\mathrm{b}}_{\mathrm{f}}(\mathrm{Gr}\ A)$ and $N^{\cdot} \in \boldsymbol{D}^+(\mathrm{Gr}\ A)$, and let $\mathcal{M}^{\cdot} = \pi(M^{\cdot})$ and $\mathcal{N}^{\cdot} = \pi(N^{\cdot})$. Then*

$$\mathrm{Ext}^q_{\mathrm{QGr}\ A}(\mathcal{M}^{\cdot}, \mathcal{N}^{\cdot}) \cong \lim_{n\to\infty} \mathrm{Ext}^q_{\mathrm{Gr}\ A}(M^{\cdot}_{\geq n}, N^{\cdot}) \tag{4-1}$$

for all $q \in \mathbb{Z}$.

Proof. There is a bi-functorial isomorphism

$$\mathrm{Hom}_{\mathrm{QGr}\ A}(\mathcal{M}, \mathcal{N}) \cong \lim_{n\to\infty} \mathrm{Hom}_{\mathrm{Gr}\ A}(M_{\geq n}, N) \tag{4-2}$$

for all $M \in \mathrm{gr}\ A$ and $N \in \mathrm{Gr}\ A$ (see [AZ, (2.2.1)]). By the exactness of the functors $M \longmapsto M_{\geq n}$ and $\lim_{n\to\infty}$, and arguing as in the proof of [Ye, 2.2], we see that the derived bi-functor $\mathrm{R}\lim_{n\to\infty} \mathrm{Hom}^{\cdot}_{\mathrm{Gr}\ A}(M^{\cdot}_{\geq n}, N^{\cdot})$ is defined for $N^{\cdot} \in \boldsymbol{D}^{+}(\mathrm{Gr}\ A)$ and $M^{\cdot} \in \boldsymbol{D}(\mathrm{Gr}\ A)$. We may assume that $N^{\cdot}$ is a bounded below complex of injectives [Ye, 4.2], and then the R can be omitted.

Now fix $N^{\cdot}$. By [H1, I.7.1(i)] the isomorphism (4-2) implies an isomorphism

$$\mathrm{RHom}^{\cdot}_{\mathrm{QGr}\ A}(\mathcal{M}^{\cdot}, \mathcal{N}^{\cdot}) \cong \mathrm{R}\lim_{n\to\infty} \mathrm{Hom}^{\cdot}_{\mathrm{Gr}\ A}(M^{\cdot}_{\geq n}, N^{\cdot})$$

for $M^{\cdot} \in \boldsymbol{D}^{\mathrm{b}}_{\mathrm{f}}(\mathrm{Gr}\ A)$. The lemma is proved by passing to cohomologies. $\square$

Observe that by taking $M^{\cdot} = A$ in (4-1) we get

$$\mathrm{H}^q(X, \mathcal{N}^{\cdot}) \cong \lim_{n\to\infty} \mathrm{Ext}^q_{\mathrm{Gr}\ A}(A_{\geq n}, N^{\cdot}).$$

By [AZ, 7.2(2)], $\mathrm{H}^q(X, \mathcal{M})$ is isomorphic to the local cohomology $\mathrm{H}^{q+1}_{\mathfrak{m}}(M)$ for $q \geq 1$, where $\underline{\mathrm{H}}^q_{\mathfrak{m}}(-)$ is defined by (3-1). By [Ye, 4.17 and 4.18], if A has a balanced dualizing complex (see [Ye, 3.3 and 4.1] for the definition), then there is a version of the local duality theorem. We will use the local duality theorem to prove the Serre duality theorem for such A.

Theorem 4.2. *Let A be a connected graded left and right noetherian ring and let $X =$ proj A. Suppose that A has a balanced dualizing complex $R^{\cdot}$.*

1. Let $M^{\cdot} \in \boldsymbol{D}^{\mathrm{b}}_{\mathrm{f}}(\mathrm{Gr}\ A)$ and $q \in \mathbb{Z}$. Then

$$\underline{\mathrm{H}}^q_{\mathfrak{m}}(M^{\cdot}) \cong \underline{\mathrm{Ext}}^{-q}_{\mathrm{Gr}\ A}(M^{\cdot}, R^{\cdot})^{*}, \tag{4-3}$$

and this is a locally finite and right bounded module.

2. Let $\mathcal{R}^{\cdot} = \pi(R^{\cdot})$ and $\mathcal{M}^{\cdot} = \pi(M^{\cdot})$. Then

$$\begin{aligned} \mathrm{H}^q(X, \mathcal{M}^{\cdot})^{*} &\cong \lim_{n\to\infty} \mathrm{H}^{q+1}_{\mathfrak{m}}(M^{\cdot}_{\geq n})^{*} \cong \mathrm{Ext}^{-(q+1)}_{\mathrm{QGr}\ A}(\mathcal{M}^{\cdot}, \mathcal{R}^{\cdot}) \\ &= \mathrm{Ext}^{-q}_{\mathrm{QGr}\ A}(\mathcal{M}^{\cdot}, \mathcal{R}^{\cdot}[-1]). \end{aligned} \tag{4-4}$$

In this sense, we call $\mathcal{R}^{\cdot}[-1]$ a dualizing complex for proj A.

3. A satisfies the condition χ and $cd(X) = d$, where $-(d+1) = \min\{\, i \mid \mathrm{H}^i(\mathcal{R}^{\cdot}) \neq 0 \,\}$.

4. Let $\omega^0 = \mathrm{H}^{-(d+1)}(\mathcal{R}^{\cdot})$. Then ω^0 is the dualizing sheaf for proj A.

5. The following conditions are equivalent:

(a) $X =$ proj A is classical Cohen-Macaulay.

(b) $\omega^0[d+1]$ is isomorphic to $\mathcal{R}^{\cdot}$ in $\boldsymbol{D}(\mathrm{QGr}\ A)$.

(c) $\mathrm{H}^q(R^{\cdot})$ is $\mathfrak{m}$-torsion for $q > -(d+1)$.

Remark. The dualizing complex $R^{\cdot}$ is a complex of bimodules, but when passing to QGr A we forget the left module structure of $R^{\cdot}$.

Proof. 1. This is [Ye, 4.18]. Note that $\underline{\mathrm{Ext}}^{-q}_{\mathrm{Gr}\ A}(M^{\cdot}, R^{\cdot})$ is a noetherian left A-module, so $\underline{\mathrm{H}}^q_{\mathfrak{m}}(M^{\cdot})$ is a locally finite, right bounded module.

2. The exact sequence

$$0 \longrightarrow A_{\geq n} \longrightarrow A \longrightarrow A/A_{\geq n} \longrightarrow 0,$$

when considered as a triangle in $\boldsymbol{D}(\mathrm{Gr}\ A)$, yields a long exact sequence

$$\cdots \to \mathrm{H}^q(M^{\cdot})_0 \to \mathrm{Ext}^q_{\mathrm{Gr}\ A}(A_{\geq n}, M^{\cdot}) \to \mathrm{Ext}^{q+1}_{\mathrm{Gr}\ A}(A/A_{\geq n}, M^{\cdot}) \to \mathrm{H}^{q+1}(M^{\cdot})_0 \to \cdots$$

for any $M^{\cdot} \in \boldsymbol{D}^{\mathrm{b}}_{\mathrm{f}}(\mathrm{Gr}\ A)$, where $\mathrm{H}^q(M^{\cdot})_0$ is the degree zero part of the q-th cohomology of $M^{\cdot}$. First taking $\lim_{n\to\infty}$ and then replacing $M^{\cdot}$ by $M^{\cdot}_{\geq l}$ in the above, we have

$$\cdots \longrightarrow \mathrm{H}^q(M^{\cdot}_{\geq l})_0 \longrightarrow \mathrm{H}^q(X, \mathcal{M}^{\cdot}) \longrightarrow \mathrm{H}^{q+1}_{\mathfrak{m}}(M^{\cdot}_{\geq l}) \longrightarrow \mathrm{H}^{q+1}(M^{\cdot}_{\geq l})_0 \longrightarrow \cdots$$

where $\mathcal{M}^{\cdot} = \pi(M^{\cdot}) = \pi(M^{\cdot}_{\geq l})$. Taking k-linear duals and using the fact that

$$\lim_{l\to\infty} \mathrm{H}^q(M^{\cdot}_{\geq l})^*_0 = 0,$$

we obtain the first isomorphism of (4-4). The second one is a consequence of part 1 and Lemma 4.1.

3. By part 1, for every noetherian right module M, $\underline{\mathrm{H}}^q_{\mathfrak{m}}(M)$ is right bounded for all q. By [AZ, 3.8(3)], A satisfies χ.

By [Ye, 4.2], we may assume $R^{\cdot}$ is a minimal injective complex (of right A-modules), i.e., each R^i is an injective hull of $ker(\partial_i)$, where $\partial_i : R^i \longrightarrow R^{i+1}$ is the coboundary map of $R^{\cdot}$. Since π is exact and has a right adjoint functor [AZ, (2.2.2)], $\mathcal{R}^{\cdot} = \pi(R^{\cdot})$ is a minimal injective complex in $\boldsymbol{D}^+(\mathrm{QGr}\ A)$. By the definition of d, we have $\mathrm{H}^q(\mathcal{R}^{\cdot}) = 0$ for all $q < -(d+1)$, and $\mathrm{H}^{-(d+1)}(\mathcal{R}^{\cdot}) \neq 0$. Hence $\mathcal{R}^q = 0$ for all $q < -(d+1)$ and, by (4-4),

$$\mathrm{H}^q(X, \mathcal{M}) \cong \mathrm{Ext}^{-(q+1)}_{\mathrm{QGr}\ A}(\mathcal{M}, \mathcal{R}^{\cdot})^* = 0.$$

Therefore $cd(X) \leq d$. By the definition of dualizing complex, $R^{\cdot} \in \boldsymbol{D}^+_{\mathrm{f}}(\mathrm{Gr}\ A)$, whence $\mathcal{R}^{\cdot} \in \boldsymbol{D}^+_{\mathrm{f}}(\mathrm{QGr}\ A)$ and $\omega^0 := \mathrm{H}^{-(d+1)}(\mathcal{R}^{\cdot}) \in \mathrm{qgr}\ A$. By (4-3)

$$\mathrm{Hom}_{\mathrm{QGr}\ A}(\mathcal{M}, \omega^0) \cong \mathrm{Ext}^{-(d+1)}_{\mathrm{QGr}\ A}(\mathcal{M}, \mathcal{R}^{\cdot}) \cong \mathrm{H}^d(X, \mathcal{M})^*. \tag{4-5}$$

Thus $\mathrm{H}^d(X, \omega^0) \neq 0$ and $cd(X) = d$.

4. Follows immediately from (4-5).

5. (a) $\Leftrightarrow$ (b) By Definiton 2.4, $X =$ proj A is classical Cohen-Macaulay if and only if $\mathrm{Ext}^i_{\mathrm{QGr}\ A}(\mathcal{M}, \omega^0) \cong \mathrm{H}^{d-i}(X, \mathcal{M})^*$ for all i and $\mathcal{M}$. By (4-4), this is equivalent to

$$\mathrm{Ext}^i_{\mathrm{QGr}\ A}(\mathcal{M}, \omega^0) \cong \mathrm{Ext}^{i-(d+1)}_{\mathrm{QGr}\ A}(\mathcal{M}, \mathcal{R}^{\cdot}) = \mathrm{Ext}^i_{\mathrm{QGr}\ A}(\mathcal{M}, \mathcal{R}^{\cdot}[-(d+1)]) \tag{4-6}$$

for all i and $\mathcal{M}$. But (4-6) holds if and only if $\mathcal{R}^{\cdot}[-(d+1)]$ is a minimal injective resolution of ω^0. This is saying that $\omega^0[d+1]$ is quasi-isomorphic to $\mathcal{R}^{\cdot}$.

(b) $\Leftrightarrow$ (c) The complex $\omega^0[d+1]$ is quasi-isomorphic to $\mathcal{R}^{\cdot}$ if and only if $\mathrm{H}^q(\mathcal{R}^{\cdot}) = 0$ for all $q > -(d+1)$. But $\mathrm{H}^q(\mathcal{R}^{\cdot}) = \pi(\mathrm{H}^q(R^{\cdot}))$. Therefore $\mathrm{H}^q(\mathcal{R}^{\cdot}) = 0$ if and only if $\mathrm{H}^q(R^{\cdot})$ is $\mathfrak{m}$-torsion. $\square$

Theorem 4.2.2 tells that if A has a balanced dualizing complex $R^{\cdot}$, then proj A has a dualizing complex $\mathcal{R}^{\cdot}[-1] \in \boldsymbol{D}^+_{\mathrm{f}}(\mathrm{QGr}\ A)$. P. Jørgensen in [Jø, 3.3] proved recently that a dualizing complex always exists in $\boldsymbol{D}^+(\mathrm{QGr}\ A)$. The question of when a dualizing complex exists in $\boldsymbol{D}^+_{\mathrm{f}}(\mathrm{QGr}\ A)$ is still open. It was proved in [Ye] that the following algebras have balanced dualizing complexes:

(a) graded noetherian AS-Gorenstein rings;
(b) graded noetherian rings finite over their centers;
(c) twisted homogeneous coordinate rings.

Finally, we state an immediate corollary of Theorem 4.2 for AS-Gorenstein rings. Recall that a connected graded ring A with injective dimension $d+1$ is called AS-Gorenstein if $\underline{\mathrm{Ext}}^{\cdot}_A(k, A) \cong k(e)[-(d+1)]$ for some $e \in \mathbb{Z}$ (see [Ye, 4.13(ii)]). By [Ye, 4.14] (which holds in this case too; the proof is exactly the same as the one of

[Ye, 4.14]), $R^{\cdot} = {}^{\sigma}A(-e)[d+1]$ is a balanced dualizing complex over A for some algebra automorphism σ.

Corollary 4.3. *Let A be a connected graded left and right noetherian AS-Gorenstein ring of injective dimension $d+1$, and let $X =$ proj A.*

1. *A satisfies the condition χ.*
2. *$cd(X) = d$, and $\omega^0 = \mathcal{A}(-e)$ is the dualizing sheaf for X.*
3. *X is classical Cohen-Macaulay.*

Remark. Presumably the Brown Representability Theorem, as used in [Jø], would imply that if the functor $(\Gamma_{\mathfrak{m}}(-))^*$, where $\Gamma_{\mathfrak{m}}(-)$ is defined to be $\underline{\mathrm{H}}^0_{\mathfrak{m}}(-)$ [Ye, page 56], has finite cohomological dimension, then it is representable on $\boldsymbol{D}(\mathrm{Gr}\ A)$ by $(\mathrm{R}\Gamma_{\mathfrak{m}}(A))^*$. However this still wouldn't make $(\mathrm{R}\Gamma_{\mathfrak{m}}(A))^*$ into a balanced dualizing complex in the sense of [Ye]. We thank the referee for calling attention to this point.

References

[AZ] M. Artin and J. J. Zhang, *Noncommutative projective schemes*, Adv. in Math. **109** (1994), 228–287. MR **96a:**14004

[H1] R. Hartshorne, *Residues and Duality*, Lecture Notes in Math. 20, Springer-Verlag, Berlin, 1966. MR **36:**5145

[H2] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York, 1977. MR **57:**3116

[Jø] P. Jørgensen, *Serre-Duality for* Tails(A), Proc. Amer. Math. Soc., this issue.

[Ro] J. J. Rotman, *An Introduction to Homological Algebra*, Academic Press, New York, 1979. MR **80k:**18001

[Ye] A. Yekutieli, *Dualizing complexes over noncommutative graded algebras*, J. Alg. **153** (1992), 41–84. MR **94a:**16077

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