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HEIGHT OF FLAT TORI

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Abstract. Relations between the height and the determinant of the Laplacian on the space of n-dimensional flat tori and the classical formulas of Kronecker and Epstein are established. Extrema of the height are shown to exist, and results for a global minimum for 2-d tori and a local minimum for 3-d tori are given, along with more general conjectures of Sarnak and Rankin.

1. Definitions and notation

The height of a compact Riemannian manifold M with smooth metric σ is the isospectral invariant defined by

$$h(M,\sigma) = Z'(0),$$

where $Z(s)$ is the zeta-regularization of the determinant of the Laplacian. More precisely, let $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues of M with respect to the Laplacian, then the determinant of the Laplacian is defined by $\det' \Delta = \prod_{\lambda_j \neq 0} \lambda_j$, and the zeta-regularization by

$$Z(s) = \sum_{\lambda_j \neq 0} \lambda_j^{-s}.$$

These are formally related through the identity $\det' \Delta = \prod_{\lambda_j \neq 0} \lambda_j = e^{-Z'(0)}$. In general, $Z(s)$ has meromorphic continuation to the whole complex plane and is regular at $s = 0$, see Sarnak [Sa].

Now let M be an n-dimensional torus and σ the flat metric, with the volume of M normalized to 1. Since such a torus can be viewed as $\mathbf{R}^n$ modulo a lattice of full rank and determinant 1, we see that the height is a function on lattices. Let $\mathbf{L}^n$ denote this space of lattices, and $\mathbf{H}^n$ be the upper-half plane defined by $\mathrm{GL}(n,\mathbf{R})/O(n)Z$ where Z is the center of $\mathrm{GL}(n,\mathbf{R})$. Then as is well known, we may identify $\mathbf{L}^n$ with $\mathrm{SL}(n,\mathbf{Z})\backslash\mathbf{H}^n$.

For a lattice $L \in \mathbf{L}^n$ with basis $v_1, \ldots, v_n \in \mathbf{R}^n$, we associate to it a matrix

$$V = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

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and a positive definite quadratic form $Q = VV^t$. The dual of a lattice L is the lattice $L^* = \{y | x \cdot y \in \mathbf{Z} \text{ for all } x \in L\}$, which has the corresponding matrix $V^* = (V^{-1})^t$ and quadratic form $Q^* = Q^{-1}$.

2. Explicit formulas

For $n = 2$, we have the following formula for the height, see Osgood-Phillips-Sarnak [OPS]. Let M be parametrized by $z = x + iy$ with $y > 0$ in $SL(2, \mathbf{Z})\backslash\mathbf{H}^2$. The quantity $Z'(0)$ is essentially given by Kronecker's Limit Formula, which is defined below in (2.4). In the present case,

$$k_2(z) = C_2 - \log(y|\eta(z)|^4), \tag{2.1}$$

where $C_2 = 2(\gamma - \log 2)$ with γ being Euler's constant, and $\eta(z)$ is the Dedekind eta-function. Then the height of a 2-d flat torus of area 1 is

$$h(M) = h(z) = \pi^{-1} k_2(z) + 2\log 2.$$

For $n = 3$, we employ GL(3) analogs of Kronecker's Limit Formula and the eta-function, see Efrat [Ef]. A 3-d torus may be parametrized by the Iwasawa decomposition

$$\tau = M = \begin{pmatrix} y_1 y_2 & y_1 x_2 & x_3 \\ & y_1 & x_1 \\ & & 1 \end{pmatrix},$$

where $x_1, x_2, x_3, y_1, y_2 \in \mathbf{R}$ with $y_1, y_2 > 0$. Also, let $x_4 = x_3 - x_1 x_2$ and $z_k = x_k + iy_k$ for $k = 1, 2$.

An analog of the eta function for GL(3) is

$$\begin{aligned} g(\tau) = \exp\left(-\frac{y_1^{1/2} y_2}{8\pi} E^*(z_1, \frac{3}{2})\right) \\ \times \prod_{(m,n)}{}' |1 - \exp(-2\pi y_2 |n z_1 - m| + 2\pi i (m x_2 + n x_4))|, \end{aligned}$$

where the product is taken over all nonzero $(m, n) \in \mathbf{Z}^2$ modulo ± 1.

For Kronecker's Limit Formula, an analog in GL(3) is

$$k_3(\tau) = C_3 - \frac{2}{3}\log(y_1 y_2^2) - 4\log g(\tau), \tag{2.2}$$

where C_3 is some constant.

Then the height of a 3-d flat torus M of volume 1 is given by

$$h(M) = h(\tau) = (2\pi)^{-1} k_3(z) + 2.$$

We proceed to derive a formula for the height in arbitrary dimension. For the moduli space of n-dimensional flat tori of volume 1, the eigenfunctions of a torus M are the exponentials and its eigenvalues are given by $Q^*[x]$ for $x \in \mathbf{Z}^n$. Hence, the zeta-regularization is

$$Z(M, s) = (2\pi)^{-2s} \sum_{x \in \mathbf{Z}^n}{}' Q^*[x]^{-s}, \qquad \operatorname{Re} s > n/2. \tag{2.3}$$

The sum on the right-hand side is essentially an Epstein zeta-function. For a positive definite quadratic form $Q[x]$, its associated Epstein zeta-function is defined by

$$Z_Q(s) = \sum_{x\in\mathbf{Z}^n}{}' Q[x]^{-s}, \qquad \operatorname{Re} s > n/2.$$

It has meromorphic continuation and functional equation, e.g. see Terras [Te1]:

Theorem 2.1 (Epstein)**.** *Let Q be the matrix of a positive definite quadratic form in n variables. Then $Z_Q(s)$ possesses the following properties*:

(i) *$Z_Q(s)$ has meromorphic continuation to the entire complex plane except for a simple pole at $s = n/2$ with residue $|Q|^{-1/2}\pi^{n/2}\Gamma(n/2)^{-1}$.*

(ii) *It satisfies the functional equation*

$$\pi^{-s}\Gamma(s)Z_Q(s) = |Q|^{-1/2}\pi^{-(n/2-s)}\Gamma(n/2-s)Z_{Q^{-1}}(n/2-s).$$

Next we state a generalization of Kronecker's Limit Formula. It gives the constant term in the Laurent expansion of $Z_Q(s)$ at the pole $s = n/2$, which is defined by

$$k_n(Q) = \lim_{s\to n/2}\left\{Z_Q(s) - \frac{|Q|^{-1/2}\pi^{n/2}\Gamma(n/2)^{-1}}{s-n/2}\right\}. \tag{2.4}$$

Employing the notation in Terras [Te2], we first decompose Q into block matrix form

$$Q = \begin{pmatrix} 1 & R^t \\ 0 & I \end{pmatrix}\begin{pmatrix} q & 0 \\ 0 & Q_{n-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ R & I \end{pmatrix},$$

where Q_{n-1} and R are $(n-1)\times(n-1)$ matrices. Set

$$z\{b\} = b^tR + i\sqrt{q(Q_{n-1}^{-1}[b])},$$

and

$$c_n = \begin{cases} \gamma - \log 2 - \frac{1}{2}\sum_{r=1}^{n/2-1} r^{-1} & \text{for } n \text{ even,} \\ \gamma - \sum_{r=0}^{(n-3)/2}(2r+1)^{-1} & \text{for } n \text{ odd.} \end{cases}$$

Then a generalization of Kronecker's Limit Formula derived by Epstein is:

Theorem 2.2 (Kronecker's Limit Formula)**.**

$$\begin{aligned} k_n(Q) &= Z_{Q_{n-1}}(n/2) + \pi^{n/2}|Q_{n-1}|^{-1/2}\Gamma(n/2)^{-1} \\ &\quad\times\left\{c_n - \log\left(q^{1/2}\left|\prod_{b\in\mathbf{Z}^{n-1}/\pm1}{}'(1-\exp(2\pi i z\{b\}))\right|^2\right)\right\}. \end{aligned}$$

We now relate $k_n(Q)$ to the height $Z'(M,0)$.

Theorem 2.3. *Let M be an n-dimensional flat torus of volume 1, and Q its associated quadratic form. Then the height of M is*

$$h(M) = Z'(M,0) = \pi^{-n/2}\Gamma(n/2)k_n(Q) + \psi(n/2) - \psi(1/2),$$

where $\psi(s) = \Gamma'(s)/\Gamma(s)$ is the logarithmic derivative of the gamma function.

Proof. First, we show that the height has the form

$$h(M) = Z'(M,0) = A_n + B_n k_n(Q),$$

for some constants A_n and B_n. Since $Z(M,s) = (2\pi)^{-2s} Z_{Q^*}(s)$ by (2.3), it suffices to show that

$$Z'_{Q^*}(0) = A'_n + B'_n k_n(Q),$$

for some constants A'_n and B'_n. Applying the functional equation in Theorem 2.1(ii),

$$Z_{Q^*}(s) = \pi^{2s-n/2} \frac{\Gamma(n/2-s)}{\Gamma(s)} Z_Q(n/2-s). \tag{2.5}$$

On the other hand, Z_Q has meromorphic continuation, so it may be written as

$$Z_Q(w) = \frac{a_{-1}}{w-n/2} + a_0 + O(w-n/2),$$

or setting $w = n/2 - s$,

$$Z_Q(n/2-s) = \frac{-a_{-1}}{s} + a_0 + O(s),$$

where a_0 is just $k_n(Q)$.

Substituting into (2.5),

$$Z_{Q^*}(s) = \pi^{2s-n/2} \frac{\Gamma(n/2-s)}{\Gamma(s)} \left\{ \frac{-a_{-1}}{s} + k_n(Q) + O(s) \right\}.$$

Because $\Gamma(s)$ has a simple pole at $s = 0$,

$$= sf(s) \left\{ \frac{-a_{-1}}{s} + k_n(Q) + O(s) \right\},$$

for some function $f(s) = b_0 + b_1 s + O(s^2)$ that is analytic at $s = 0$,

$$\begin{aligned} &= s(b_0 + b_1 s + O(s^2)) \left\{ \frac{-a_{-1}}{s} + k_n(Q) + O(s) \right\} \\ &= -b_0 a_{-1} + (-b_1 a_{-1} + b_0 k_n(Q))s + O(s^2). \end{aligned}$$

Differentiating,

$$Z'_{Q^*}(0) = -b_1 a_{-1} + b_0 k_n(Q).$$

Computing the constants is a straightforward exercise; their values are:

$$\begin{gathered} a_{-1} = \pi^{n/2} \Gamma(n/2)^{-1}, \\ b_0 = 1/a_{-1}, \\ b_1 = \pi^{-n/2} \Gamma(n/2) \{2 \log \pi - \psi(n/2) - \gamma\}, \end{gathered}$$

where γ is Euler's constant. From these we obtain the values of A_n and B_n in the theorem. □

3. Existence of extrema

In this section, we prove the following existence theorem:

Theorem 3.1. *The height on the moduli space of n-dimensional flat tori of volume 1 attains a minimum.*

Rather than using the explicit formula of Theorem 2.3—which is more suitable for computational purposes—we express the Epstein zeta function in another form via an integral representation. This is an old technique due to Riemann, see [Te1].

Lemma 3.2 (Riemann)**.** *The Epstein zeta function $Z_Q(s)$ may be expressed as*

$$\pi^{-s}\Gamma(s)Z_Q(s) = \frac{|Q|^{-1/2}}{s-n/2} - \frac{1}{s} + \sum_{a\in\mathbf{Z}^n-0}\left(\int_1^\infty e^{-\pi Q[a]t}t^s\,\frac{dt}{t} + \int_1^\infty e^{-\pi Q^{-1}[a]t}t^{n/2-s}\frac{dt}{t}\right).$$

Proof. (Theorem 3.1) We view the moduli space of flat tori as the space of lattices $\mathbf{L}^n$ described in section 1. Given a lattice $L \in \mathbf{L}^n$, its minimal vector m_L is defined to be the vector $x \in \mathbf{Z}^n$ that minimizes its associated quadratic form $Q[x]$. A set of lattices whose minimal vector is of bounded length has compact closure, so on such a set the height attains a minimum. Therefore, it suffices to show that the height becomes arbitrarily large for lattices whose minimal vector is sufficiently short.

Now the volume of the lattices is normalized to 1, so $|m_L^*| \to 0$ as $|m_L| \to 0$. Hence by (2.3), all we need to prove is that $Z'_{Q^*}(0) \to +\infty$ as $|m_L^*| \to 0$.

By Lemma 3.2,

$$Z_{Q^*}(s) = Z_{Q^{-1}}(s) = \frac{\pi^s}{\Gamma(s)}\left(\frac{1}{s-n/2} - \frac{1}{s}\right) + \frac{\pi^s}{\Gamma(s)}\left\{\sum_{a\in\mathbf{Z}^n-0}\left(\int_1^\infty e^{-\pi Q^{-1}[a]t}t^s\,\frac{dt}{t} + \int_1^\infty e^{-\pi Q[a]t}t^{n/2-s}\,\frac{dt}{t}\right)\right\}.$$

At $\frac{d}{dt}|_{s=0}$, the first term is analytic at $s=0$ and independent of Q, so it contributes a constant C. Let $F_Q(s)$ denote the function in the braces. Then

$$\frac{d}{ds}Z_{Q^*}(s)\bigg|_{s=0} = C + \left[\frac{\pi^s}{\Gamma(s)}F'_Q(s) + \frac{\pi^s\log\pi}{\Gamma(s)}F_Q(s) + \frac{\pi^s}{\Gamma(s)}\left(\frac{-\Gamma'(s)}{\Gamma(s)}\right)F_Q(s)\right]_{s=0}.$$

Because $F_Q(0)$ and $F'_Q(0)$ clearly converge and $1/\Gamma(0) = 0$, the second and third terms vanish. In the last term, the factor $\frac{\pi^s}{\Gamma(s)}(\frac{-\Gamma'(s)}{\Gamma(s)})$ is a positive real number. Hence it suffices to check that $F_Q(0)$ tends to $+\infty$ as the shortest vector tends to zero.

By positivity of the terms,

$$F_Q(0) = \sum_{a\in\mathbf{Z}^n-0}\int_1^\infty e^{-\pi Q^{-1}[a]t}\,\frac{dt}{t} + \sum_{a\in\mathbf{Z}^n-0}\int_1^\infty e^{-\pi Q[a]t}t^{n/2}\,\frac{dt}{t} \geq \int_1^\infty \sum_{a\in\mathbf{Z}^n-0} e^{-\pi Q^{-1}[a]t}\,\frac{dt}{t}.$$

Now let r be the length of the minimal vector of the lattice L^*, and fix $T > 1$. Discarding all but multiples of the shortest vector, the last integral is

$$\begin{aligned}
&\geq \int_1^T \sum_{n\geq 1} e^{-\pi(rn)^2 t}\,\frac{dt}{t} \\
&\geq \int_1^T \int_1^\infty e^{-\pi t(rx)^2}\,dx\frac{dt}{t} \\
&\geq \int_1^T \int_{r\sqrt{\pi t}}^\infty e^{-u^2}\,\frac{du}{r\sqrt{\pi t}}\frac{dt}{t}.
\end{aligned}$$

Let B be the constant defined by $\int_T^\infty e^{-t^2}\,dt = B\sqrt{\pi}$, then as soon as $r < \sqrt{T/\pi}$ and upon integrating, we obtain

$$= \frac{2B}{r}\left(1 - \frac{1}{\sqrt{T}}\right).$$

And this goes to $+\infty$ as $r \to 0$. □

4. Some results on the extrema

In this section we give some results on the extrema of the height and $Z_Q(s)$ for $n = 2$ and $n = 3$, and conclude with a couple of conjectures for the general case.

Let L_0^n denote the lattice in $\mathbf{L}^n$ with the longest minimal vector, and denote its quadratic form by Q_0^n. It is well known that L_0^2 is the hexagonal lattice with basis matrix

$$\begin{pmatrix} \sqrt{3}/2 & 1/2 \\ 0 & 1 \end{pmatrix},$$

and L_0^3 is the face-centered cubic lattice with basis matrix

$$\begin{pmatrix} 1/\sqrt{2} & 1/2 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

These distinguished lattices possess a number of interesting properties; among them is that they provide the densest lattice packing of spheres in $\mathbf{R}^2$ and $\mathbf{R}^3$, respectively. See Conway and Sloane [CS].

For $n = 2$, by considering the properties of the Dedekind eta-function in Kronecker's Limit Formula in (2.1), Osgood-Phillips-Sarnak [OPS] arrived at the following result:

Theorem 4.1 ([OPS])**.** *The height on the moduli space of* 2*-d flat tori of area* 1 *has a global minimum at the torus corresponding to the hexagonal lattice.*

Independent and earlier work on minimizing the Epstein zeta-function began by Rankin [Ra] and carried further by Cassels [Ca], Ennola [En1] and Diananda [Di] produced the following result which encompasses Theorem 4.1.

Theorem 4.2 (CDER)**.** *Let Q be a* 2*-d positive definite quadratic form with determinant* 1*. Then for s with* $\mathrm{Re}(s) > 0$*,*

$$Z_Q(S) - Z_{Q_0^2}(S) \geq 0,$$

and equality holds iff Q is the quadratic form Q_0^2 of the hexagonal lattice.

By the definition of $k_n(z)$ in (2.4), it is clear that at the point $s = 1$, Theorem 4.2 is equivalent to Theorem 4.1.

For $n = 3$, Ennola [En2] managed a partial generalization of Theorem 4.2.

Theorem 4.3 (Ennola)**.** *Let Q be a 3-d positive definite quadratic form with determinant 1. Then for s with $\mathrm{Re}(s) > 0$, and Q in a sufficiently small neighborhood of Q_0^3,*

$$Z_Q(S) - Z_{Q_0^3}(S) \geq 0,$$

and equality holds iff Q is the quadratic form Q_0^3 of the face-centered cubic lattice.

Again, at the pole $s = 3/2$, this is equivalent to the height having a local minimum.

Corollary 4.4. *The height on the moduli space of 3-d flat tori of volume 1 has a local minimum at the torus corresponding to the face-centered cubic lattice.*

The author has checked the height of L_0^3 against a set of points that are equidistributed in $\mathbf{L}^3$. The Hecke points provide such a set, see Chiu [Ch]. For $n = 3$ and p prime, the Hecke points are

$$S(p) = \left\{ \begin{pmatrix} p & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & a & \\ & p & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & a \\ & 1 & b \\ & & p \end{pmatrix} \middle| \, 0 \leq a, b < p \right\}.$$

For $S(101)$, which has $10,303$ points (some of which have the same height value due to symmetries), the height as computed using formula (2.2) is strictly greater than the height of L_0^3.

We conclude with some general conjectures on the minimum of the height and the Epstein zeta-function. Heuristically, from examining the sum

$$Z_Q(s) = \sum_{x \in \mathbf{Z}^n}{}' Q[x]^{-s},$$

it is apparent that the shorter vectors play a greater role, so it is reasonable to speculate that the quadratic form of lattices with longer minimal vectors would have smaller values for $Z_Q(s)$ for fixed s. This should certainly be the case for large values of $|s|$, and a result of Ryshkov [Ry] confirms it. For small values of $|s|$, the question becomes more delicate and is yet to be resolved. In light of the results in this section, we state the following conjectures on the height and $Z_Q(s)$, which we attribute to Sarnak and Rankin, respectively.

Conjecture 4.5 (Sarnak)**.** *The height on the moduli space of n-d flat tori of volume 1 has a global minimum at the torus corresponding to the lattice with the longest minimal vector.*

Conjecture 4.6 (Rankin)**.** *Let Q be an n-d positive definite quadratic form with determinant 1. Then for s with $\mathrm{Re}(s) > 0$,*

$$Z_Q(S) - Z_{Q_0^n}(S) \geq 0,$$

and equality holds iff Q is the quadratic form Q_0^n of the lattice with the longest minimal vector.

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