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NOTE ON CLARK'S THEOREM FOR p-ADIC CONVERGENCE

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Abstract. We must read Clark's statement under the hypothesis that the *negative* of each zero of the indicial polynomial is non-Liouville. In this note we shall give the example for which under the original hypothesis the statement does not hold.

Introduction

Clark's article [1] is referred to by many authors because he first studied the p-adic convergence of power series solutions at a singular point and pointed out the importance of p-adic non-Liouvilleness of exponents. However, his result is not properly stated. We must read his statement under the hypothesis that the *negative* of each zero of the indicial polynomial is non-Liouville. In this note we shall give the example for which under the original hypothesis the statement does not hold. For another approach to p-adic convergence using the notion of p-adic Liouvilleness in the sense of Schikhof [2], see [3].

Throughout this note $\mathbb{C}_p$ denotes the completion of the algebraic closure of $\mathbb{Q}_p$, and the p-adic valuation $|\ \ |_p$ is simply written $|\ \ |$. By $\mathbb{N}$ we denote the positive integers, and by $\mathbb{N}^0$ the non-negative integers. $\mathbb{Z}$ and $\mathbb{Z}_p$ denote the rational integers and the ring of p-adic integers of $\mathbb{Q}_p$, respectively.

1. Some definitions

The statement in question is Clark's Theorem 3 [1]. We shall give a summary of this below. To understand it we refer to some definitions in his article.

Definition 1. An element $\alpha \in \mathbb{C}_p$ is said to be (p-adically) *non-Liouville* if for $s \in \mathbb{N}$, we have

$$\operatorname{ord}(\alpha + s) = O(\log s) \quad \text{as} \quad s \longrightarrow \infty.$$

Definition 2. For each non-Liouville number α the *weight* $w(\alpha)$ is defined by Clark. In this note we only use

$$w(\alpha) = \frac{1}{p-1}$$

for $\alpha \in \mathbb{Z}_p$.

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Definition 3. The *weight* $w(g)$ of a polynomial $g \in \mathbb{C}_p[X]$ whose roots are non-Liouville is the sum of the weights of the roots.

The *ordinal* of a polynomial $\Phi \in \mathbb{C}_p[X]$, say $\sum_{i=0}^{n} a_i X^i$, is defined by

$$\text{ord } \Phi = \min_i \{\text{ord } a_i\}.$$

(Notice that "minimal" is carelessly mistaken for "maximal" in his article.)

We can summarize Clark's statement as follows:
Consider a linear differential equation

$$A_n(x)y^{(n)} + A_{n-1}(x)y^{(n-1)} + \cdots + A_0(x)y = 0 \tag{1}$$

where coefficients lie in $\mathbb{C}_p[[x]]$ with nonzero radius of convergence. We may write $A_i = x^i \sum_{j=0}^{\infty} a_{ij}x^j$ $(0 \leq i \leq n)$ and $a_{i0} \neq 0$ for some i. Let $\Phi_j(s) = \sum_{i=0}^{n} a_{ij} \binom{s}{i} i!$ for $j \geq 0$; then Φ_0 is the indicial polynomial of (1) at $x = 0$.

"Statement". *If every zero of the indicial polynomial of* (1) *is non-Liouville, then each power series solution converges for*

$$\text{ord } x > \sup_{j \geq 1} \left\{ \frac{\text{ord } \Phi_0 - \text{ord } \Phi_j}{j} \right\} + w(\Phi_0).$$

2. A non-Liouville number

In order to give the example for which the Statement does not hold, we define a special non-Liouville number in the following lemma.

Lemma. *Let $(e_k)_{k=1}^{\infty}$ be a sequence of positive integers given by $e_1 = 1$ and $e_{k+1} = kp^{e_k}$ $(k \geq 1)$, and let $\lambda = \sum_{k=1}^{\infty} p^{e_k}$. Then we have*

(i) $\underline{\lim}_{n\to\infty} \sqrt[n]{|n - \lambda|} = 0$;
(ii) $\lambda \in \mathbb{Z}_p \setminus \mathbb{Z}$;
(iii) $\text{ord}(\lambda + n) = O(\log n)$ *as* $n \longrightarrow \infty$.

Proof. (i) Let $n_k = \sum_{j=1}^{k} p^{e_j}$. Then $(n_k)_{k=1}^{\infty}$ is a strictly increasing sequence of positive integers, and we have

$$\sqrt[n_k]{|\lambda - n_k|} = p^{-e_{k+1}/n_k}.$$

Since $p^{e_k} \leq n_k < 2p^{e_k}$, it follows that

$$-\frac{e_{k+1}}{n_k} < -\frac{e_{k+1}}{2p^{e_k}} = -\frac{k}{2},$$

and hence

$$\sqrt[n_k]{|\lambda - n_k|} < p^{-\frac{k}{2}}.$$

Thus we have

$$\lim_{k\to\infty} \sqrt[n_k]{|\lambda - n_k|} = 0,$$

and therefore

$$\underline{\lim}_{n\to\infty} \sqrt[n]{|\lambda - n|} = 0.$$

(ii) Clearly $\lambda \in \mathbb{Z}_p \setminus \mathbb{N}^0$. If $-\lambda \in \mathbb{N}$, that is, there is a positive integer m such that $-\lambda = m$, then we have

$$\sqrt[n]{|n-\lambda|} = |n+m|^{\frac{1}{n}} \geq \left(\frac{1}{n+m}\right)^{\frac{1}{n+m}\frac{n+m}{n}},$$

and therefore

$$\lim_{n\to\infty} \sqrt[n]{|\lambda - n|} = 1,$$

which is contrary to (i).

(iii) Let $n \geq p$, and let

$$n = a_0 + a_1 p + \cdots + a_l p^l$$

be the p-adic expansion of n, where $a_l \neq 0$. Then, as $p^l \leq n < p^{l+1}$, we have

$$0 \leq \operatorname{ord}(\lambda + n) \leq l + 2 \leq \frac{\log n}{\log p} + 2,$$

and therefore

$$0 \leq \frac{\operatorname{ord}(\lambda+n)}{\log n} \leq \frac{3}{\log p}.$$

Hence

$$\operatorname{ord}(\lambda + n) = O(\log n) \quad \text{as} \quad n \longrightarrow \infty.$$

□

3. Example

Now, let λ be the non-Liouville number defined above. Consider the linear differential equation

$$(1-x)x^2 y'' + \{(1-\lambda) - (2-\lambda)x\}xy' + \lambda x y = 0. \tag{2}$$

In this equation, we have

$$\begin{aligned} A_2 &= x^2(1-x), \\ A_1 &= x\{(1-\lambda) - (2-\lambda)x\}, \\ A_0 &= \lambda x, \end{aligned}$$

and so

$$\begin{aligned} &a_{20} = 1, && a_{21} = -1, \\ &a_{10} = 1-\lambda, && a_{11} = -(2-\lambda), \\ &a_{00} = 0, && a_{01} = \lambda, \\ &a_{ij} = 0 \quad \text{if } j \geq 2. \end{aligned}$$

As $\Phi_j(s) = a_{0j} + a_{1j}s + a_{2j}s(s-1)$, we have

$$\begin{aligned} \Phi_0(s) &= (1-\lambda)s + s(s-1) = s(s-\lambda), \\ \Phi_1(s) &= \lambda - (2-\lambda)s - s(s-1) = -s^2 - (1-\lambda)s + \lambda, \\ \Phi_j(s) &= 0 \quad \text{if } j \geq 2. \end{aligned}$$

The roots of the indicial equation are 0 and λ, and so the requirement of the Statement is satisfied. Since ord $\lambda \geq 0$ by Lemma (ii), we have

$$\begin{aligned}
&\mathrm{ord}\ \Phi_0 = \min\{\mathrm{ord}\ 1,\ \mathrm{ord}\ \lambda\} = 0,\\
&\mathrm{ord}\ \Phi_1 = \min\{\mathrm{ord}\ 1,\ \mathrm{ord}(1-\lambda),\ \mathrm{ord}\ \lambda\} = 0,\\
&\mathrm{ord}\ \Phi_j = \infty \quad \text{if } j \geq 2.
\end{aligned}$$

This gives

$$\begin{aligned}
\sup_{j\geq 1}\left\{\frac{\mathrm{ord}\ \Phi_0 - \mathrm{ord}\ \Phi_j}{j}\right\} + w(\Phi_0) &= \sup_{j\geq 1}\left\{\frac{-\ \mathrm{ord}\ \Phi_j}{j}\right\} + w(0) + w(\lambda)\\
&= \frac{2}{p-1}.
\end{aligned}$$

According to the Statement, the power series solution of (2) must converge for

$$\mathrm{ord}\ x > \frac{2}{p-1},$$

but the equation (2) has the power series solution

$$y = \sum_{n=0}^{\infty} \frac{x^n}{n-\lambda},$$

whose radius of convergence is zero by virtue of Lemma (i).

4. “Corollary”

Clark’s original proof of Theorem 3 needs the Corollary on p.266 of his article:

“Corollary”. *Let h be a monic polynomial whose roots are non-Liouville and in $\{\alpha \in \mathbb{C}_p : |\alpha| \leq 1\}$, and let t be a fixed positive integer greater than all roots of h in $\mathbb{Z}$. If $s, s' \in \mathbb{Z}, s > s' > t$, then*

$$\sum_{j=s'}^{s} \mathrm{ord}\ h(j) = w(h)(s-s') + O(\log s).$$

Professor Dwork suggested to the author that a similar error appears in this statement. Indeed, if it were true we should have the following claim, which is contrary to our Lemma.

Claim. Let $\alpha \in \mathbb{Z}_p \setminus \mathbb{Z}$ such that $\mathrm{ord}(\alpha + s) = O(\log s)$. Then

$$\underline{\lim}_{s\to\infty} \sqrt[s]{|s-\alpha|} \geq p^{-\frac{1}{p-1}}.$$

In fact, let $h(X) = X - \alpha$. This is the monic polynomial whose root is only α. Therefore

$$w(h) = w(\alpha) = \frac{1}{p-1}.$$

By the Corollary we have

$$\sum_{j=s'}^{s} \mathrm{ord}(j-\alpha) = \frac{s-s'}{p-1} + O(\log s),$$

where $s > s' > t$. Thus

$$0 \leq \operatorname{ord}(s-\alpha) \leq \sum_{j=s'}^{s} \operatorname{ord}(j-\alpha) \leq \frac{s-t}{p-1} + M\log s$$

for some $M > 0$, so that

$$|s-\alpha| \geq p^{-\frac{s-t}{p-1} - M\log s},$$

and therefore

$$\varliminf_{s\to\infty} \sqrt[s]{|s-\alpha|} \geq p^{-\frac{1}{p-1}}.$$

However, this claim contradicts our Lemma when $\alpha = \lambda$.

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