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CONJUGACY CLASSES OF SYMMETRIES IN ORTHOGONAL GROUPS

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ABSTRACT. The number of conjugacy classes of symmetries in the integral orthogonal group of an indefinite $\mathbb{Z}$-lattice is determined. The results are applied to the extended Bianchi groups.

1. Introduction

Let V be a regular quadratic space, of finite dimension $n \geq 4$, over the rational field $\mathbb{Q}$ with quadratic form $Q : V \to \mathbb{Q}$ and associated bilinear form $2f(x,y) = Q(x+y) - Q(x) - Q(y)$. Let L be a $\mathbb{Z}$-lattice on V, let $O(L)$ be the orthogonal group of L, and let $O'(L)$ be its spinor kernel. Then both $O(L)$ and $O'(L)$ act on the symmetries

$$\Psi(x) : y \to y - 2f(x,y)Q(x)^{-1}x, \qquad x,\ y \in L,$$

in $O(L)$ by conjugation, with $\phi\Psi(x)\phi^{-1} = \Psi(\phi(x))$. We study the number of conjugacy classes under both actions, and then apply the results to the extended Bianchi groups and Hilbert modular groups.

The lattice L represents $c \in \mathbb{Z}$ if there exists an $x \in L$ with $Q(x) = c$. The representation is primitive if $\mathbb{Z}x$ is a direct summand of L. Since the symmetry $\Psi(x)$ is to be integral, only primitive representations that satisfy the extra condition $2f(x,L) \subseteq c\mathbb{Z}$ will be considered. Let $N(L,c)$ be the number of these primitive representations of c modulo the action of $O(L)$. Then $N(L,c)$ also counts the number of conjugacy classes of symmetries $\Psi(x)$ with x primitive and $Q(x) = c$. Let $N'(L,c)$ be the number of these primitive representations of c modulo the action of $O'(L)$, and let $N'(L_p,c)$ be the corresponding number of primitive representations by the local lattice L_p over the p-adic integers $\mathbb{Z}_p$. Using Kneser's Strong Approximation Theorem to set up a bijection between the global and corresponding set of local orbits, as in the proof of Theorem 2.3 in [5], or Theorem 4.1 in [6], gives the following product formula.

Theorem 1.1. *Let L be a lattice on V with $f(L,L) \subseteq \mathbb{Z}$ and discriminant D. Assume $c \neq 0$ and the Witt index $i_\infty(V \perp \langle -c\rangle) \geq 2$. Then, for $n \geq 4$,*

$$N'(L,c) = \prod_{p|2D} N'(L_p,c) < \infty.$$

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The number of conjugacy classes of symmetries under the action of $O'(L)$ is then determined from the action of $-I$ on the $O'(L)$-orbits. The value $N(L,c)$ can be studied via the action of the quotient group $O(L)/O'(L)$ on the $O'(L)$-orbits. These methods will be used to study the conjugacy classes of symmetries in $O(L)$ in a special case corresponding to the Bianchi groups and Hilbert modular groups $PSL(2,\mathcal{O}_d)$. Related methods were used in [6] to classify the maximal non-elementary Fuchsian subgroups of the Bianchi groups up to conjugacy. In [11], Vulakh studied the conjugacy classes of reflections in the extended Bianchi group RB_d. Via the identification of RB_d with a suitable group $O(L)$ (see [2, §11] or [10]), this is essentially the same as studying the conjugacy classes of symmetries for this $O(L)$. Theorem 3.5 is a generalization of one of Vulakh's results. Theorem 3.6 gives an analogue for the conjugacy classes in RB_d under the action of the Bianchi group. A Hilbert modular group analogue is simultaneously established. See also [2, §11] for an earlier special case obtained using Siegel's analytic theory of indefinite forms.

The next section contains some general results evaluating $N'(L_p,c)$. They can easily be modified to give a function field analogue where $\mathbb{Z}$ is replaced by the polynomial ring $\mathbb{F}[X]$ (with $2 \neq 0$).

2. Local spinor orbits

Local isometry invariants on $x \in L_p$ under the action of $O(L_p)$ are given in [7] and [9]. In general they are complicated; however, for odd p, the restriction $f(x,L_p) \subseteq Q(x)\mathbb{Z}_p$ means that $\mathbb{Z}_p x$ splits L_p as a rank one orthogonal summand (see [8, §92:6]). For each prime p let

$$L_p = J_1 \perp \cdots \perp J_t$$

be a Jordan splitting of L_p (see [8, §91C]), where each J_i is a p^{r_i}-modular lattice, $r_1 < \cdots < r_t$, and $n_i = \text{rank } J_i$ are invariants of L_p. For odd p the discriminants $d_i = (\det J_i)\mathfrak{u}_p^2$, where $\mathfrak{u}_p$ denotes the p-adic units, are also invariants. Note, for p odd, $[SO(L_p) : O'(L_p)] = 4$ whenever some $n_i \geq 2$, and the r_i are not all even or not all odd (see [8, §92:5]). When all r_i have the same parity, $[SO(L_p) : O'(L_p)] \leq 2$. Also, $SO(L_p) = O'(L_p)$ if and only if all $n_i = 1$, and all the square classes $d_i\mathbb{Q}_p^{*2}$ are the same (in particular, the r_i all have the same parity).

If $\Psi(x) \in O(L)$ with x primitive and $Q(x) = c$, then $\mathbb{Z}_p x$ splits L_p as an orthogonal summand for each odd p. In particular, this implies that a Jordan splitting can be chosen with $\mathbb{Z}_p x \subseteq J_i$ for some i. Hence $cp^{-r_i} \in \mathfrak{u}_p$, and this is therefore a necessary condition for $N(L_p,c) > 0$.

Theorem 2.1. *Let p be odd. Assume $cp^{-r_i} \in \mathfrak{u}_p$ for some i. Then*

1. $N'(L_p,c) = 0$ *if* $n_i = 1$ *and* $(\frac{c/d_i}{p}) = -1$.
2. $N'(L_p,c) = 2$ *if* $n_i = 1$, $(\frac{c/d_i}{p}) = 1$, *and* $r_j - r_i$ *is odd for* $j \neq i$.
3. $N'(L_p,c) = 2$ *if all* $n_j = 1$, $(\frac{c/d_i}{p}) = 1$, *and the* $d_j\mathbb{Q}_p^{*2}$, *for* $j \neq i$, *take at most two values, and these values are not* $c\mathbb{Q}_p^{*2}$.
4. $N'(L_p,c) = 2$ *if* $n_i = 2$, *all other* $n_j = 1$, *and the* $d_j\mathbb{Q}_p^{*2}$, *for* $j \neq i$, *and* $d_i c^{-1}\mathbb{Q}_p^{*2}$ *take at most two values, and the product of any two values is not* $\epsilon\mathbb{Q}_p^{*2}$ *with* ϵ *a non-square unit.*
5. $N'(L_p,c) = 1$ *otherwise.*

Proof. Assume first $n_i = 1$. Let $x \in L_p$ be primitive with $Q(x) = c$. Then we may choose $J_i = \mathbb{Z}_p x$ and hence $c \in d_i \mathfrak{u}_p^2$. In this case, the representations of c by x and $-x$ are not spinor equivalent when $r_j - r_i$ is odd for all $j \neq i$. For assume $\phi \in O'(L_p)$ with $\phi(x) = -x$. Then $\Psi(x)\phi$ fixes x, and hence can be viewed as an isometry on the orthogonal complement of J_i. Therefore, by [8, §92:4], $\Psi(x)\phi$ is a product of an odd number of symmetries $\Psi(y)$ with each $Q(y)p^{-r_j}$ a unit for some j. Calculating spinor norms then gives a contradiction. A similar argument holds if all $n_j = 1$ as in case 3. Any two representations are $SO(L_p)$-equivalent (see [7]), and it follows that $N'(L_p, c) \leq 2$ if some $n_j \geq 2$, since the orthogonal complement of x then admits isometries with all spinor norms in $\mathfrak{u}_p \mathbb{Q}_p^{*2}$. When all $n_j = 1$, it still follows that $N'(L_p, c) \leq 2$, because if all the $d_j \mathbb{Q}_p^{*2}, j \neq i$, are the same, then $[SO(L_p) : O'(L_p)] \leq 2$. In case 4, x and $\theta(x)$ are not spinor equivalent for $\theta \in SO(J_i)$ with spinor norm $\epsilon \mathbb{Q}_p^{*2}$; take $\theta(x) = -x$ if $d_i \notin \mathbb{Q}_p^{*2}$.

Now consider the case $n_i \geq 3$, or $n_i = 2$ and $n_j \geq 2$ for some $j \neq i$. If x and y both represent c, by [7] there exists $\phi \in SO(L_p)$ with $\phi(x) = y$. It is easy to adjust and get $\phi \in O'(L_p)$ using the orthogonal complement of x. The remaining cases are similar. □

Corollary 2.2. *The two spinor orbits in case* 4 *with* $d_i \notin \mathbb{Q}_p^{*2}$, 2 *or* 3*, are interchanged by* $-I$*. The two spinor orbits in case* 2 *are interchanged by any* $\phi \in SO(L_p)$ *with spinor norm* $\eta p \mathbb{Q}_p^{*2}$*,* η *a unit, and are fixed by all* $\psi \in SO(L_p)$ *with spinor norm* $\eta \mathbb{Q}_p^{*2}$*.*

Proof. Assume $Q(\pm x) = c$. There exists $\theta \in O(L_p)$ fixing x and with $\phi\theta\Psi(x) \in O'(L_p)$, using the orthogonal complement of x. Hence x is spinor equivalent to $\phi(-x)$. For the remaining part choose $\theta \in SO(L_p)$ fixing x and with $\psi\theta \in O'(L_p)$. Then x is spinor equivalent to $\psi(x)$. □

A dyadic unimodular lattice L_2 is *even* when $Q(L_2) \subseteq 2\mathbb{Z}_2$. Otherwise L_2 is *odd* and has an orthogonal basis $e_1, \ldots, e_n$. Then $x = \sum_i a_i e_i \in L_2$ is *characteristic* if all coefficients a_i are units (see [5], [9]).

Theorem 2.3. *Let* L_2 *be a dyadic unimodular lattice. Then*

1. $N'(L_2, c) = 0$ *when* $c \in 4\mathbb{Z}_2$*, or* L_2 *is even with* c *a unit.*
2. $N'(L_2, c) = 3$ *when* L_2 *is odd and* $c \equiv \sum_i Q(e_i) \not\equiv 0, 4 \mod 8$.
3. $N'(L_2, c) = 1$ *otherwise.*

Proof. Let $x \in L_2$ be primitive with $Q(x) = c$. Then $2f(x, L_2) \subseteq c\mathbb{Z}_2$ forces $c|2$. When L_2 is even, it is split by a hyperbolic plane $\mathbb{Z}_2 u + \mathbb{Z}_2 v$, and x is spinor equivalent to $u + 2^{-1}cv$ (see [5] or [9, §2]). Now assume that L_2 is odd and $x = \sum_i a_i e_i$ is characteristic. Then $c = Q(x) = \sum_i a_i^2 Q(e_i) \equiv \sum_i Q(e_i) \mod 8$. By Hensel's lemma, this is a necessary and sufficient condition for the existence of a characteristic representation of c. When $c \equiv 0 \mod 4$ the restriction $c|2$ is violated. The result follows by strengthening the arguments in Lemmas 4.3 and 5.7 in [5], or as in Theorem 2.1 above using [9, §2]. □

Corollary 2.4. *When* $N'(L_2, c) = 3$*, the two orbits corresponding to the characteristic representations are interchanged by* $-I$ *if* n *is odd, but are fixed by* $-I$ *if* n *is even.*

Proof. The action of each $\Psi(e_i)$ interchanges the two characteristic orbits (see Theorem 2.2 in [5]). □

For the final two theorems in this section we assume, as in Theorem 1.1, that L is a $\mathbb{Z}$-lattice with $f(L,L) \subseteq \mathbb{Z}$ and $n \geq 4$, that $c \neq 0$ and the index $i_\infty(V \perp \langle -c \rangle) \geq 2$. Also assume L_2 is unimodular.

Theorem 2.5. *Necessary and sufficient conditions for $N(L,c) > 0$ are:*

1. $c \not\equiv 0 \mod 4$, *and* $2|c$ *when* L_2 *is even,*
2. *at each odd prime,* $cp^{-r_i} \in \mathfrak{u}_p$ *for some* i, *and moreover, if* $n_i = 1$ *then* $(\frac{c/d_i}{p}) = 1$.

By Theorem 1 in [4], it suffices to assume V indefinite in Theorem 2.5 (instead of $i_\infty(V \perp \langle -c \rangle) \geq 2$).

Let m be the number of odd primes p where $N'(L_p, c) = 2$.

Theorem 2.6. *Assume $N(L,c) > 0$. When $m \geq 1$ and not only case 4 with $d_i \in \mathbb{Q}_p^{*2}$ occurs, the number of conjugacy classes of symmetries $\Psi(x)$ with x primitive and $Q(x) = c$ under the action of $O'(L)$ is $2^{m-1}N'(L_2,c)$. If $m = 0$ and $N'(L_2,c) = 3$, the number of conjugacy classes is 2 when n is odd, and 3 when n is even.*

Proof. When $m \geq 1$, this follows since $\Psi(x) = \Psi(-x)$, but x and $-x$ are in different $O'(L)$-orbits by Corollary 2.2. Use 2.4 when $m = 0$. □

The result is different if $m \geq 1$ and only case 4 with $d_i \in \mathbb{Q}_p^{*2}$ occurs, since $-I$ now fixes all local p-adic orbits for p odd.

3. Extended Bianchi groups

Let $\mathcal{O}_d$ be the ring of integers in $\mathbb{Q}(\sqrt{d})$, where d is a square-free integer. It was shown in [6] that the Bianchi group $PSL(2, \mathcal{O}_d)$, for $d < 0$ and $d \equiv 2, 3 \mod 4$, is isomorphic to $O'(L)$, where

$$L = \mathbb{Z}r \perp \mathbb{Z}s \perp (\mathbb{Z}u + \mathbb{Z}v) = B \perp H$$

is the lattice with $Q(r) = 2$, $Q(s) = -2d$, and u, v are isotropic with $f(u,v) = d$. The extended Bianchi group B_d, namely, the maximal discrete extension of $PSL(2, \mathcal{O}_d)$ in $PSL(2, \mathbb{C})$, is isomorphic to $PSO(L)$. The extended Bianchi group RB_d is B_d with the action of complex conjugation adjoined; it is isomorphic to $PO(L)$. The Hilbert modular group, where $d > 0$, is also isomorphic to $O'(L)$. For $d \equiv 1 \mod 4$, L must be replaced by $M = L + \mathbb{Z}2^{-1}(r - s)$. These isomorphisms are related to those constructed over commutative rings in [3, §7.3B].

Let

$$J = \{x \in L \mid Q(x) \in 2d\mathbb{Z}\} = \mathbb{Z}dr \perp \mathbb{Z}s \perp H$$

and

$$K = \{x \in L \mid f(x, J) \subseteq 2d\mathbb{Z}\} = \mathbb{Z}r \perp \mathbb{Z}s \perp 2H.$$

Then J and K are sublattices of L that are invariant under the action of $O(L)$. Note that $O(J) = O(L)$. Clearly $O(L) \subseteq O(J)$. Conversely, let $\phi \in O(J)$ with $\phi(r) \in V$. Since $f(\phi(r), J) = f(r, J) = 2d\mathbb{Z}$, it follows that $\phi(r) \in K$ and $\phi \in O(L)$. Therefore, scaling J by d^{-1}, $O(L)$ is isomorphic to the orthogonal group of the integral form $dX_1{}^2 - X_2^2 + X_3X_4$ used in [2].

We now study the conjugacy classes of symmetries in $O(L)$ and $O(M)$. For the symmetry $\Psi(x)$, with x primitive in L and $Q(x) = 2c$, to be integral we need $f(x, L) \subseteq c\mathbb{Z}$, and hence $Q(x)$ divides $4d$. First determine $N'(L, 2c) = \prod_{p|2d} N'(L_p, 2c)$ for each $Q(x) = 2c|4d$. Then $N(L, 2c)$ is obtained from the action of the quotient group $O(L)/O'(L)$ on the $O'(L)$-orbits. From [6], $[O(L):O'(L)] = 2^{t+2}$ where t is the number of distinct prime divisors of the discriminant of $\mathbb{Q}(\sqrt{d})$.

For $w \in \mathbb{Z}r \perp \mathbb{Z}d^{-1}s$ and $x \in L$, let

$$E(u,w)(x) = x - f(u,x)w + f(w,x)u - 2^{-1}Q(w)f(u,x)u.$$

The Eichler transformation $E(u,w)$ lies in $O'(L)$.

Theorem 3.1. *Let d be even. Then*

1. $N'(L_2, 2c) = 0$ *for* $c \equiv 0 \mod 4$.
2. $N'(L_2, 2c) = 1$ *for* $c \equiv 1 + d, -3 \mod 8$.
3. $N'(L_2, 2c) = 2$ *for* $c \equiv \pm 2 \mod 8$.
4. $N'(L_2, 2c) = 3$ *for* $c \equiv 1, 1 - d \mod 8$.

Proof. Let $x = a_1 r + a_2 s + bu + b'v \in L_2$ with $Q(x) = 2c$. When $4|c$ the condition $f(x, L_2) \subseteq 4\mathbb{Z}_2$ forces $x \in 2L_2$ so that $N'(L_2, 2c) = 0$. Assume $c \not\equiv 0 \mod 4$. If $x \notin K_2$, we may assume b is a unit and then x is spinor equivalent, via a suitable $E(v,w) \in O'(L)$, to $bu + (bd)^{-1}cv$ for c even, or to $r + bu + (bd)^{-1}(c-1)v$ for c odd. The map τ fixing r and s, and sending u to $b^{-1}u$, has spinor norm $b\mathbb{Q}_2^{*2}$. In both cases there exists $\sigma \in SO(L)$ fixing x with spinor norm $b\mathbb{Q}_2^{*2}$ (use Theorem 3.14 in [1] on the orthogonal complement of x when c is odd). Then $\tau\sigma \in O'(L_2)$, so we may assume $b = 1$. Thus, up to spinor equivalence, there is only one representation $x \notin K_2$ with $Q(x) = 2c$.

If $x \in K_2$, then $c \equiv a_1^2 - da_2^2 \equiv 1, 1-d, \pm 2 \mod 8$. Then, by modifying the previous argument, get $b = 2$. When c is even, a suitable $E(v,w)$ sends x to a uniquely determined $ar + s + 2u + 2b'v$ with $a = 0$ or 2. When c is odd, x can be sent to $\pm r + as + 2u + 2b'v$, with $a = 0$ or 1 unique for x. The two sign choices lie in different spinor orbits by Theorem 3.14(i) in [1], since the orthogonal complement of x only allows isometries with spinor norm a unit. □

Corollary 3.2. $N(L_2, 2c) = 2$ *when* $c \equiv 1, 1-d, \pm 2 \mod 8$.

Proof. The $x \in K_2$ are analogous to the characteristic x in the dyadic unimodular case. They determine an orbit that cannot be interchanged by $O(L_2)$ with an orbit given by an $x \notin K_2$. The two characteristic orbits are interchanged by $-I$ when $c \equiv 1, 1-d \mod 8$. □

Theorem 3.3. *Let $d \equiv 3 \mod 4$. Then*

1. $N'(L_2, 2c) = 0$ *for* $c \equiv 0 \mod 4$.
2. $N'(L_2, 2c) = 1$ *for* $c \equiv 2, 3 \mod 4$.
3. $N'(L_2, 2c) = 3$ *for* $c \equiv 1 \mod 4$.

Proof. Let $x = a_1 r + a_2 s + b_1 u + b_2 v \in L_2$ be primitive with $Q(x) = 2c$ and $f(x, L_2) \subseteq c\mathbb{Z}_2$. Then $c|2, c|b_1$ and $c|b_2$. Hence there are no solutions when $c \equiv 0 \mod 4$, and x is spinor equivalent to $r + s + 2u + 2bv \in K_2$ when $c \equiv 2 \mod 4$. Now assume c is a unit. Then $u + cd^{-1}v$ is a representation of $2c$, and $r + 2u + (2d)^{-1}(c-1)v$ and $s + 2u + (2d)^{-1}(c+d)v \in K_2$ give spinor inequivalent representations when $c \equiv 1 \mod 4$. These are distinct by [1], and are the only ones by arguments similar to those in Theorem 3.1. □

Corollary 3.4. $N(L_2, 2c) = 2$ *when* $c \equiv 1 \mod 4$.

Proof. The two orbits corresponding to the $x \in K_2$ are interchanged by $\Psi(r-s) \in O(L_2)$. □

For p a prime dividing d, put $d = pq$ and choose $a, b \in \mathbb{Z}$ with $aq + bp = 1$. Then $\sigma(p) = -\Psi(u + bv)\Psi(bpr + au + bv) \in SO(L)$ has spinor norm $p\mathbb{Q}^{*2}$. Let $m = m(c)$ be the number of odd primes dividing d with $(p, c) = 1$ and $(\frac{c}{p}) = 1$. The following gives the number $N(L, c)$ of conjugacy classes under $O(L)$ of symmetries $\Psi(x)$ with x primitive and $Q(x) = 2c$.

Theorem 3.5. *Assume that $c > 0$ when $d < 0$, that $N(M, 2c) > 0$ when $d \equiv 1 \mod 4$, and $N(L, 2c) > 0$ when $d \equiv 2, 3 \mod 4$. Then*

1. $N(M, 2c) = 2^{-m}N'(M, 2c) = 1$ *for* $d \equiv 1 \mod 4$.
2. $N(L, 2c) = 2^{-m}N'(L, 2c) = N'(L_2, 2c)$ *for* $c \not\equiv 1, 1-d \mod 8$ *when* d *is even.*
3. $N(L, 2c) = 1$ *and* $N'(L, 2c) = 2^m$ *for* $c \equiv 2, 3 \mod 4$ *when* $d \equiv 3 \mod 4$.
4. $N(L, 2c) = 2$ *and* $N'(L, 2c) = 2^m 3$ *for* $c \equiv -d \equiv 1 \mod 4$, *and for* $c \equiv 1, 1-d \mod 8$ *when* d *is even.*

Proof. When $d \equiv 1 \mod 4$, M_2 is an even unimodular lattice and hence, by Theorems 1.1 and 2.1, $N'(M, 2c) = 2^m$. The group $O(M)$ is generated over $O'(M)$ by $\sigma(p_i), 1 \leq i \leq t$, $\Psi(u - v)$ and $\Psi(u + v)$. The result now follows since $\sigma(p_i)$ with $(p_i, c) = 1$ interchanges the two $O'(M_{p_i})$-orbits and leaves all other $O'(M_{p_j})$-orbits unchanged by Corollary 2.2. Each $\sigma(p)$ corresponding to a p counted in m thus independently halves the total number of $O'(M)$-orbits. Thus $N(M, 2c) = 1$.

The argument for d even is essentially the same, with $\sigma(p)$ having no effect on the local dyadic orbits for odd p. By Corollary 3.2 the $O'(L_2)$-orbits are only effected by $O(L)$ when $c \equiv 1, 1 - d \mod 8$, and then $-I$ reduces the number of dyadic orbits from 3 to 2.

For $d \equiv 3 \mod 4$ the dyadic orbits must again be considered when $c \equiv 1 \mod 4$. Now

$$\Psi(adr + as - 2abu - 2v)\Psi(v - au) \in SO(L),$$

where $2q = 1-d$ and $aq-2b = 1$, has spinor norm $2\mathbb{Q}^{*2}$, interchanges the two dyadic orbits corresponding to the representatives from K_2, and fixes the local orbits at odd primes by Corollary 2.2. □

This is essentially a refined version of Theorem 11 in [11] where only an upper bound is given for the total number of conjugacy classes of reflections in RB_d. Explicit examples, similar to those given in Theorem 11.3 of [2] and Theorem 12 of [11], can be constructed from the local information. That $N(M, 2c) = 1$ also follows from Theorems 2 and 4 in [4].

Theorem 3.6. *Assume $d \equiv 2, 3 \mod 4$, that $c > 0$ when $d < 0$, and $N(L, 2c) > 0$. When $m \geq 1$, the number of conjugacy classes of symmetries $\Psi(x)$ with x primitive and $Q(x) = 2c|4d$, under the action of $O'(L)$, is $2^{m-1}N'(L_2, 2c)$. When $m = 0$, this number is $N(L_2, 2c)$, except for $c \equiv -d \equiv 1 \mod 4$ where it is 3.*

Proof. The action of $-I$ halves the number of orbits by interchanging in pairs the local $O'(L)$-orbits at odd primes when $m > 0$. Use Corollary 3.2 when $m = 0$. □

This essentially gives the number of conjugacy classes of reflections in RB_d under the action of $PSL(2, O_d), d < 0$. The corresponding result for $O(M)$, $d \equiv 1 \mod 4$, is already covered in Theorem 2.6.

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