CONJUGACY CLASSES OF SYMMETRIES
IN ORTHOGONAL GROUPS

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Abstract. The number of conjugacy classes of symmetries in the integral
orthogonal group of an indefinite \( \mathbb{Z} \)-lattice is determined. The results are
applied to the extended Bianchi groups.

1. Introduction

Let \( V \) be a regular quadratic space, of finite dimension \( n \geq 4 \), over the rational
field \( \mathbb{Q} \) with quadratic form \( Q : V \rightarrow \mathbb{Q} \) and associated bilinear form
\( 2f(x, y) = Q(x + y) - Q(x) - Q(y) \). Let \( L \) be a \( \mathbb{Z} \)-lattice on \( V \), let \( O(L) \) be the orthogonal
group of \( L \), and let \( O'(L) \) be its spinor kernel. Then both \( O(L) \) and \( O'(L) \) act on
the symmetries
\[ \Psi(x) : y \mapsto y - 2f(x, y)Q(x)^{-1}x, \quad x, y \in L, \]
in \( O(L) \) by conjugation, with \( \phi \Psi(x) \phi^{-1} = \Psi(\phi(x)) \). We study the number of
conjugacy classes under both actions, and then apply the results to the extended
Bianchi groups and Hilbert modular groups.

The lattice \( L \) represents \( c \in \mathbb{Z} \) if there exists an \( x \in L \) with \( Q(x) = c \). The
representation is primitive if \( \mathbb{Z}x \) is a direct summand of \( L \). Since the symmetry
\( \Psi(x) \) is to be integral, only primitive representations that satisfy the extra condi-
tion \( 2f(x, L) \subseteq c\mathbb{Z} \) will be considered. Let \( N(L, c) \) be the number of these primitive
representations of \( c \) modulo the action of \( O(L) \). Then \( N(L, c) \) also counts the num-
ber of conjugacy classes of symmetries \( \Psi(x) \) with \( x \) primitive and \( Q(x) = c \). Let
\( N'(L, c) \) be the number of these primitive representations of \( c \) modulo the action of
\( O'(L) \), and let \( N'(L_p, c) \) be the corresponding number of primitive representations
by the local lattice \( L_p \) over the \( p \)-adic integers \( \mathbb{Z}_p \). Using Kneser's Strong Approxima-
tion Theorem to set up a bijection between the global and corresponding set of
local orbits, as in the proof of Theorem 2.3 in [5], or Theorem 4.1 in [6], gives the
following product formula.

**Theorem 1.1.** Let \( L \) be a lattice on \( V \) with \( f(L, L) \subseteq \mathbb{Z} \) and discriminant \( D \).
Assume \( c \neq 0 \) and the Witt index \( i_\infty(V \perp (-c)) \geq 2 \). Then, for \( n \geq 4 \),
\[ N'(L, c) = \prod_{p \mid 2D} N'(L_p, c) < \infty, \]

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The number of conjugacy classes of symmetries under the action of \( O'(L) \) is then determined from the action of \( -I \) on the \( O'(L) \)-orbits. The value \( N(L, c) \) can be studied via the action of the quotient group \( O(L)/O'(L) \) on the \( O'(L) \)-orbits. These methods will be used to study the conjugacy classes of symmetries in \( O(L) \) in a special case corresponding to the Bianchi groups and Hilbert modular groups \( \text{PSL}(2, O_d) \). Related methods were used in [6] to classify the maximal non-elementary Fuchsian subgroups of the Bianchi groups up to conjugacy. In [11], Vulakh studied the conjugacy classes of reflections in the extended Bianchi group \( RB_d \). Via the identification of \( RB_d \) with a suitable group \( O(L) \) (see [2, §11] or [10]), this is essentially the same as studying the conjugacy classes of symmetries for this \( O(L) \). Theorem 3.5 is a generalization of one of Vulakh’s results. Theorem 3.6 gives an analogue for the conjugacy classes in \( RB_d \) under the action of the Bianchi group. A Hilbert modular group analogue is simultaneously established. See also [2, §11] for an earlier special case obtained using Siegel’s analytic theory of indefinite forms.

The next section contains some general results evaluating \( N'(L_p, c) \). They can easily be modified to give a function field analogue where \( L \) is replaced by the polynomial ring \( F[X] \) (with \( 2 \neq 0 \)).

2. LOCAL SPINOR ORBITS

Local isometry invariants on \( x \in L_p \) under the action of \( O(L_p) \) are given in [7] and [9]. In general they are complicated; however, for odd \( p \), the restriction \( f(x, L_p) \subseteq Q(x)Z_p \) means that \( Z_p \cdot x \) splits \( L_p \) as a rank one orthogonal summand (see [8, §92.6]). For each prime \( p \) let

\[
L_p = J_1 \perp \cdots \perp J_t
\]

be a Jordan splitting of \( L_p \) (see [8, §91C]), where each \( J_i \) is a \( p^{r_i} \)-modular lattice, \( r_1 < \cdots < r_t \), and \( n_i = \text{rank } J_i \) are invariants of \( L_p \). For odd \( p \) the discriminants \( d_i = (\det J_i)u_p^2 \), where \( u_p \) denotes the \( p \)-adic units, are also invariants. Note, for \( p \) odd, \( |SO(L_p) : O'(L_p)| = 4 \) whenever some \( n_i \geq 2 \), and the \( r_i \) are not all even or not all odd (see [8, §92.5]). When all \( r_i \) have the same parity, \( |SO(L_p) : O'(L_p)| \leq 2 \). Also, \( SO(L_p) = O'(L_p) \) if and only if all \( n_i \) is 1, and all the square classes \( d_i Q_p^{*2} \) are the same (in particular, the \( r_i \) all have the same parity).

If \( \Psi(x) \in O(L) \) with \( x \) primitive and \( Q(x) = c \), then \( Z_p \cdot x \) splits \( L_p \) as an orthogonal summand for each odd \( p \). In particular, this implies that a Jordan splitting can be chosen with \( Z_p \cdot x \subseteq J_i \) for some \( i \). Hence \( c \cdot p^{-r_i} \in u_p \), and this is therefore a necessary condition for \( N(L_p, c) > 0 \).

**Theorem 2.1.** Let \( p \) be odd. Assume \( c^{-r_i} \in u_p \) for some \( i \). Then

1. \( N'(L_p, c) = 0 \) if \( n_i = 1 \) and \( (c/d_i)^{r_i} = -1 \).
2. \( N'(L_p, c) = 2 \) if \( n_i = 1 \), \( (c/d_i)^{r_i} = 1 \), and \( r_j - r_i \) is odd for \( j \neq i \).
3. \( N'(L_p, c) = 2 \) if all \( n_j = 1 \), \( (c/d_i)^{r_i} = 1 \), and the \( d_j Q_p^{*2} \), for \( j \neq i \), take at most two values, and these values are not \( c Q_p^{*2} \).
4. \( N'(L_p, c) = 2 \) if \( n_i = 2 \), all other \( n_j \) is 1, and the \( d_j Q_p^{*2} \), for \( j \neq i \), and \( d_i c^{-1} Q_p^{*2} \) take at most two values, and the product of any two values is not \( c Q_p^{*2} \) with \( \epsilon \) a non-square unit.
5. \( N'(L_p, c) = 1 \) otherwise.
Proof. Assume first $n_i = 1$. Let $x \in L_p$ be primitive with $Q(x) = c$. Then we may choose $J_i = \mathbb{Z}_p x$ and hence $c \in d_i \mathbb{Z}_p^2$. In this case, the representations of $c$ by $x$ and $-x$ are not spinor equivalent when $r_j - r_i$ is odd for all $j \neq i$. For assume $\phi \in O'(L_p)$ with $\phi(x) = -x$. Then $\Psi(x)\phi$ fixes $x$, and hence can be viewed as an isometry on the orthogonal complement of $J_i$. Therefore, by [8, §92.4], $\Psi(x)\phi$ is a product of an odd number of symmetries $\Psi(y)$ with each $Q(y) \equiv r_j \mod{r_i}$ a unit for some $j$. Calculating spinor norms then gives a contradiction. A similar argument holds if all $n_j = 1$ as in case 3. Any two representations are $SO(L_p)$-equivalent (see [7]), and it follows that $N'(L_p, c) \leq 2$ if some $n_j \geq 2$, since the orthogonal complement of $x$ then admits isometries with all spinor norms in $u_j \mathbb{Q}_p^2$. When all $n_j = 1$, it still follows that $N'(L_p, c) \leq 2$, because if all the $d_j \mathbb{Q}_p^2, j \neq i$, are the same, then $\left[SO(L_p) : O'(L_p)\right] \leq 2$. In case 4, $x$ and $\theta(x)$ are not spinor equivalent for $\theta \in SO(J_i)$ with spinor norm $c \mathbb{Q}_p^2$, take $\theta(x) = -x$ if $d_i \not\equiv \mathbb{Q}_p^2$.

Now consider the case $n_i \geq 3$, or $n_i = 2$ and $n_j \geq 2$ for some $j \neq i$. If $x$ and $y$ both represent $c$, by [7] there exists $\phi \in SO(L_p)$ with $\phi(x) = y$. It is easy to adjust and get $\phi \in O'(L_p)$ using the orthogonal complement of $x$. The remaining cases are similar.

Corollary 2.2. The two spinor orbits in case 4 with $d_i \not\equiv \mathbb{Q}_p^2$, 2 or 3, are interchanged by $-I$. The two spinor orbits in case 2 are interchanged by any $\phi \in SO(L_p)$ with spinor norm $\eta \mathbb{Q}_p^2$, $\eta$ a unit, and are fixed by all $\psi \in SO(L_p)$ with spinor norm $\eta \mathbb{Q}_p^2$.

Proof. Assume $Q(\pm x) = c$. There exists $\theta \in O(L_p)$ fixing $x$ and with $\phi \theta \Psi(x) \in O'(L_p)$, using the orthogonal complement of $x$. Hence $x$ is spinor equivalent to $\phi(-x)$. For the remaining part choose $\theta \in SO(L_p)$ fixing $x$ and with $\psi \theta \in O'(L_p)$. Then $x$ is spinor equivalent to $\psi(x)$.

A dyadic unimodular lattice $L_2$ is even when $Q(L_2) \subseteq 2\mathbb{Z}_2$. Otherwise $L_2$ is odd and has an orthogonal basis $e_1, \ldots, e_n$. Then $x = \sum a_i e_i \in L_2$ is characteristic if all coefficients $a_i$ are units (see [5], [9]).

Theorem 2.3. Let $L_2$ be a dyadic unimodular lattice. Then

1. $N'(L_2, c) = 0$ when $c \in 4\mathbb{Z}_2$, or $L_2$ is even with $c$ a unit.
2. $N'(L_2, c) = 3$ when $L_2$ is odd and $c \equiv \sum Q(e_i) \not\equiv 0, 4 \mod{8}$.
3. $N'(L_2, c) = 1$ otherwise.

Proof. Let $x \in L_2$ be primitive with $Q(x) = c$. Then $2f(x, L_2) \subseteq c\mathbb{Z}_2$ forces $c|2$. When $L_2$ is even, it is split by a hyperbolic plane $\mathbb{Z}_2 u + \mathbb{Z}_2 v$, and $x$ is spinor equivalent to $u + 2^{-1}cv$ (see [5] or [9, §2]). Now assume that $L_2$ is odd and $x = \sum a_i e_i$ is characteristic. Then $c = Q(x) = \sum a_i^2 Q(e_i) = \sum Q(e_i) \mod{8}$. By Hensel’s lemma, this is a necessary and sufficient condition for the existence of a characteristic representation of $c$. When $c \equiv 0 \mod{4}$ the restriction $c|2$ is violated. The result follows by strengthening the arguments in Lemmas 4.3 and 5.7 in [5], or as in Theorem 2.1 above using [9, §2].

Corollary 2.4. When $N'(L_2, c) = 3$, the two orbits corresponding to the characteristic representations are interchanged by $-I$ if $n$ is odd, but are fixed by $-I$ if $n$ is even.

Proof. The action of each $\Psi(e_i)$ interchanges the two characteristic orbits (see Theorem 2.2 in [5]).
For the final two theorems in this section we assume, as in Theorem 1.1, that \( L \) is a \( \mathbb{Z} \)-lattice with \( f(L,L) \subseteq \mathbb{Z} \) and \( n \geq 4 \), that \( c \neq 0 \) and the index \( i_{\infty}(V \perp (-c)) \geq 2 \). Also assume \( L_2 \) is unimodular.

**Theorem 2.5.** Necessary and sufficient conditions for \( N(L,c) > 0 \) are:
1. \( c \neq 0 \mod 4 \), and \( 2|c \) when \( L_2 \) is even,
2. at each odd prime, \( cp^{-r_i} \in u_p \) for some \( i \), and moreover, if \( n_i = 1 \) then \( \left( \frac{c}{d} \right) = 1 \).

By Theorem 1 in [4], it suffices to assume \( V \) indefinite in Theorem 2.5 (instead of \( i_{\infty}(V \perp (-c)) \geq 2 \)).

Let \( m \) be the number of odd primes \( p \) where \( N'(L_p,c) = 2 \).

**Theorem 2.6.** Assume \( N(L,c) > 0 \). When \( m \geq 1 \) and not only case 4 with \( d_i \in \mathbb{Q}_p^2 \) occurs, the number of conjugacy classes of symmetries \( \Psi(x) \) with \( x \) primitive and \( Q(x) = c \) under the action of \( O'(L) \) is \( 2^{m-1}N'(L_2,c) \). If \( m = 0 \) and \( N'(L_2,c) = 3 \), the number of conjugacy classes is 2 when \( n \) is odd, and 3 when \( n \) is even.

**Proof.** When \( m \geq 1 \), this follows since \( \Psi(x) = \Psi(-x) \), but \( x \) and \( -x \) are in different \( O'(L) \)-orbits by Corollary 2.2. Use 2.4 when \( m = 0 \).

The result is different if \( m \geq 1 \) and only case 4 with \( d_i \in \mathbb{Q}_p^2 \) occurs, since \(-I\) now fixes all local \( p \)-adic orbits for \( p \) odd.

3. Extended Bianchi groups

Let \( \mathcal{O}_d \) be the ring of integers in \( \mathbb{Q} \big( \sqrt{d} \big) \), where \( d \) is a square-free integer. It was shown in [6] that the Bianchi group \( \text{PSL}(2, \mathcal{O}_d) \), for \( d < 0 \) and \( d \equiv 2, 3 \mod 4 \), is isomorphic to \( O'(L) \), where

\[
L = \mathbb{Z}r \perp \mathbb{Z}s \perp (\mathbb{Z}u + \mathbb{Z}v) = B \perp H
\]

is the lattice with \( Q(r) = 2 \), \( Q(s) = -2d \), and \( u, v \) are isotropic with \( f(u,v) = d \). The extended Bianchi group \( B_d \), namely, the maximal discrete extension of \( \text{PSL}(2, \mathcal{O}_d) \) in \( \text{PSL}(2, \mathbb{C}) \), is isomorphic to \( \text{PSO}(L) \). The extended Bianchi group \( RB_d \) is \( B_d \) with the action of complex conjugation adjoined; it is isomorphic to \( \text{PO}(L) \). The Hilbert modular group, where \( d > 0 \), is also isomorphic to \( O'(L) \). For \( d \equiv 1 \mod 4 \), \( L \) must be replaced by \( M = L + \mathbb{Z}2(1-r-s) \). These isomorphisms are related to those constructed over commutative rings in [3, §7.3B].

Let \( J = \{ x \in L \mid Q(x) \in 2d\mathbb{Z} \} = \mathbb{Z}dr \perp \mathbb{Z}s \perp H \)
and
\[
K = \{ x \in L \mid f(x, J) \subseteq 2d\mathbb{Z} \} = \mathbb{Z}r \perp \mathbb{Z}s \perp 2H.
\]

Then \( J \) and \( K \) are sublattices of \( L \) that are invariant under the action of \( O(L) \). Note that \( O(J) = O(L) \). Clearly \( O(L) \subseteq O(J) \). Conversely, let \( \phi \in O(J) \) with \( \phi(r) \in V \).
Since \( f(\phi(r), J) = f(r, J) = 2d\mathbb{Z} \), it follows that \( \phi(r) \in K \) and \( \phi \in O(L) \). Therefore, scaling \( J \) by \( d^{-1} \), \( O(L) \) is isomorphic to the orthogonal group of the integral form \( dx_1^2 - x_2^2 + x_3x_4 \) used in [2].

We now study the conjugacy classes of symmetries in \( O(L) \) and \( O(M) \). For the symmetry \( \Psi(x) \), with \( x \) primitive in \( L \) and \( Q(x) = 2c \), to be integral we need \( f(x, L) \subseteq c\mathbb{Z} \), and hence \( Q(x) \) divides \( 4d \). First determine \( N'(L,2c) = \prod_{p \mid 2d} N'(L_p, 2c) \) for each \( Q(x) = 2c|4d \). Then \( N(L,2c) \) is obtained from the action
of the quotient group $O(L)/O'(L)$ on the $O'(L)$-orbits. From [6], $[O(L) : O'(L)] = 2^{t+2}$ where $t$ is the number of distinct prime divisors of the discriminant of $\mathbb{Q}(\sqrt{d})$.

For $w \in \mathbb{Z}r \perp \mathbb{Z}d^{-1}s$ and $x \in L$, let

$$E(u, w)(x) = x - f(u, x)w + f(w, x)u - 2^{-1}Q(w)f(u, x)u.$$  

The Eichler transformation $E(u, w)$ lies in $O'(L)$.

**Theorem 3.1.** Let $d$ be even. Then

1. $N'(L_2, 2c) = 0$ for $c \equiv 0 \mod 4$.
2. $N'(L_2, 2c) = 1$ for $c \equiv 1 + d, -3 \mod 8$.
3. $N'(L_2, 2c) = 2$ for $c \equiv \pm 2 \mod 8$.
4. $N'(L_2, 2c) = 3$ for $c \equiv 1, 1 - d \mod 8$.

**Proof.** Let $x = a_1t + a_2s + bu + b'v \in L_2$ with $Q(x) = 2c$. When $4|c$ the condition $f(x, L_2) \subseteq 4\mathbb{Z}_2$ forces $x \in 2L_2$ so that $N'(L_2, 2c) = 0$. Assume $c \not\equiv 0 \mod 4$. If $x \notin K_2$, we may assume $b$ is a unit and then $x$ is spinor equivalent, via a suitable $E(u, w) \in O'(L)$, to $bu + (bd)^{-1}cv$ for $c$ even, or to $r + bu + (bd)^{-1}(c-1)v$ for $c$ odd.

The map $\tau$ fixing $r$ and $s$, and sending $u$ to $b^{-1}u$, has spinor norm $b\mathbb{Q}_2^2$. In both cases there exists $\sigma \in SO(L)$ fixing $x$ with spinor norm $b\mathbb{Q}_2^2$ (use Theorem 3.14 in [1] on the orthogonal complement of $x$ when $c$ is odd). Then $\tau\sigma \in O'(L_2)$, so we may assume $b = 1$. Thus, up to spinor equivalence, there is only one representation $x \notin K_2$ with $Q(x) = 2c$.

If $x \in K_2$, then $c \equiv a_1^2 - da_2^2 \equiv 1, 1 - d, \pm 2 \mod 8$. Then, by modifying the previous argument, get $b = 2$. When $c$ is even, a suitable $E(v, w)$ sends $x$ to a uniquely determined $ar + s + 2u + 2b'v$ with $a = 0$ or 2. When $c$ is odd, $x$ can be sent to $\pm r + as + 2u + 2b'v$, with $a = 0$ or 1 unique for $x$. The two sign choices lie in different spinor orbits by Theorem 3.14(i) in [1], since the orthogonal complement of $x$ only allows isometries with spinor norm a unit.

**Corollary 3.2.** $N(L_2, 2c) = 2$ when $c \equiv 1, 1 - d, \pm 2 \mod 8$.

**Proof.** The $x \in K_2$ are analogous to the characteristic $x$ in the dyadic unimodular case. They determine an orbit that cannot be interchanged by $O(L_2)$ with an orbit given by an $x \notin K_2$. The two characteristic orbits are interchanged by $-I$ when $c \equiv 1, 1 - d \mod 8$.

**Theorem 3.3.** Let $d \equiv 3 \mod 4$. Then

1. $N'(L_2, 2c) = 0$ for $c \equiv 0 \mod 4$.
2. $N'(L_2, 2c) = 1$ for $c \equiv 2, 3 \mod 4$.
3. $N'(L_2, 2c) = 3$ for $c \equiv 1 \mod 4$.

**Proof.** Let $x = a_1t + a_2s + b_1u + b_2v \in L_2$ be primitive with $Q(x) = 2c$ and $f(x, L_2) \subseteq c\mathbb{Z}_2$. Then $c|2, c|b_1$ and $c|b_2$. Hence there are no solutions when $c \equiv 0 \mod 4$, and $x$ is spinor equivalent to $r + s + 2u + 2b'v \in K_2$ when $c \equiv 2 \mod 4$. Now assume $c$ is a unit. Then $u + cv$ is a representation of $2c$, and $r + 2u + (2d)^{-1}(c-1)v$ and $s + 2u + (2d)^{-1}(c+d)v \in K_2$ give spinor inequivalent representations when $c \equiv 1 \mod 4$. These are distinct by [1], and are the only ones by arguments similar to those in Theorem 3.1.

**Corollary 3.4.** $N(L_2, 2c) = 2$ when $c \equiv 1 \mod 4$.

**Proof.** The two orbits corresponding to the $x \in K_2$ are interchanged by $\Psi(r - s) \in O(L_2)$.  

For $p$ a prime dividing $d$, put $d = pq$ and choose $a, b \in \mathbb{Z}$ with $aq + bp = 1$. Then $\sigma(p) = -\Psi(u + bv)\Psi(bpv + au + bv) \in SO(L)$ has spinor norm $pQ^{a,b}$. Let $m = m(c)$ be the number of odd primes dividing $d$ with $(p, c) = 1$ and $(\frac{c}{p}) = 1$. The following gives the number $N(L, c)$ of conjugacy classes under $O(L)$ of symmetries $\Psi(x)$ with $x$ primitive and $Q(x) = 2c$.

**Theorem 3.5.** Assume that $c > 0$ when $d < 0$, that $N(M, 2c) > 0$ when $d \equiv 1 \mod 4$, and $N(L, 2c) > 0$ when $d \equiv 2, 3 \mod 4$. Then

1. $N(M, 2c) = 2^{-m}N'(M, 2c) = 1$ for $d \equiv 1 \mod 4$.
2. $N(L, 2c) = 2^{-m}N'(L, 2c) = N'(L_2, 2c)$ for $c \not\equiv 1, 1 - d \mod 8$ when $d$ is even.
3. $N(L, 2c) = 1$ and $N'(L, 2c) = 2^m$ for $c \equiv 2, 3 \mod 4$ when $d \equiv 3 \mod 4$.
4. $N(L, 2c) = 2$ and $N'(L, 2c) = 2^m 3$ for $c \equiv -d \equiv 1 \mod 4$, and for $c \equiv 1, 1 - d \mod 8$ when $d$ is even.

**Proof.** When $d \equiv 1 \mod 4$, $M_2$ is an even unimodular lattice and hence, by Theorems 1.1 and 2.1, $N'(M, 2c) = 2^m$. The group $O(M)$ is generated over $O'(M)$ by $\sigma(p_i), 1 \leq i \leq t$, $\Psi(u - v)$ and $\Psi(u + v)$. The result now follows since $\sigma(p_i)$ with $(p_i, c) = 1$ interchanges the two $O'(M_{p_i})$-orbits and leaves all other $O'(M_{p_i})$-orbits unchanged by Corollary 2.2. Each $\sigma(p)$ corresponding to a $p$ counted in $m$ thus independently halves the total number of $O'(M)$-orbits. Thus $N(M, 2c) = 1$.

The argument for $d$ even is essentially the same, with $\sigma(p)$ having no effect on the local dyadic orbits for odd $p$. By Corollary 3.2 the $O'(L_2)$-orbits are only effected by $O(L)$ when $c \equiv 1, 1 - d \mod 8$, and then $-I$ reduces the number of dyadic orbits from 3 to 2.

For $d \equiv 3 \mod 4$ the dyadic orbits must again be considered when $c \equiv 1 \mod 4$. Now

$$\Psi(adr + as - 2abu - 2v)\Psi(v - au) \in SO(L),$$

where $2q = 1 - d$ and $aq - 2b = 1$, has spinor norm $2Q^{a,b}$, interchanges the two dyadic orbits corresponding to the representatives from $K_2$, and fixes the local orbits at odd primes by Corollary 2.2.

This is essentially a refined version of Theorem 11 in [11] where only an upper bound is given for the total number of conjugacy classes of reflections in $RB_d$. Explicit examples, similar to those given in Theorem 11.3 of [2] and Theorem 12 of [11], can be constructed from the local information. That $N(M, 2c) = 1$ also follows from Theorems 2 and 4 in [4].

**Theorem 3.6.** Assume $d \equiv 2, 3 \mod 4$, that $c > 0$ when $d < 0$, and $N(L, 2c) > 0$. When $m \geq 1$, the number of conjugacy classes of symmetries $\Psi(x)$ with $x$ primitive and $Q(x) = 2c|4d$, under the action of $O'(L)$, is $2^{m-1}N'(L_2, 2c)$. When $m = 0$, this number is $N(L_2, 2c)$, except for $c \equiv -d \equiv 1 \mod 4$ where it is 3.

**Proof.** The action of $-I$ halves the number of orbits by interchanging in pairs the local $O'(L)$-orbits at odd primes when $m > 0$. Use Corollary 3.2 when $m = 0$. This essentially gives the number of conjugacy classes of reflections in $RB_d$ under the action of $PSL(2, O_d), d < 0$. The corresponding result for $O(M), d \equiv 1 \mod 4$, is already covered in Theorem 2.6.
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