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GROUP ALGEBRAS WHOSE UNITS SATISFY A GROUP IDENTITY. II

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ABSTRACT. Let $K[G]$ be the group algebra of a torsion group G over an infinite field K, and let $U = U(G)$ denote its group of units. A recent paper of A. Giambruno, S. K. Sehgal, and A. Valenti proved that if U satisfies a group identity, then $K[G]$ satisfies a polynomial identity, thereby confirming a conjecture of Brian Hartley. Here we add a footnote to their result by showing that the commutator subgroup G' of G must have bounded period. Indeed, this additional fact enables us to obtain necessary and sufficient conditions for $U(G)$ to satisfy an identity.

§1. INTRODUCTION

Let $K[G]$ be the group algebra of a torsion group G over an infinite field K, and let $U = U(G)$ denote its group of units. Then U is said to satisfy a *group identity* if there exists a nontrivial word $w = w(x_1, \dots, x_m)$ in the free group $\langle x_1, \dots, x_m \rangle$ such that $w(u_1, \dots, u_m) = 1$ for all $u_i \in U$. Recently, [GSV] confirmed a conjecture of Brian Hartley by showing that if U satisfies a group identity, then $K[G]$ satisfies a polynomial identity. In particular, in view of [P, Corollaries 5.3.8 and 5.3.10], G must have a large abelian section. But more can be said in this circumstance. For example, if $\operatorname{char} K = 0$, then [GSV, Lemma 2.3] implies that G is abelian. In other words, the commutator subgroup G' has bounded period equal to 1, and $U(G)$ satisfies the identity $(x, y) = x^{-1}y^{-1}xy = 1$. We show here that a similar phenomenon occurs in characteristic $p > 0$. Note that Hartley's conjecture for p'-groups in characteristic p was verified in the earlier paper [GJV].

For the remainder of this paper, let p be a fixed prime and let K denote a fixed infinite field of characteristic p. Recall that a group A is said to be *p-abelian* if its commutator subgroup A' is a finite p-group, and that, by [P, Corollary 5.3.10], the group algebra $K[G]$ satisfies a polynomial identity if and only if G has a normal p-abelian subgroup of finite index. This explains some of the group theoretic conditions which occur in part (ii) of our main result below.

Theorem 1.1. *Let $K[G]$ be the group algebra of a torsion group G over an infinite field K of characteristic $p > 0$. If $U = U(G)$ denotes the group of units of $K[G]$, then the following are equivalent*:

(i) *U satisfies a group identity.*

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(ii) *G has a normal p-abelian subgroup of finite index, and G' is a p-group of bounded period.*
(iii) *U satisfies $(x, y)^{p^k} = 1$ for some $k \geq 0$.*

Of course, (iii) $\Rightarrow$ (i) is trivial. Furthermore, most of (i) $\Rightarrow$ (ii) is the main result of [GSV]. Thus, our goal here is to fine-tune the latter paper to obtain precise necessary and sufficient conditions for U to satisfy a group identity.

§2. The implication (i) $\Rightarrow$ (ii)

Here we assume that G is torsion and that U satisfies the group identity $w = 1$. Thus, in view of [GSV, Theorem] and [P, Corollary 5.3.10], we know that G has a normal p-abelian subgroup of finite index. In particular, G is locally finite, and [GSV, Lemma 2.3] implies that G' is a p-group.

Lemma 2.1. *Suppose $U(G)$ satisfies the group identity $w = 1$. If H is any subgroup of G or if G/N is any homomorphic image of G, then $U(H)$ and $U(G/N)$ also satisfy $w = 1$.*

Proof. The result for H is obvious since $U(H) \subseteq U(G)$. For G/N, let us suppose first that N is a finite p-group. Then the kernel of the epimorphism $K[G] \to K[G/N]$ is a nilpotent ideal and therefore $U(G)$ maps onto $U(G/N)$. With this, it is clear that $U(G/N)$ satisfies $w = 1$. Next, let N be a finite p'-group, and set $e = \hat{N}/|N|$ where $\hat{N}$ is the sum of the elements of N in $K[G]$. Then e is a central idempotent of $K[G]$ and $eK[G] \cong K[G/N]$. As a consequence, $U(G/N)$ is isomorphic to a subgroup of $U(G)$, and therefore $U(G/N)$ also satisfies $w = 1$.

Now suppose that N is merely finite. Since N' is a p-group, it follows that N has a normal Sylow p-subgroup P. But then, $G/N = (G/P)/(N/P)$, so the result follows here by applying the preceding two special cases in turn. The general case is now a consequence of the fact that G is locally finite. Indeed, if $\bar{u}_1, \dots, \bar{u}_m$ are units of $K[G/N]$, then there exists a finite subgroup L of G such that these units and their inverses are in the image of $K[L]$. But, by the above, we know that $U(L)$ and $U(L/(N \cap L))$ both satisfy $w = 1$, so we conclude that $w(\bar{u}_1, \dots, \bar{u}_m) = 1$, as required. $\square$

It is convenient to record the following well-known observation.

Lemma 2.2. *Let A be a normal abelian subgroup of G and suppose that G/A is cyclic of finite order q. If $G = \langle A, t\rangle$, then $G' = (A, t) = \{ (a, t) \mid a \in A \}$ and $G' \cap \mathbb{C}_G(t)$ has period dividing q.*

Proof. Since A is abelian, the map $a \mapsto (a, t) = a^{-1}a^t$ is easily seen to be an endomorphism of A with image (A, t) as given above. Note that (A, t) is normalized by A and by t, so $(A, t) \triangleleft G$. Now we can mod out by (A, t) and, since $G/(A, t)$ is center-by-cyclic, it follows that this factor group is abelian and therefore that $(A, t) = G'$. Finally, if $b = (a, t) \in (A, t) \cap \mathbb{C}_G(t)$, then

$$b^q = b^{1+t+\cdots+t^{q-1}} = a^{(t-1)(1+t+\cdots+t^{q-1})} = a^{t^q-1} = 1$$

since $t^q \in A$. $\square$

We now come to the heart of the argument.

Lemma 2.3. *Suppose that $G = \langle A, t\rangle$ where A is a normal abelian subgroup and where t has order q. If $U(G)$ satisfies a group identity, then G' has finite period.*

Proof. We proceed by induction on q, implicitly using Lemmas 2.1 and 2.2 throughout. Suppose first that $\langle t\rangle$ has a proper subgroup $\langle s\rangle$. Then $H = \langle A, s\rangle$ is a subgroup of G with the same structure and, by induction, $H' = (A, s)$ has finite period. Of course, $B = \langle H', s\rangle$ also has finite period, and note that B is normalized by both A and t since $(A, s) = H' \subseteq B$. Now consider $\bar{G} = G/B = \langle \bar{A}, \bar{t}\rangle$. Since $\bar{t}$ has smaller order than that of t, induction again implies that $\bar{G}'$ has finite period. Thus $G'/(B\cap G')$ has finite period and the result clearly follows. In other words, it suffices to assume that $\langle t\rangle$ has no proper subgroup and therefore that q is a prime. Furthermore, we can assume that $t \notin A$ since otherwise $G = A$ is abelian and $G' = \langle 1\rangle$. This implies that $G = A \rtimes \langle t\rangle$ is the semidirect product of A by $\langle t\rangle$.

Suppose now that $G = A \rtimes \langle t\rangle$ and that $\langle t\rangle$ has prime order q. Note that t acts on $K[A]$ by conjugation and that this action determines a trace map from $K[A]$ to $K[A] \cap \mathbb{Z}(K[G])$ given by

$$\mathrm{tr}(\sigma) = \sigma + \sigma^t + \cdots + \sigma^{t^{q-1}} \text{ for all } \sigma \in K[A].$$

Here, of course, $\mathbb{Z}(K[G])$ denotes the center of $K[G]$. Observe that $\mathrm{tr}(\sigma)^p = \mathrm{tr}(\sigma^p)$ and that, if $\zeta \in K[A]\cap\mathbb{Z}(K[G])$, then $\mathrm{tr}(\sigma\zeta) = \mathrm{tr}(\sigma)\zeta$ and $\mathrm{tr}(\zeta) = q\zeta$. Furthermore, if we set

$$\tau = 1 + t + \cdots + t^{q-1} \in K[G],$$

then $t^i\tau = \tau$, $\tau\sigma\tau = \mathrm{tr}(\sigma)\tau$ and $\tau^2 = q\tau$.

By [GJV, Proposition 1] with $b = c$, it follows that if $\alpha, \beta \in K[G]$ with $\alpha^2 = 0 = \beta^2$, then $(\alpha\beta)^n = 0$ for some integer n depending on the word w. We can, of course, assume that $n = p^k$ is a fixed power of p. There are now two cases to consider.

Case 1. $q \neq p$.

Proof. Let $a \in A$ and observe that $\alpha = \tau a^{-1}(1-t^{-1})$ has square 0 since $(1-t^{-1})\tau = 0$. Furthermore, $qa - \mathrm{tr}(a)$ has trace 0, so it follows that $\beta = (qa - \mathrm{tr}(a))\tau$ also has square 0. Thus, by the above mentioned result of [GJV], we have $(\alpha\beta)^n = 0$ with $n = p^k$. Now $\mathrm{tr}(a)$ is central and $(1 - t^{-1})\tau = 0$, so

$$\begin{aligned}\alpha\beta &= \tau a^{-1}(1 - t^{-1}) \cdot (qa - \mathrm{tr}(a))\tau = \tau a^{-1}(1 - t^{-1})qa\tau\\ &= q\tau(1 - a^{-1}a^t t^{-1})\tau = q\tau(1 - a^{-1}a^t)\tau = q(q - \mathrm{tr}(b))\tau\end{aligned}$$

where $b = a^{-1}a^t = (a, t)$. Thus, since $q(q - \mathrm{tr}(b))$ is central and $\tau^2 = q\tau$, we have

$$\begin{aligned}0 = (\alpha\beta)^{p^k} &= q^{p^k}(q - \mathrm{tr}(b))^{p^k}\tau^{p^k}\\ &= q^{p^k}(q^{p^k} - \mathrm{tr}(b^{p^k}))q^{p^k-1}\tau = q(q - \mathrm{tr}(b^{p^k}))\tau\end{aligned}$$

and hence $q = \mathrm{tr}(b^{p^k})$. Finally, note that the group element 1 occurs in the support of the left-hand side of the latter equation, so it must also occur in the right-hand expression. But all group elements in $\mathrm{tr}(b^{p^k})$ are conjugate to b^{p^k}, and therefore we conclude that $1 = b^{p^k} = (a, t)^{p^k}$, as required. □

Case 2. $q = p$.

Proof. Again let $a \in A$ and note that both τ and $a^{-1}\tau a$ have square 0 since $q = p$. Consider $\alpha = a^{-1}\tau a \cdot \tau = a^{-1}\mathrm{tr}(a)\tau$ and observe that

$$\alpha^2 = a^{-1}\mathrm{tr}(a)\tau \cdot a^{-1}\mathrm{tr}(a)\tau = \mathrm{tr}(a^{-1})\mathrm{tr}(a) \cdot a^{-1}\mathrm{tr}(a)\tau = \mathrm{tr}(a^{-1})\mathrm{tr}(a)\alpha.$$

Thus, by induction, we have

$$\alpha^i = [\mathrm{tr}(a^{-1})\mathrm{tr}(a)]^{i-1}\alpha,$$

and hence, if p^k is as above, then

$$\begin{aligned} 0 = \tau\alpha^{p^k} &= [\mathrm{tr}(a^{-1})\mathrm{tr}(a)]^{p^k-1}\tau\cdot\alpha \\ &= [\mathrm{tr}(a^{-1})\mathrm{tr}(a)]^{p^k-1}\tau\cdot a^{-1}\mathrm{tr}(a)\tau = [\mathrm{tr}(a^{-1})\mathrm{tr}(a)]^{p^k}\tau. \end{aligned}$$

Therefore we conclude that

$$0 = [\mathrm{tr}(a^{-1})\mathrm{tr}(a)]^{p^k} = \mathrm{tr}(b^{-1})\mathrm{tr}(b)$$

where $b = a^{p^k}$.

Now observe that

$$\begin{aligned} 0 = \mathrm{tr}(b^{-1})\mathrm{tr}(b) &= (b^{-1} + b^{-t} + \cdots + b^{-t^{p-1}})(b + b^t + \cdots + b^{t^{p-1}}) \\ &= \sum_{i=0}^{p-1} \mathrm{tr}(b^{-1}b^{t^i}) \end{aligned}$$

and hence, since $\mathrm{tr}(b^{-1}b^{t^0}) = p1 = 0$, we have

$$0 = \sum_{i=1}^{p-1} \mathrm{tr}(b^{-1}b^{t^i}) = \sum_{i\neq j} b^{-t^i}b^{t^j}.$$

Note that the right-hand expression above is the sum of $p(p-1)$ formally different group elements. Thus, for this sum to be zero in $K[G]$, these support elements must be equal in groups of size p. In particular, they can take on at most $p-1$ distinct values. But the conjugates of $b^{-1}b^t$ which appear in $\mathrm{tr}(b^{-1}b^t)$ are either all equal or they take on p distinct values and, as we just observed, the latter situation cannot occur. Thus $b^{-1}b^t \in (A,t) \cap \mathbb{C}_G(t)$ and the preceding lemma yields

$$1 = (b^{-1}b^t)^p = (a^{-p^k}a^{p^k t})^p = (a^{-1}a^t)^{p^{k+1}} = (a,t)^{p^{k+1}}.$$

In other words, every element of $G' = (A,t)$ has period dividing p^{k+1}, and the lemma is proved. $\square$

The remainder of the proof is now routine and quite quick.

Lemma 2.4. (i) $\Rightarrow$ (ii).

Proof. Here we assume that $U(G)$ satisfies a group identity, so [GSV, Theorem] and [P, Corollary 5.3.10] imply that G has a normal p-abelian subgroup A of finite index. Furthermore, by [GSV, Lemma 2.3], G' is a p-group. Thus the goal here is to show that G' has bounded period. Since A' is a finite normal p-subgroup of G, it clearly suffices to consider G/A', or equivalently we can assume that A is abelian. Now let

$$B = \langle L' \mid A \subseteq L \subseteq G \text{ and } L/A \text{ is cyclic}\rangle.$$

Then, by Lemma 2.3, B is a subgroup of A generated by a finite number of groups each of finite period. Hence, since A is abelian, B also has finite period. Furthermore, $B \triangleleft G$ and $B \subseteq G'$. Finally, observe that A/B is a central subgroup of G/B of finite index, and therefore G/B has a finite commutator subgroup by [P, Lemma 4.1.4]. In other words, G'/B is finite, and this obviously implies that G' has finite period. $\square$

§3. The implication (ii) ⇒ (iii)

The goal now is to show that the group theoretic conditions of part (ii) imply that $U(G)$ satisfies a particular group identity, namely $1 = (x, y)^{p^k} = (x^{-1}y^{-1}xy)^{p^k}$ for some $k \geq 0$. If R is any K-algebra, let $U(R)$ denote its group of units. For convenience, we first observe

Lemma 3.1. *Let R be a K-algebra and let I be an ideal of R which is nil of bounded degree $\leq p^k$. If $U(R/I)$ satisfies $(x, y)^{p^j} = 1$, then $U(R)$ satisfies $(x, y)^{p^{j+k}} = 1$.*

Proof. The map $^-\colon R \to R/I$ yields a group homomorphism $^-\colon U(R) \to U(R/I)$ which is onto since I is nil. If $x, y \in U(R)$, then $\bar{x}, \bar{y} \in U(R/I)$, so $(\bar{x}, \bar{y})^{p^j} = 1$. Hence $(x, y)^{p^j} - 1 \in I$, so this element is nilpotent of degree $\leq p^k$, and consequently

$$(x, y)^{p^{j+k}} - 1 = [(x, y)^{p^j} - 1]^{p^k} = 0,$$

as required. □

Next, we need

Lemma 3.2. *Let A be a normal abelian subgroup of G of finite index n and let I be a G-stable ideal of $K[A]$ which is nil of bounded degree $\leq p^k$. Then $I \cdot K[G]$ is an ideal of $K[G]$ which is nil of bounded degree $\leq np^k$.*

Proof. Let $g_1, g_2, \ldots, g_n$ be coset representatives for A in G and, for any $\alpha \in K[G]$, write $g_i\alpha = \sum_j \alpha_{i,j} g_j$ with $\alpha_{i,j} \in K[A]$. Then we know from [P, Lemma 5.1.10] that the map $\alpha \mapsto [\alpha_{i,j}]$ is an algebra embedding of $K[G]$ into the ring $\mathrm{M}_n(K[A])$ of $n \times n$ matrices over the commutative algebra $K[A]$. Furthermore, if $\alpha \in I \cdot K[G]$, then $g_i\alpha \in I \cdot K[G]$, so each $\alpha_{i,j} \in I$. In other words, $I \cdot K[G]$ embeds in $\mathrm{M}_n(I)$. As a consequence, it suffices to show that $\mathrm{M}_n(I)$ is nil of bounded degree $\leq np^k$. To this end, let $\tau \in \mathrm{M}_n(I)$. Then τ satisfies its characteristic polynomial, so

$$\tau^n = \gamma_0 + \gamma_1\tau + \cdots + \gamma_{n-1}\tau^{n-1}$$

for suitable scalars $\gamma_i \in I$. Thus, since all these elements commute and since $\gamma_i^{p^k} = 0$, we have

$$\tau^{np^k} = \gamma_0^{p^k} + \gamma_1^{p^k}\tau^{p^k} + \cdots + \gamma_{n-1}^{p^k}\tau^{(n-1)p^k} = 0,$$

and the result follows. □

Finally, we can prove

Lemma 3.3. (ii) ⇒ (iii).

Proof. By assumption, G has a normal p-abelian subgroup A of finite index and G' is a p-group of bounded period $\leq p^j$. The goal is to show that $U(G)$ satisfies an identity of the form $(x, y)^{p^k} = 1$ for some $k \geq 0$. To start with, $A' \triangleleft G$ and we know that the kernel of the homomorphism $K[G] \to K[G/A']$ is nilpotent since A' is a finite p-group. Thus, by Lemma 3.1, it suffices to consider G/A', or equivalently we can assume that A is abelian.

Next, if $B = (A, G)$, then B is a normal subgroup of G contained in $A \cap G'$. Thus B is a p-group of period $\leq p^j$ and, since A is abelian, it is easy to see that the kernel I of the homomorphism $K[A] \to K[A/B]$ is nil of bounded degree $\le p^j$. Indeed, if $b_i \in B$ and if $\alpha_i \in K[A]$, then

$$\Big[\sum_i (1-b_i)\alpha_i\Big]^{p^j} = \sum_i (1-b_i^{p^j})\alpha_i^{p^j} = 0$$

since $b_i^{p^j} = 1$. Furthermore, I is a G-stable ideal of $K[A]$ and therefore, by Lemma 3.2, $I \cdot K[G]$ is a nil ideal of $K[G]$ of bounded degree. In particular, since $I \cdot K[G]$ is the kernel of the homomorphism $K[G] \to K[G/B]$, it suffices to consider G/B, or equivalently we can now assume that A is central.

Finally, it follows from [P, Lemma 4.1.4] that G' is finite and therefore that G is a p-abelian group. Again, this implies that the kernel of the map $K[G] \to K[G/G']$ is nilpotent so, by Lemma 3.1, we can now assume that $G' = \langle 1 \rangle$. But then $K[G]$ is a commutative algebra, and consequently $U(G)$ satisfies $(x, y) = 1$. With this, the lemma is proved. □

Since the implication (iii) ⇒ (i) is trivial, Lemmas 2.4 and 3.3 combine to yield Theorem 1.1.

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