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THE OPERATOR $a(x)\frac{d}{dx}$ ON BANACH SPACE

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ABSTRACT. The operator $a(x)\frac{d}{dx}$ on $C(I)$, where I is an interval contained in the real line, is considered in many places. In this paper, we attempt to reconsider it in the subspace of $C_0(-\infty,\infty)$ containing all even functions, and show that it generates a strongly continuous semigroup. It is interesting that our main conditions seem contradictory to previous ones. It is due to the symmetry of the functions and the different domain of the operator than usual.

INTRODUCTION AND MAIN RESULTS

Let $a(x)$ be a continuous function on $(-\infty,\infty)$. Define

$$X \equiv \{f \in C_0(-\infty,\infty), f(-x) = f(x)\}$$

with the norm

$$\|f\| \equiv \max_{-\infty<x<\infty} |f(x)|.$$

From now on, let us assume the following:
(1) $a(x)$ is continuous, $a(-x) = -a(x)$;
(2) $m(\{x : a(x) = 0\}) = 0$; here m is Lebesgue measure;
(3) $\frac{1}{a(x)} \in L^1_{\mathrm{loc}}(-\infty,\infty)$.

Now, let $\mathrm{AC}_{\mathrm{loc}}$ be the set of all functions f in X that are absolutely continuous in any bounded interval. Define an operator A on X by

$$D(A) \equiv \{f \in \mathrm{AC}_{\mathrm{loc}} \,|\, f'(x) \text{ exists when } a(x) \neq 0, \\ \lim_{x\to x_0} a(x)f'(x) \text{ exists when } a(x_0) = 0\},$$

$$(Af)(x) \equiv \begin{cases} a(x)\frac{df(x)}{dx} & \text{if } a(x) \neq 0, \\ \lim_{x\to x_0} a(x)\frac{df(x)}{dx} & \text{if } a(x_0) = 0. \end{cases}$$

The domain of our operator is the analogue, in our space, of the operator D_a in [1, pp. 304–305] on $C[0,1]$.

Theorem 1. *If* (1)–(3) *are satisfied, then A generates a strongly continuous semigroup if and only if $a(x)$ is nonnegative on $[0,\infty)$. The semigroup is then a semigroup of contractions.*

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It is interesting to compare Theorem 1 with previous results. In [2, Proposition 2.7], it is shown that an operator very similar to ours,

$$B \equiv \overline{aD|_{C_c^\infty(\mathbb{R})}}$$

on $C_0(\mathbb{R})$, generates a strongly continuous semigroup if and only if certain conditions on the local integrability of $\frac{1}{a(x)}$ are satisfied. In particular, if $a(x_0) = 0$ and $a(x)$ is positive for x in a neighborhood $(x_0, x_0 + \varepsilon)$, for some $\varepsilon > 0$, then, in order for B to generate a strongly continuous semigroup, it is necessary that $\int_{x_0}^{x_0+\varepsilon} \frac{1}{a(x)} dx = \infty$. Also, see the similar result for $[0, 1]$ in [5]. But, Theorem 1, especially our condition (3), seems contradictory to those results just mentioned. In our Theorem 1, $a(x)$ could have infinitely many zeroes, yet condition (2) has $\frac{1}{a(x)}$ locally integrable on both sides of all zeroes of $a(x)$. This seeming contradiction is probably due to the symmetry of $a(x)$ and the different domain of the operator.

Theorem 1 is directly related to the following Cauchy problem:

$$\frac{\partial u}{\partial t} = a(x)\frac{\partial u}{\partial x} \qquad (t \geq 0, x \in \mathbb{R}),$$

with initial condition $u(0, x) = f(x)$. Let $a : \mathbb{R} \mapsto \mathbb{R}$ be locally Lipschitzian. Let $y(t) = \gamma(t, x)$ be the unique solution of the ODE

$$(+) \qquad y'(t) = a(y(t)), \qquad y(0) = x;$$

then the solution of the above PDE is given by

$$u(t, x) = f(\gamma(t, x)).$$

This follows from the classical method of characteristics (see [4]). A Trotter product formula approach was given in [6]. The main thing is that $\gamma(t, x)$ must exist for all $t \geq 0$. This is very restrictive in practice; for instance, it suffices to assume that $a(x)$ is globally Lipschitzian.

Thus, assuming the global existence of $\gamma(t, x)$, one has a semigroup on any space Y (supremum norm) of bounded continuous functions on $\mathbb{R}$. If $Y = C_0(\mathbb{R})$, one must also check that

$$(++) \qquad \gamma(t, x) \to \infty, \qquad t \to \infty,$$

for each x. Integrating (+), one has $\int \frac{dy}{a(y)} = t + c$, and (++) requires $\frac{1}{a} \in L^1_{\text{loc}}$. This is just our condition (3).

If a is merely continuous (as assumed in our condition (1)), the classical theory of characteristics can not be applied because the uniqueness of (+) typically fails. To understand our conditions (1) and (3) better, let us take $a(x) = x^\alpha$ for $x \geq 0$, and $-(-x)^\alpha$ for $x < 0$. Then $\frac{1}{a} \in L^1_{\text{loc}}$ implies $\alpha < 1$, and a local Lipschitz near 0 requires $\alpha \geq 1$. Thus the integrability condition (3) implies that one should not demand too much regularity on a. In (1) only continuity is needed. By assuming a is odd and $\{f \in C_0(\mathbb{R}) : f \text{ is even}\}$ we are working on $C_0([0, \infty))$ with a generalized Neumann boundary condition at 0 (see [3] for an explanation why).

The same kind of argument can be applied to the case in which $a(x)$ has a jump discontinuity at 0. Here we give an outline of the approach. First, let us modify the domain of A. Besides the requirements in the definition of $D(A)$ given above, we require that $\lim_{x\to 0} a(x)f'(x)$ exists.

Define

$$(\widetilde{A}f)(x) = \begin{cases} a(x)f'(x) & \text{if } a(x) \neq 0, \text{ and } x \neq 0, \\ \lim_{y\to x} a(y)f'(y) & \text{if } a(x) = 0, \text{ or } x = 0. \end{cases}$$

It is not difficult to verify that the following set is in the domain of $\widetilde{A}$:

$$\tilde{\omega}(-\infty,\infty) \equiv \{f \in C^\infty(-\infty,\infty) | f \text{ is even and has compact support}\}.$$

So $\widetilde{A}$ is a densely defined closed operator. Furthermore, we have the following theorem.

Theorem 2. *Assume that* (1)–(3) *are satisfied except that* $a(x)$ *has a jump discontinuity at* 0. *Then* $\widetilde{A}$ *generates a contraction strongly continuous semigroup if and only if* $a(x)$ *is nonnegative in* $[0,\infty)$.

Now let us apply Theorem 2 to the half unbounded interval $[0,\infty)$. Define

$$C_0[0,\infty) \equiv \{f \in C[0,\infty) | \lim_{x\to\infty} f(x) = 0\},$$

$$\begin{aligned} D(B) = \{f \in C_0[0,\infty) | f \in \mathrm{AC}_{\mathrm{loc}}, f'(x) &\text{ exists when } a(x) \neq 0 \\ &\text{and } x \neq 0, \lim_{x\to x_0} a(x)f'(x) \text{ exists} \\ &\text{when } a(x_0) = 0, \text{ and } \lim_{x\to 0} a(x)f'(x) \text{ exists}\} \end{aligned}$$

and

$$(Bf)(x) \equiv \begin{cases} a(x)f'(x) & \text{if } a(x) \neq 0, \text{ and } x \neq 0, \\ \lim_{y\to x} a(y)f'(y) & \text{if } a(x) = 0, \\ \lim_{y\to 0} a(y)f'(y) & \text{if } x = 0. \end{cases}$$

Theorem 3. *Assume that* $a(x)$ *is continuous in* $[0,\infty)$ *and satisfies* (2), (3) *in* $[0,\infty)$. *Then* B *generates a contraction strongly continuous semigroup on* $C_0[0,\infty)$ *if and only if* $a(x)$ *is nonnegative in* $[0,\infty)$.

The proof of the main results

First of all, we need some lemmas before proving Theorem 1.

Lemma 1. *If* (2) *and* (3) *are satisfied, then* A *is a closed operator.*

Proof. Let $\{f_n\}_{n=1}^\infty \subset D(A)$ such that

$$f_n \to f \quad \text{and} \quad Af_n \to g$$

in X, that is,

$$f_n(x) \to f(x) \text{ uniformly in } (-\infty,\infty)$$

and

$$a(x)f_n'(x) \to g(x) \text{ uniformly in } (-\infty,\infty).$$

So

$$f_n'(x) \text{ converges to a finite limit } \frac{g(x)}{a(x)} \text{ a.e.}$$

But, for n large enough,

$$|a(x)f_n'(x)| \leq |g(x)| + 1, \quad \text{for any } x \in \mathbb{R},$$

which implies that

$$|f_n'(x)| \le \frac{|g(x)|+1}{|a(x)|} \in L^1_{\text{loc}}(-\infty,\infty) \quad \text{a.e.}$$

Thus, by the Lebesgue dominated convergence theorem, for any $x \in \mathbb{R}$,

$$f(x) - f(0) = \lim_{n\to\infty}(f_n(x) - f_n(0)) = \lim_{n\to\infty}\int_0^x f_n'(s)\,ds = \int_0^x \frac{g(s)}{a(s)}\,ds.$$

So,

$$f(x) = f(0) + \int_0^x \frac{g(s)}{a(s)}\,ds.$$

Clearly, $f \in \text{AC}_{\text{loc}}$ and is differentiable at x when $a(x) \neq 0$, with

$$a(x)f'(x) = g(x).$$

Since $g(x)$ is continuous, we have

$$\lim_{x\to x_0} a(x)f'(x) = g(x_0)$$

when $a(x_0) = 0$. So, f is in the domain of A, and $Af = g$. □

Lemma 2. *If* $\lim_{x\to+\infty}\int_0^x \frac{1}{a(s)}\,ds \neq -\infty$, *or the limit does not exist, then* $\ker(\lambda - A) = \{0\}$ *for* $\text{Re}\,\lambda > 0$.

Proof. Let $(\lambda - A)f = 0$, i.e. $\lambda f(x) - a(x)f'(x) = 0$ a.e. Then

$$(e^{-\lambda\int_0^x \frac{1}{a(s)}ds} f(x))' = 0 \quad \text{a.e.}$$

So, there exists a constant c such that

$$e^{-\lambda\int_0^x \frac{1}{a(s)}ds} f(x) = c, \quad \text{for all real } x.$$

Since $f \in C_0(-\infty,\infty)$, $C = 0$ if $\lim_{x\to+\infty}\int_0^x \frac{1}{a(s)}\,ds \neq -\infty$, or the limit does not exist. □

Lemma 3. $\bigcap_{\text{Re}\,\lambda>0}\text{Im}(\lambda - A)$ *is dense in* X.

Proof. Let Q be the set of all functions in X which have compact support. Clearly, it is sufficient to check that $Q \subset \bigcap_{\text{Re}\,\lambda>0}\text{Im}(\lambda - A)$. For any g in Q, define

$$f(x) \equiv -\int_{-\infty}^x \frac{g(t)}{a(t)} e^{\int_t^x \frac{\lambda}{a(s)}\,ds}\,dt. \tag{$*$}$$

Note that

$$f(x) = -e^{\int_0^x \frac{\lambda\,ds}{a(s)}}\left[\int_{-\infty}^x \frac{g(t)}{a(t)} e^{-\int_0^t \frac{\lambda\,ds}{a(s)}}\,dt\right].$$

Since $\int_0^t \frac{\lambda\,ds}{a(s)}$ is an even function of t, $\frac{g(t)}{a(t)}e^{-\int_0^t \frac{\lambda\,ds}{a(s)}}$ is an odd function of t. Thus $f(x)$ is a continuous even function. Let M be a positive number such that $\text{supp}(g) \subset [-M,M]$. Then $\text{supp}(f)$ is also contained in $[-M,M]$, because the integrand is odd, and $\int_{-\infty}^M \frac{g(t)}{a(t)}e^{-\int_0^t \frac{\lambda\,ds}{a(s)}}\,dt = 0$. So $f \in X$.

Since $\frac{1}{a(x)} \in L^1_{\text{loc}}(-\infty,\infty)$ and g has compact support, $f(x) \in \text{AC}_{\text{loc}}$. When $a(x) \neq 0$, by a simple calculation, we have $(\lambda - a(x)\frac{d}{dx})f(x) = g(x)$. Since g is continuous, $\lim_{x-x_0} a(x)f'(x)$ exists if $a(x_0) = 0$. Thus f is in $D(A)$ and $(\lambda - A)f = g$. □

Remark 4. From the proof of Lemma 3, we can see that $f(x)$ defined in $(*)$ is also in the domain of $\widetilde{A}$. So, Lemma 3 is still true for $\widetilde{A}$.

Lemma 5. *For any g in Q, if $a(x)$ is nonnegative in $[0,\infty)$, then*

$$\|(\lambda - A)^{-1}g\| \le \frac{1}{\text{Re }\lambda}\|g\| \quad \textit{for } \text{Re }\lambda > 0.$$

Proof. Again choose $M > 0$ such that $\text{supp}(g) \subset [-M, M]$. Then

$$\begin{aligned}
\|(\lambda - A)^{-1}g\| &= \sup_{-\infty<x\le 0}\left\{\left|\int_{-M}^{x} \frac{g(t)}{a(t)} e^{\lambda \int_t^x \frac{1}{a(s)}\,ds}\,dt\right|\right\} \\
&\le \sup_{-\infty<x\le 0}\left(-\int_{-M}^{x} \frac{1}{a(t)} e^{\text{Re }\lambda \int_t^x \frac{1}{a(s)}\,ds}\,dt\right)\|g\| \\
&= \sup_{-\infty<x\le 0}\left\{\frac{1}{\text{Re }\lambda}\left(1 - e^{\text{Re }\lambda \int_{-M}^x \frac{1}{a(s)}\,ds}\right)\right\}\|g\| \\
&\le \frac{1}{\text{Re }\lambda}\|g\|. \quad \square
\end{aligned}$$

Lemma 6. (a) *If* (1)–(3) *are satisfied, then* $\{\lambda \in \mathbb{C} : \text{Re }\lambda > 0\} \subset \rho(A)$, *and*

$$(**) \qquad \|(\lambda - A)^{-1}\| \le \frac{1}{\text{Re }\lambda}$$

if $a(x)$ is nonnegative in $[0,\infty)$.

(b) *If there exists $x_0 \in (0,\infty)$ such that $a(x_0) < 0$, then $\|(\lambda-A)^{-1}\|$ is unbounded as* $\text{Re}(\lambda) \to \infty$.

Proof. (a) Suppose (1)–(3) are satisfied, $a(x)$ is nonnegative in $[0,\infty)$ and $\text{Re}(\lambda) > 0$. By Lemma 3, $\lambda - A$ has dense range. Thus for any $g \in X$, there exists a sequence $\{f_n\} \subset D(A)$ such that $(\lambda - A)f_n \to g$. By Lemma 5, there exists $h \in X$ such that $f_n \to h$. Since A is closed, $h \in D(A)$ and $(\lambda - A)h = g$. Thus, $(\lambda - A)$ is surjective. Since the hypothesis of Lemma 2 is satisfied if $a(x)$ is nonnegative, $(\lambda - A)$ is also injective. By the closed graph theorem, $(\lambda - A)^{-1}$ is bounded. Using Lemma 5 again, we get the norm estimate $(**)$.

(b) Assume that $a(x)$ is negative in some maximal interval $(a, b) \subseteq [0,\infty)$. If $b \ne \infty$, let y be a fixed number in $(-b,-a)$, and choose M to be a number greater than b and close enough to b so that $a(x) < 0$ in $(-M,-b)$ and $\int_{-M}^{y} \frac{1}{a(s)}\,ds > 0$. This is possible because $a(x) > 0$ in $(-b,-a)$. Let g be an even function supported in $[-M, M]$ such that $g(x) = 1$ in $[-b,-a]$, and $g(x) < 1$ in $(-M,-b)$. Note that $a(x)$ is negative in $(-M,-b)$; then

$$\begin{aligned}
\|(\lambda - A)^{-1}g\| &= \max_{x\in[-M,0]}\left|\int_0^x \frac{g(t)}{a(t)} e^{\lambda \int_t^x \frac{1}{a(s)}\,ds}\,dt\right| \\
&\ge \int_{-b}^{y} \frac{1}{a(t)} e^{\text{Re }\lambda \int_t^y \frac{1}{a(s)}\,ds}\,dt + \int_{-M}^{-b} \frac{1}{a(t)} e^{\text{Re }\lambda \int_t^y \frac{1}{a(s)}\,ds}\,dt \\
&= \frac{1}{\text{Re }\lambda}\left[\left(e^{\text{Re }\lambda \int_{-b}^y \frac{1}{a(s)}\,ds} - 1\right) + \left(e^{\text{Re }\lambda \int_{-M}^y \frac{1}{a(s)}\,ds} - e^{\text{Re }\lambda \int_{-b}^y \frac{1}{a(s)}\,ds}\right)\right] \\
&= \frac{1}{\text{Re }\lambda}\left(e^{\text{Re }\lambda \int_{-M}^y \frac{1}{a(s)}\,ds} - 1\right).
\end{aligned}$$

Thus, since $\int_{-M}^{y} \frac{1}{a(s)}\,ds > 0$, $e^{\text{Re }\lambda \int_{-M}^y \frac{1}{a(s)}\,ds}$ can be arbitrarily large as $\text{Re }\lambda$ becomes large.

If $b = \infty$, we can choose M to be any positive number greater than a and $g(x)$ to be a positive even function such that $g(x) = 1$ in $[a, \frac{M}{2}]$. Let y be an arbitrary number in $(-\frac{M}{2}, -a)$; then

$$\begin{aligned}\|(\lambda - A)^{-1} g\| &= \max_{x \in [-M,0]} \left| \int_0^x \frac{g(t)}{a(t)} e^{\lambda \int_t^x \frac{1}{a(s)}\, ds}\, dt \right| \\ &\geq \int_{-\frac{M}{2}}^{y} \frac{1}{a(t)} e^{\mathrm{Re}\, \lambda \int_t^y \frac{1}{a(s)}\, ds}\, dt = \frac{1}{\mathrm{Re}\, \lambda} (e^{\mathrm{Re}\, \lambda \int_{-M/2}^{y} \frac{1}{a(s)}\, ds} - 1).\end{aligned}$$

Thus, since $a(x)$ is positive in $(-\frac{M}{2}, -a)$, $e^{\mathrm{Re}\, \lambda \int_{-M/2}^{y} \frac{1}{a(s)}\, ds}$ can be arbitrarily large as Re λ becomes large. □

Proof of Theorem 1. Just apply the Hille-Yosida theorem (e.g. see [7] or [8]) and Lemma 6. □

Proof of Theorem 2. By Remark 4, Lemma 6 is also true for $\widetilde{A}$. So, Theorem 2 follows form the Hille-Yosida theorem. □

Proof of Theorem 3. Extend $a(x)$ to the whole real line as an odd function, and the function in $C_0[0, \infty)$ to the whole real line as an even function, then apply Theorem 2 to get a contraction strongly continuous semigroup $T(t)$ in X (defined in the previous section). Clearly $T(t)$ leaves $C_0[0, \infty)$ invariant (just check the definition of $C_0[0, \infty)$). It is easy to check that the restriction of $T(t)$ on $C_0[0, \infty)$ is the contraction strongly continuous semigroup generated by B. □

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References

1. C. J. K. Batty, *Derivations on compact spaces*, Proc. London Math. Soc. **42** (1981), 299–330. MR **82k:**46078
2. ______, *Derivations on the line and flows along orbits*, Pacific J. Math. **126** (1987), 209–225. MR **88f:**46125
3. B. C. Burch and J. A. Goldstein, *Some boundary value problems for the Hamilton-Jacobi equation*, Hiroshima Math. J. **8** (1978), 223–233. MR **58:**1558
4. R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Interscience, New York, 1962. MR **25:**4216
5. R. deLaubenfels, *Well-behaved derivations on* $C[0,1]$, Pacific J. Math. **115** (1984), 73–80. MR **86a:**46059
6. J. A. Goldstein, *Abstract evolution equations*, Tran. Amer. Math. Soc. **141** (1969), 159–185. MR **40:**789
7. ______, *Semigroups of Linear Operators and Applications*, Oxford, New York, 1985. MR **87c:**47056
8. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, New York, 1983. MR **85g:**47061
9. D. W. Robinson, *Smooth derivations on Abelian* C^**-dynamical systems*, J. Austral. Math. Soc. **42** (1987), 247–264. MR **87m:**46139

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