

## Q.U.P. AND PALEY-WIENER PROPERTIES OF UNIMODULAR, ESPECIALLY NILPOTENT, LIE GROUPS

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ABSTRACT. We give a new proof of a weak Paley-Wiener theorem for nilpotent Lie groups due to Lipsman and Rosenberg and we introduce a general notion of Q.U.P for any unimodular locally compact group.

### 0. INTRODUCTION

0.0. Let  $G$  be a locally compact group equipped with left Haar measure  $m$ .  $\widehat{G}$  will denote the dual of  $G$  (i.e. a maximal set of pairwise disjoint inequivalent continuous irreducible unitary representations of  $G$ ). The Fourier transform  $\hat{f}$  of a  $L^1(G)$  function  $f$  is defined by:

$$\hat{f}(\pi) = \pi(f) = \int_G f(x)\pi(x) dm(x), \quad \pi \in \widehat{G}.$$

If  $f, g$  are measurable functions on  $G$ , we define their convolution  $f * g$  by

$$f * g(x) = \int_G f(y)g(y^{-1}x) dm(y)$$

whenever the integral exists.

For a function  $f$  on  $G$ , let

$$f^*(x) = \overline{f(x^{-1})}, \quad x \in G,$$

and

$${}_x f(y) = f(x^{-1}y), \quad f_x(y) = f(yx^{-1}), \quad x, y \in G.$$

For  $f$  in  $L^1(G)$ , let  $A_f = \{x \in G; f(x) \neq 0\}$  and  $B_f = \{\pi \in \widehat{G}; \hat{f}(\pi) \neq 0\}$ .

In 1973, Matolcsi and Szücs [Mat,Szu] showed that if  $G$  is a locally compact abelian group, then with notation as above,  $m(A_f)\hat{m}(B_f) < 1$  implies  $f = 0$  almost everywhere (here  $\hat{m}$  is the Haar measure on the dual group  $\widehat{G}$ , normalized so that the Plancherel identity is valid). In a 1974 preprint (which was published much later), Benedicks [Ben] showed that for  $f \in L^1(\mathbb{R}^n)$ ,  $m(A_f)\hat{m}(B_f) < \infty$  implies  $f = 0$  almost everywhere, and in 1977 Amrein and Berthier [Amr,Ber] reached the same conclusion using Hilbert space techniques.

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0.1. Now let  $G$  be a unimodular group of type I. Then there exists a measure  $\hat{m}$  on  $\hat{G}$  such that

$$L^2(G) \simeq L^2(\hat{G}) = \int_{\hat{G}}^{\oplus} \text{End}_2(\mathcal{H}_\pi) d\hat{m}(\pi),$$

i.e. the Fourier transform  $f \mapsto \hat{f}$  may be extended to  $L^2(G)$  to furnish an isometric isomorphism between  $L^2(G)$  and the space  $L^2(\hat{G})$  of the measurable fields  $h : \pi \mapsto h(\pi)$ , with  $h(\pi)$  in  $\text{End}_2(\mathcal{H}_\pi)$  and  $\int_{\hat{G}} \|h(\pi)\|_2^2 d\hat{m}(\pi) < \infty$ .

Here  $\text{End}_2(\mathcal{H})$  denotes the space of the Hilbert-Schmidt operators on the Hilbert space  $\mathcal{H}$  and  $\|T\|_2$  is the Hilbert-Schmidt norm

$$\|T\|_2^2 = \sum_k \|T\phi_k\|_{\mathcal{H}}^2, \quad T \in \text{End}_2(\mathcal{H}),$$

where  $\{\phi_k\}$  is any orthonormal basis of  $\mathcal{H}$ .

If  $E \subset \hat{G}$  is a measurable subset, we define  $\mu(E)$  according to Hogan (see [Hog]) by

$$\mu(E) = \int_E (\dim \mathcal{H}_\pi) d\hat{m}(\pi).$$

In [Hog], a unimodular locally compact group  $G$  of type I is said to satisfy the qualitative uncertainty principle (Q.U.P.) if, for each  $f \in L^2(G)$ ,

$$m(A_f) < m(G) \text{ and } \mu(B_f) < \mu(\hat{G}) \Rightarrow f = 0 \quad m\text{-a.e.}$$

It is shown in [Hog], Theorem 2.2, that any unimodular type I group with noncompact identity component satisfies the Q.U.P.

0.2. The condition  $\mu(B_f) < \infty$  implies that for almost all  $\pi \in B_f$ ,  $\dim(\mathcal{H}_\pi) < \infty$ , which is rather restrictive, and that

$$(0.2.1) \quad \int_{\hat{G}} \text{rank}(\pi(f)) d\hat{m}(\pi) < \infty,$$

where  $\text{rank}(\pi(f)) = \dim(\text{image}(\pi(f)))$ .

In this paper, we replace the condition  $\mu(B_f) < \infty$  by the condition (0.2.1). More precisely, we define for  $f$  in  $L^2(G) \cap L^1(G)$  the number  $\hat{\nu}(f)$  by

$$(0.2.2) \quad \hat{\nu}(f) = \int_{\hat{G}} \text{rank}(\pi(f)) d\hat{m}(\pi),$$

and we say that a locally compact unimodular type I group  $G$  satisfies (Q.U.P.) if for any function  $f \in L^2(G)$ ,

$$(0.2.3) \quad m(A_f) < m(G) \text{ and } \hat{\nu}(f) < \infty \Rightarrow f = 0 \quad m\text{-a.e.}$$

There is a third less restrictive candidate in the literature for the second inequality in the definition of Q.U.P., namely

$$\hat{m}(B_f) < \infty.$$

But this last definition leads to very difficult still unsolved problems (see §1.6).

0.3. In the first section, we prove that the property  $\hat{\nu}(f) < \infty$  is equivalent to the existence of an  $h \in L^2(G)$  such that  $h = h^*$  and  $h * f = f$  (Proposition 1.1). In fact, for a unimodular group  $G$  “many” functions  $f$  in  $L^2(G)$  with the property  $\hat{\nu}(f) < \infty$  do exist (Theorem 1.4). We show then that any unimodular type I group  $G$ , which contains a closed noncompact connected subset satisfies (Q.U.P.) (see Theorem 1.3). The proof of this theorem is of course an adaptation of the proofs in [Amr,Ber] (as in [Hog]).

We conclude section 1 with another type of weak Q.U.P. property for some particular group (Theorem 1.8).

0.4. We say that a locally compact unimodular type I group  $G$  has the weak Paley-Wiener property, if any distribution  $T$  on  $G$  with compact support, whose Fourier transform vanishes on a subset of positive Plancherel measure, must be 0.

Recently, Ronald L. Lipsman and Jonathan Rosenberg have shown that nilpotent Lie groups have the weak Paley-Wiener property (see [Lip,Ros]). We shall present in our second section a different proof of this result, based on the notion of adapted Fourier transform (§2.2).

1. Q.U.P. FOR UNIMODULAR GROUPS

a. A general Q.U.P.

**1.1. Proposition.** *Let  $G$  be a locally compact unimodular group of type I. Let  $f$  be in  $L^2(G) \cap L^1(G)$ . Then the number  $\hat{\nu}(f) = \int_{\hat{G}} \text{rank}(\pi(f)) d\hat{m}(\pi)$  is finite if and only if there exists  $h = h^*$  in  $L^2(G)$ , such that  $h * f = f$ .*

*Proof.* Suppose that  $\hat{\nu}(f)$  is finite. Then  $\text{rank}(\pi(f))$  must be finite for almost all  $\pi$ . Hence  $\text{image}(\pi(f * f^*)) = \text{image}(\pi(f))$ ,  $\pi$ -a.e. in  $\hat{G}$ . Let  $h_0 = f * f^*$  and for  $k$  in  $\mathbb{N}$ , let  $h_k = (h_{k-1})^{1/2}$ , where we consider  $h_k$  as an element of the  $C^*$ -algebra  $C^*(G)$  of  $G$ . The limit  $h = \lim_{k \rightarrow \infty} h_k$  exists in the von Neumann algebra  $Vn(G)$  of  $G$  and its Fourier transform  $\hat{h}$  is given for almost all  $\pi$  in  $\hat{G}$  by

$$\hat{h}(\pi) = \lim_{k \rightarrow \infty} \hat{h}_k(\pi) = \lim_{k \rightarrow \infty} (\pi(h_{k-1}))^{1/2} = H(\pi),$$

where  $H(\pi)$  is the orthogonal projection onto the image of  $\pi(f)$ .

The field of operators  $H : \pi \mapsto H(\pi) \in \text{End}(\mathcal{H}_\pi)$  is measurable and is of course contained in the von Neumann algebra of the decomposable operators on  $L^2(\hat{G})$  (see [Dix(1)], 18.8). Furthermore, since

$$\int_{\hat{G}} \text{tr}(H(\pi)^2) d\hat{m}(\pi) = \int_{\hat{G}} \text{rank}(\pi(f)) d\hat{m}(\pi) < \infty,$$

we see that  $H$  belongs to  $L^2(\hat{G})$  and so  $h$  to  $L^2(G)$ . Since  $H(\pi) \circ \pi(f) = \pi(f)$  for almost all  $\pi \in \hat{G}$ , we have that  $h * f = f$ .

Conversely, suppose that  $h * f = f$ , for some  $h = h^*$  in  $L^2(G)$ . Then since  $\hat{h}(\pi) \circ \hat{f}(\pi) = \pi(f)$  for almost all  $\pi$  in  $\hat{G}$ , it follows that  $\text{image}(\pi(f))$  is contained in the eigenspace of  $\hat{h}(\pi)$  for the eigenvalue 1. Hence  $\text{rank}(\pi(f)) \leq \text{tr}(\hat{h}(\pi)^2)$  and

$$\hat{\nu}(f) = \int_{\hat{G}} \text{rank}(\pi(f)) d\hat{m}(\pi) \leq \int_{\hat{G}} \text{tr}(\hat{h}(\pi)^2) d\hat{m}(\pi) = \|h\|_2^2 < \infty. \quad \square$$

The following lemma can also be found in a slightly different form in [Hog].

**1.2. Lemma.** *Let  $G$  be a unimodular locally compact group which contains a closed noncompact connected subset. Let  $V \subset W$  be two measurable subsets of  $G$ , such that  $0 \neq m(V)$ ,  $m(W) < \infty$ . Then for any  $\varepsilon > 0$ , there exists  $x \in G$ , such that*

$$m(V) - 2\varepsilon < m(V \cdot x \cap W) < m(V) - \varepsilon.$$

*Proof.* Clearly the connected component of the identity  $G_0$  is not compact. Denoting the characteristic function of a subset  $A$  of  $G$  by  $1_A$ , we have for  $x$  in  $G$ ,

$$m(V \cdot x \cap W) = \int_G 1_{V \cdot x}(y) 1_W(y) dm(y) = (1_V)^* * 1_W(x).$$

The functions  $(1_V)^*$  and  $1_W$  are in  $L^2(G)$  and so  $\phi = (1_V)^* * 1_W$  is a continuous function which vanishes at infinity. This tells us, that  $\lim_{G_0 \ni x \rightarrow \infty} \phi(x) = 0$ . Since  $G_0$  is connected, the image of the restriction of  $\phi$  to  $G_0$  contains the connected subset  $]0, m(V)[$  of  $\mathcal{R}$ .  $\square$

**1.3. Theorem.** *Let  $G$  be a locally compact unimodular group containing a closed noncompact connected subset. Let  $f$  be in  $L^2(G)$  such that  $m(A_f) < m(G)$  and such that there exists  $h$  in  $L^2(G)$  with  $h * f = f$ . Then  $f = 0$  m-a.e.*

*Proof.* Suppose that  $f \neq 0$ . We observe that for any  $x \in G$ ,  $h * f_x = f_x$ .

Now let  $C = \text{supp} f = A_f$  and let  $x_0 = e$ , the identity element of  $G$ . Using Lemma 1.2 we can easily find a sequence  $x_0, x_1, \dots$ , in  $G$ , such that

$$(1.3.1) \quad m(C) - \frac{1}{2^{k-1}} < m(C \cdot x_k \cap C_{k-1}) < m(C) - \frac{1}{2^k}, \quad k \in \mathbb{N},$$

where for  $k \geq 1$ ,  $C_{k-1} = \bigcup_{i=0}^{k-1} C \cdot x_i \supset C$ .

Let  $C_\infty = \bigcup_{k=0}^\infty C_k$ . We have that

$$\begin{aligned} m(C_{k+1}) &= m(C_k \cup C \cdot x_{k+1}) = m(C_k) + m(C \cdot x_{k+1} \setminus C_k) \\ &= m(C_k) + m(C \cdot x_{k+1}) - m(C \cdot x_{k+1} \cap C_k) \leq m(C_k) + \frac{1}{2^k} \end{aligned}$$

by (1.3.1) and the identity  $m(C \cdot x) = m(C)$  for all  $x$  in  $G$ . Hence,  $m(C_{k+1}) \leq m(C) + \sum_{i=0}^k \frac{1}{2^i}$ , for all  $k$  in  $\mathbb{N}$  and so  $m(C_\infty) < \infty$ .

Let  $\phi = 1_{C_\infty}$  and consider the operator  $K : L^2(G) \rightarrow L^2(G)$ ,  $g \mapsto \phi \cdot h * g$ . Since

$$K(g)(x) = \phi(x) \int_G h(y) g(y^{-1}x) dm(y) = \int_G \phi(x) h(xy^{-1}) g(y) dm(y),$$

$K$  is a kernel operator with kernel function:  $(x, y) \mapsto \phi(x) h(xy^{-1})$ . But this kernel function is in  $L^2(G \times G)$  and thus  $K$  is a Hilbert-Schmidt operator and hence is compact.

We have that  $\text{supp} f_{x_k} = C \cdot x_k \subset C_\infty$ . Whence the functions  $f_{x_k}$  are all eigenvectors of  $K$  for the eigenvalue 1. On the other hand, again by (1.3.1),

$$m(\text{supp} f_{x_k} \setminus \bigcup_{i=0}^{k-1} \text{supp} f_{x_i}) \neq 0.$$

The functions  $f_{x_j}$  are thus all linearly independent in  $L^2(G)$  and so the compact operator  $K$  has an eigenspace of infinite dimension, which is impossible. We see that  $f$  must be the zero function in  $L^2(G)$ .  $\square$

1.4. If  $G$  is unimodular, it is easy to construct by the method of [Dix(2)] elements  $h, f$  in  $L^2(G)$  such that  $h \neq 0$  and  $h * f = f$ .

**1.4. Theorem.** *Let  $G$  be a locally compact unimodular group. The functions  $f$  in  $L^2(G)$  such that there exists  $h = h^*$  in  $L^2(G)$  so that  $h * f = f$  are dense in  $L^2(G)$ .*

*Proof.* Indeed, let  $g = g^*$  in  $L^1(G) \cap L^2(G)$ . Define for  $\lambda \in \mathbb{R}$ ,  $e(i\lambda g)$  by

$$e(i\lambda g) = i\lambda g + \frac{1}{2}(i\lambda g)^{2*} + \frac{1}{3!}(i\lambda g)^{3*} + \dots = \sum_{j=1}^{\infty} \frac{1}{j!}(i\lambda L_g)^{j-1}(i\lambda g)$$

where  $L_g$  denotes the selfadjoint operator  $L_g(k) = g * k$  on  $L^2(G)$ . Let  $q(\lambda)$  be the function  $\frac{e^{i\lambda} - 1}{i\lambda}$ , for  $\lambda$  in  $\mathbb{R}$ . Then the operator  $\sum_{j=1}^{\infty} \frac{1}{j!}(i\lambda L_g)^{j-1}$  is  $q\{L_g\}$  in the sense of functional calculus and its norm  $\|\sum_{j=1}^{\infty} \frac{1}{j!}(i\lambda L_g)^{j-1}\|_{\text{op}}$  is less than  $\|q\|_{\infty}$ . Whence:

$$e(i\lambda g) \in L^2(G) \quad \text{and} \quad \|e(i\lambda g)\|_2 \leq \|g\|_2 \|q\|_{\infty} (1 + |\lambda|), \quad \forall \lambda \in \mathbb{R}.$$

For any function  $\phi$  in  $C^3(\mathbb{R})$ , with compact support, such that  $\phi(0) = 0$ , we have  $|\hat{\phi}(\lambda)| \|e(i\lambda g)\|_2 \leq C(1 + |\lambda|)^{-2}$  for some constant  $C$  independent of  $\lambda$  and so the integral

$$\phi\{g\} = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\phi}(\lambda) e(i\lambda g) d\lambda$$

converges in  $L^2(G)$ . Furthermore we have

$$(1.4.1) \quad \phi\{g\} * k = \phi\{L_g\}(k), \quad k \in L^2(G).$$

Indeed,  $e(i\lambda g) * k = (\exp(L_{i\lambda g}) - 1)(k)$  and  $\|\exp(L_{i\lambda g})\|_{\text{op}} = 1$ , for all  $\lambda$  in  $\mathbb{R}$ , hence for any character  $\mu$  of the (abelian)  $C^*$ -algebra generated by  $L_g$ , we have

$$\mu \left( \int_{\mathbb{R}} \hat{\phi}(\lambda) e(L_{i\lambda g}) d\lambda \right) = \int_{\mathbb{R}} \hat{\phi}(\lambda) \exp(i\lambda \mu(L_g)) d\lambda = 2\pi \phi(\mu(L_g)),$$

by Fourier's inversion formula.

Choose now  $C^3$  functions  $\varphi, \phi : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\varphi$  and  $\phi$  vanish on a neighbourhood of 0 and such that  $\phi\varphi = \varphi$ . Let

$$h = \int_{\mathbb{R}} \hat{\phi}(\lambda) \exp(i\lambda g) d\lambda, \quad f = \int_{\mathbb{R}} \hat{\varphi}(\lambda) \exp(i\lambda g) d\lambda \in L^2(G).$$

Then  $h * f * k = \phi\{L_g\} \circ \varphi\{L_g\}(k) = \phi\varphi\{L_g\}(k) = \varphi\{L_g\}(k) = f * k$ , for all  $k$  in  $L^2(G)$ . Whence  $h * f = f$ . In order to prove density, let  $k$  be any element in  $L^2(G)$  and let  $\varepsilon > 0$ . Choose  $g = g^*$  in  $L^1(G)$  continuous with compact support, such that  $\|g\|_1 = 1$  and  $\|g^{3*} * k - k\|_2 \leq \varepsilon$ . Take a function  $\varphi_0$  in  $C^3(\mathbb{R})$  with  $\varphi_0(t) = t^3$  in a neighbourhood of  $] -1, 1[$ . Then  $\varphi_0\{g\} = g^{*3}$ . Choose  $\varphi$  in  $C^3(\mathbb{R})$  so that  $\|\varphi - \varphi_0\|_{\infty} < \varepsilon$  and so that  $\varphi$  vanishes in a neighbourhood of 0. Then

$$\begin{aligned} \|\varphi\{g\} * k - g^{3*} * k\|_2 &= \|\varphi\{g\} * k - \varphi_0\{g\} * k\|_2 = \|(\varphi\{L_g\} - \varphi_0\{L_g\})(k)\|_2 \\ &\leq \|\varphi\{L_g\} - \varphi_0\{L_g\}\|_{\text{op}} \|k\|_2 \leq \|\varphi - \varphi_0\|_{\infty} \|k\|_2 < \varepsilon \|k\|_2. \end{aligned}$$

Let  $f = \varphi\{g\}$ ,  $f$  is in  $L^2(G)$ . We then have

$$\|f * k - k\|_2 \leq (1 + \|k\|_2)\varepsilon, \quad h * f * k = f * k, \quad \text{for } h = \phi\{g\} \in L^2(G). \quad \square$$

**1.5. Corollary.** *Let  $G$  be a unimodular group containing a closed noncompact connected subset. Let  $f = f^*$  be in  $L^2(G) \cap L^1(G)$  and let  $\varphi$  be in  $C^3(\mathbb{R})$  which is zero in a neighbourhood of 0. If  $m(A_{\varphi\{f\}})$  is finite, then  $\varphi\{f\} = 0$ .*

The proof is an immediate consequence of Theorem 1.3 and the proof of Theorem 1.4.

**b. A weak Q.U.P. for special groups.**

**1.6.** Let  $G$  be a connected simply connected nilpotent Lie group. The problem whether for such groups the condition  $f \in L^2(G)$ ,  $m(A_f)\hat{m}(B_f) < \infty$  implies  $f = 0$  is still not settled, not even in low dimensions except for some special Heisenberg type groups (see [Ech,Kan,Kum]). Nevertheless, we shall close this section with a weaker result, for a relatively larger class of groups.

From now on,  $G$  is a locally compact group containing a nontrivial central vector group  $Z$ . Every  $\pi$  in  $\widehat{G}$  restricts on  $Z$  to a character  $\chi_\pi$ .

**1.7. Definition.** We say that a subset  $\mathcal{B}$  of  $\widehat{G}$  is *bounded in  $\widehat{Z}$* , if there exists a bounded subset  $\mathcal{B}_0$  in  $\widehat{Z}$  such that for every  $\pi$  in  $\mathcal{B}$ ,  $\chi_\pi$  is in  $\mathcal{B}_0$ .

We shall prove here the vanishing of any function  $f$  in  $L^2(G)$ , such that  $m(A_f) < \infty$  and  $B_f$  bounded in  $\widehat{Z}$ . This requirement on  $f$  is clearly stronger than the condition  $m(A_f)\hat{m}(B_f) < \infty$ , thus we say that Theorem 1.8 is a weak Q.U.P. for  $G$ .

**1.8. Theorem.** *Let  $G$  be a unimodular amenable separable group containing a nontrivial central vector group  $Z$ . Let  $f$  be in  $L^2(G)$  such that  $m(A_f) < \infty$  and such that  $B_f$  is bounded in  $\widehat{Z}$ . Then  $f = 0$ .*

*Proof.* Since, for any  $g$  in  $L^1(G)$ ,

$$\int_G g(x) dm(x) = \int_{G/Z} \left( \int_Z g(xz) dm_Z(z) \right) dm_{G/Z}(x),$$

it follows from Fubini's theorem, that for almost every  $x$  in  $G$ , the function  $z \mapsto {}_x f|_Z(z) := f(xz)$  is in  $L^2(Z)$  and  $A({}_x f|_Z)$  is of bounded Lebesgue measure. The fact that  $B_f$  is bounded in  $\widehat{Z}$  implies that there exists a bounded subset  $\mathcal{B}_0$  in  $\widehat{Z}$ , such that  $B_f \subset \{\pi \in \widehat{G}; \chi_\pi \in \mathcal{B}_0\}$ . Let

$$\widehat{G}_{\mathcal{B}_0} := \{\pi \in \widehat{G}, \text{ such that } \chi_\pi \notin \mathcal{B}_0\}.$$

Then  $\widehat{G}_{\mathcal{B}_0}$  has empty intersection with  $B_f$ . Fix now  $\chi$  in  $\widehat{Z} \setminus \mathcal{B}_0$  and let  $\rho = \text{ind}_Z^G \chi$ . Since  $G$  is amenable, the kernel of  $\rho$  in  $C^*(G)$  is the intersection of all the  $\ker_{C^*(G)} \pi$  for which  $\chi_\pi = \chi$ . It follows that  $f$  is in  $\ker_{C^*(G)} \rho$ , i.e.  $\rho(f) = 0$ . The kernel of  $\rho$  in  $L^1(G)$  is also given by

$$\ker_{L^1(G)} \rho = \{g \in L^1(G); \text{ for almost every } x \in G, \\ {}_x g|_Z \in L^1(Z) \text{ and } ({}_x g|_Z)^\wedge(\chi) = 0\}$$

(see [Lud]). Hence for almost all  $x$  in  $G$ ,  $({}_x f|_Z)^\wedge(\chi) = 0$ . If we choose a countable dense subset  $\mathcal{S}$  in  $\widehat{Z} \setminus \mathcal{B}_0$ , then we see that there exists a null set  $\mathcal{N}$  in  $G$ , such that

- (i) for every  $x \notin \mathcal{N}$ , the function  ${}_x f|_Z$  is in  $L^2(Z)$ ,
- (ii) the support of  ${}_x f|_Z$  is of bounded Haar measure and
- (iii)  $({}_x f|_Z)^\wedge(\chi) = 0$  for all  $\chi$  in  $\mathcal{S}$ , hence for all  $\chi$  in  $\widehat{Z} \setminus \mathcal{B}_0$ .

Thus for any  $x \notin \mathcal{N}$ ,  $({}_x f|_Z)^\wedge$  has its support contained in the bounded subset  $\mathcal{B}_0$ , it is in  $L^2(\widehat{Z})$  and its Fourier transform  ${}_x f|_Z$  vanishes on a subset of positive Lebesgue measure, since  $m(A_{{}_x f|_Z}) < \infty$ . Whence Q.U.P. for the abelian group  $Z$  tells us that  ${}_x f|_Z = 0$ , for all  $x \notin \mathcal{N}$ . Finally  $f = 0$ .  $\square$

2. A WEAK PALEY-WIENER PROPERTY FOR NILPOTENT LIE GROUPS

2.0. Let  $G$  be a nilpotent Lie group and let  $T$  be a distribution on  $G$  with compact support. For any  $f$  in the Schwartz space  $S(G)$  of the rapidly decreasing  $C^\infty$  functions on  $G$ , the function  $T * f$ , defined by  $T * f(x) = \langle T, l_x(\tilde{f}) \rangle$  ( $x \in G$ ), where  $\tilde{f}(x) = f(x^{-1})$ , is also in  $S(G)$  and  $f \mapsto T * f$  defines a continuous endomorphism of  $S(G)$ . We can define for  $\pi$  in  $\widehat{G}$  the operator  $\pi(T)$  which acts on the space  $\mathcal{H}_\pi^\infty$  of the  $C^\infty$ -vectors of  $\mathcal{H}_\pi$  in the following way.

By a result of Howe (see [Cor,Gre] 4.2.1), the convolution algebra  $S(G)$  acts algebraically irreducibly on  $\mathcal{H}_\pi^\infty$  and so it is easy to see that

$$\pi(T)(\pi(f)\xi) := \pi(T * f)\xi, \quad f \in S(G), \xi \in \mathcal{H}_\pi,$$

is well defined and that  $\pi(T)$  is even a continuous linear endomorphism of  $\mathcal{H}_\pi^\infty$ . We can thus define the Fourier transform  $\widehat{T}$  of  $T$  by setting  $\widehat{T}(\pi) = \pi(T)$  in  $\text{End}(\mathcal{H}_\pi^\infty)$ ,  $\pi$  in  $\widehat{G}$ .

Lipsman and Rosenberg ([Lip,Ros]) have recently proved a conjecture of Moss (see [Mos]), which said that if  $\widehat{T}$  vanishes on a subset of positive Plancherel measure, then  $T = 0$ . Recently a new proof of this result has appeared in a preprint of G. Garimella (see [Gar]). We shall present here another proof of this conjecture which is based on the notion of the adapted Fourier transform (see [Arn,Cor], [Arn,Gut] and [Lud,Zah]). Let us recall that Kirillov's theory allows us to identify  $\widehat{G}$  with the space  $\mathfrak{g}^*/G$  of the coadjoint orbits of  $G$  on the dual vector space  $\mathfrak{g}^*$  of the Lie algebra  $\mathfrak{g}$  of  $G$ . Denote by  $K$  the Kirillov mapping

$$K : \mathfrak{g}/G \rightarrow \widehat{G}, \quad l \mapsto \pi_l = K(\text{Ad}^*(G)l).$$

The relation between  $\Omega = \text{Ad}^*(G)l$  and  $\pi = K(\Omega)$  is given by Kirillov's character formula

$$(2.0.1) \quad \text{tr}(\pi(f)) = \int_\Omega (f \circ \exp)^\wedge(\xi) d\mu(\xi)$$

for any  $f$  in  $S(G)$ , where  $d\mu$  denotes the  $G$ -invariant Kirillov measure on  $\Omega$ .

2.1. Pukanszky has given the following description of the orbit space  $\mathfrak{g}^*/G$ . Let  $\mathfrak{B} = \{q_1, \dots, q_n\}$  be a Jordan-Hölder basis of the  $\mathfrak{g}$ -module  $\mathfrak{g}^*$ . There exists a family of  $r$  polynomial functions  $p_k$ ,  $1 \leq k \leq r$ , on  $\mathfrak{g}^*$  and for every  $j$  in  $\{1, \dots, r\}$  an index set  $e_j = \{i_1(j), \dots, i_{d(j)}(j)\} \subset \{1, \dots, n\}$  with the following property:

Let  $\mathfrak{g}_j^* = \{l \in \mathfrak{g}^*; p_1(l) = \dots = p_{j-1}(l) = 0, p_j(l) \neq 0\}$  ( $j = 1, \dots, r$ ) (here  $p_0 = 0$ ). The subsets  $\mathfrak{g}_j^*$  form a  $G$ -invariant partition of  $\mathfrak{g}^*$ . There exists for every  $j$  a family of functions  $p_1^j, \dots, p_n^j$  on  $\mathfrak{g}_j^* \times \mathbb{R}^{d(j)}$  which are polynomials in  $Z \in \mathbb{R}^{d(j)}$  with rational coefficients in  $l \in \mathfrak{g}_j^*$  and the denominators of these rational functions in  $l$  are powers of  $p_j$ . These  $p_i^j$ ,  $i = 1, 2, \dots$ , have the following properties:

- (i)  $p_k^j(l, (z_1, \dots, z_{d(j)}))$  does only depend on  $l$  and on  $z_1, \dots, z_{k'}$ , where  $i_{k'}(j) \leq k < i_{k'+1}(j)$ .
- (ii)  $p_{i_k}^j = z_k$ ,  $k = 1, \dots, d(j)$ ,  $i_k \in e_j$ .
- (iii)  $\text{Ad}^*(G)l = \{P^j(l, z) := \sum_{k=0}^n p_k^j(l, z)q_k, z \in \mathbb{R}^{d(j)}\}$ ,  $l \in \mathfrak{g}_j^*$ .

Now let  $d = d(1)$ ,  $e = e_1 = \{i_1, \dots, i_d\}$  and let  $\mathfrak{Y} = \sum_{i \notin e} \mathbb{R}q_i$  and  $\mathfrak{Y}_1 = \mathfrak{Y} \cap \mathfrak{g}_1^*$ . Then for any  $l \in \mathfrak{Y}_1$  we have  $l = P^1(l, 0)$  and the mapping

$$\mathfrak{Y}_1 \rightarrow \mathfrak{g}_1^*/G, \quad l \mapsto \text{Ad}^*(G)P^1(l, 0)$$

is a bijection. In particular  $\mathfrak{Y}_1$  is Zariski-open in  $\mathfrak{Y}$ ,  $\widehat{G}_1 := K(\mathfrak{g}_1^*/G) \subset \widehat{G}$  carries the Plancherel measure and there exists a polynomial function  $Q$  on  $\mathfrak{Y}$  such that the measure  $|Q|dl$  corresponds to the Plancherel measure on  $\widehat{G}$  under the bijection  $\mathfrak{Y}_1 \rightarrow \widehat{G}_1 : l \mapsto \pi_l$ . In fact for any measurable function  $\phi$  on  $\mathfrak{g}^*$ , we have

$$(2.1.1) \quad \int_{\mathfrak{g}^*} \phi(l) dl = \int_{\mathfrak{Y}_1} \left( \int_{\mathbb{R}^{d(1)}} \phi(P^1(l, Z)) dZ \right) |Q(l)| dl.$$

2.2. Let us now define the adapted Fourier transform (see [Lud,Zah]).

There exists a  $G$ -invariant Zariski-open subset  $\mathfrak{g}_0^*$  of  $\mathfrak{g}_1^*$  and a continuous function

$$a : G \times \mathfrak{g}_0^* \rightarrow \mathbb{R}, \quad (x, l) \mapsto a(x, l) = \sum_{\alpha} a_{\alpha}(l)x^{\alpha}$$

which is polynomial in  $x \in G$  with rational coefficients  $a_{\alpha} = \frac{Q_{\alpha}}{Q_0}$ , whose common denominator  $Q_0$  does not vanish on  $\mathfrak{g}_0^*$  and which replaces the usual bracket  $\langle l, x \rangle$  on  $\mathfrak{g}^* \times \mathfrak{g}$ , i.e., if we define the adapted Fourier transform  $\hat{f}^a$  of an  $L^1(G)$ -function  $f$  by

$$\hat{f}^a(l) = \int_G f(x)e^{-ia(x,l)} dm(x), \quad l \in \mathfrak{g}_0^*,$$

then  $\hat{f}^a$  is a continuous bounded function on  $\mathfrak{g}_0^*$  and, for any  $\pi = K(\text{Ad}^*(G)l)$  in  $\widehat{G}_0 = K(\mathfrak{g}_0^*/G)$ , for any  $C^\infty$ -function  $f$  with compact support, we have

$$(2.2.1) \quad \text{tr}(\pi(f) \circ \pi(f)^*) = \int_{\text{Ad}^*(G)l} |\hat{f}^a(\xi)|^2 d\mu(\xi),$$

where  $d\mu$  is as in (2.0.1). Formula (2.2.1) allows us to deduce from  $\hat{f}^a = 0$  on  $\text{Ad}^*(G)l$  that  $\pi(f) = 0$ , whereas  $\hat{f}(\text{Ad}^*(G)l) = 0$  generally does not imply the vanishing of  $\pi(f)$  (see [Lud]).

2.3. We shall give now a new proof of theorem 2.16 in [Lip,Ros]. We need first a technical result on analytic functions, for which we could not find a reference.

**2.3 Lemma.** *Let  $\mathcal{U}$  be an open connected subset of  $\mathbb{R}^n$ , let  $f : \mathcal{U} \rightarrow \mathbb{C}$  be an analytic function. If  $f$  vanishes on a subset of positive Lebesgue measure, then  $f = 0$ .*

*Proof.* (Induction on  $n$ ) The lemma is of course trivial for  $n = 1$ . Suppose now  $n > 1$ . Let  $N = \{x \in \mathcal{U}, f(x) = 0\}$ . Choose a box  $V = [a, b] \times \overline{V}$  in  $\mathcal{U}$  such that  $\int_V 1_N(y) dy \neq 0$ . Fubini's theorem implies that  $\int_{\overline{V}} 1_N(y_1, \overline{y}) d\overline{y} \neq 0$  for all  $y_1$  in a subset  $I$  of positive Lebesgue measure in  $[a, b]$ . For such a  $y_1$ , the analytic function  $\overline{y} \mapsto f(y_1, \overline{y})$  vanishes on a set of positive Lebesgue measure and so must be 0 by the induction hypothesis. Hence  $f(y_1, \overline{y}) = 0$  for any  $y_1$  in  $I$ , for every  $\overline{y}$  in  $\overline{V}$ . Since  $I$  is of positive Lebesgue measure, we see that  $f$  is identically zero on  $V$ .  $\square$

**2.4 Theorem.** *Let  $G$  be a connected simply connected nilpotent Lie group. Let  $T$  be a distribution on  $G$  with compact support such that  $\widehat{T}$  vanishes in  $\widehat{G}$  on a subset of positive Plancherel measure. Then  $T = 0$ .*

*Proof.* Let  $\mathfrak{W}'$  be a measurable subset of positive Plancherel measure in  $\widehat{G}$  on which  $\widehat{T}$  vanishes. Let  $\mathfrak{W} = \{l \in \mathfrak{Y}_1; \pi_l \in \mathfrak{W}'\}$ . Since the Plancherel measure is concentrated on  $K(\mathfrak{g}_0^*/G)$ , and since it corresponds to  $|Q|$  times Lebesgue measure on  $\mathfrak{Y}_1$ , we see that  $\mathfrak{W}$  is of positive Lebesgue measure and so  $\mathcal{W} = \text{Ad}^*(G)\mathfrak{W}$  is of positive Lebesgue measure in  $\mathfrak{g}^*$  by (2.1.1).



Let  $g$  be any  $C^\infty$  function with compact support on  $G$  and let  $f = T * g$ . The function  $f$  is itself a  $C^\infty$  function whose support is contained in  $\text{supp } T \cdot \text{supp } g$ , whence is compact. Since  $\pi(f) = \pi(T * g) = \pi(T) \circ \pi(g)$ , for any  $\pi$  in  $\widehat{G}$ , we see that  $\hat{f}$  vanishes on a set which contains  $\mathfrak{W}'$ . If we show that  $f = 0$ , then  $\langle T, \tilde{g} \rangle = T * g(e) = 0$ , and so  $T$  must be 0.

In order to prove that  $f = 0$ , we consider  $\mathfrak{g}_0^* \subset \mathfrak{g}_{0\mathbb{C}}^* = \{\xi \in \mathfrak{g}_{\mathbb{C}}^*; Q_0(\xi) \neq 0\}$ . It is easy to see that  $\mathfrak{g}_{0\mathbb{C}}^*$  is pathwise connected and so is connected. Since the support of  $f$  is compact, the integral

$$\hat{f}^a(\xi) = \int_G f(x) e^{-ia(x,\xi)} dx$$

exists for any  $\xi$  in  $\mathfrak{g}_{0\mathbb{C}}^*$  and the function  $\hat{f}^a$  is even holomorphic on  $\mathfrak{g}_{0\mathbb{C}}^*$ , indeed, for any  $\xi, \eta$  in  $\mathfrak{g}_{0\mathbb{C}}^*$ ,

$$\frac{d}{dz} \hat{f}^a(\xi + z\eta)|_{z=0} = \int_G \left( -i \frac{d}{dz} a(x, \xi + z\eta)|_{z=0} \right) f(x) e^{-ia(x,\xi)} dx$$

does exist and is continuous in  $\xi$ . Since  $\hat{f}$  vanishes on  $\mathfrak{W}'$ , we have  $\pi_l(f) = 0$  for any  $l$  in  $\mathfrak{W}$  and so by (2.2.1)  $\hat{f}^a$  is 0 on  $\mathcal{W}$ , which is of positive Lebesgue measure. Hence, by Lemma 2.3, the holomorphic function  $\hat{f}^a$  is identically 0 in  $\mathfrak{g}_{0\mathbb{C}}^*$ , since  $\mathfrak{g}_{0\mathbb{C}}^*$  is connected.

Thus  $\hat{f}$  vanishes on  $K(\mathfrak{g}_0^*/G)$  which is dense in  $\widehat{G}$ . The injectivity of the Fourier transform implies that  $f = 0$ .  $\square$

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