PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 4, April 1997, Pages 1209–1214
S 0002-9939(97)03622-8

INTERSECTION OF SETS WITH n-CONNECTED UNIONS

CHARLES D. HORVATH AND MARC LASSONDE

(Communicated by Peter Li)

Abstract. We show that if n sets in a topological space are given so that all the sets are closed or all are open, and for each $k \leq n$ every k of the sets have a $(k-2)$-connected union, then the n sets have a point in common. As a consequence, we obtain the following starshaped version of Helly's theorem: If every $n+1$ or fewer members of a finite family of closed sets in $\mathbb{R}^n$ have a starshaped union, then all the members of the family have a point in common. The proof relies on a topological KKM-type intersection theorem.

1. Introduction

The purpose of this paper is to generalize to an arbitrary finite family of sets the following elementary fact: *If in a topological space two nonempty sets, both closed or both open, have a pathwise connected union, then they have a point in common.* To this end, we use the notion of n-connectedness which is a natural generalization of pathwise connectedness. Let us recall the definition.

For any integer $n \geq -1$, we denote by Δ_{n+1} the unit $(n+1)$-simplex and by $\partial\Delta_{n+1}$ its boundary. A topological space C is said to be *n-connected* if every continuous map $f : \partial\Delta_{n+1} \to C$ has a continuous extension $g : \Delta_{n+1} \to C$.

Clearly, (-1)-connected means nonempty and 0-connected means pathwise connected. It is also easily seen that for a pathwise connected space, 1-connected is equivalent to simply connected. Convex sets and starshaped sets in topological vector spaces, and, more generally, contractible spaces are n-connected for every n.

Our main result (Theorem 3) reads as follows: *If n sets in a topological space are given so that all the sets are closed or all are open and for each $k \leq n$ every k of the sets have a $(k-2)$-connected union, then the n sets have a point in common.* For $n = 2$ we exactly recover the above-mentionned fact. This extension is equivalent to Brouwer's fixed point theorem.

Applications to Helly-type theorems are considered. In particular, the following generalization of results of Breen [2, 3] is established (Theorem 5): *If every $n+1$ or fewer members of a finite family of closed sets in $\mathbb{R}^n$ have a starshaped union, then all the members of the family have a point in common.*

The basic tool for our study is an intersection theorem (Theorem 2) obtained by combining the well-known theorem of Knaster-Kuratowski-Mazurkiewicz with a selection-extension property for families of n-connected sets (Theorem 1). No

Received by the editors August 14, 1995 and, in revised form, October 25, 1995.
1991 *Mathematics Subject Classification.* Primary 52A30, 54C99; Secondary 52A35, 52A07.
Key words and phrases. n-connected sets, starshaped sets, Helly's theorem, KKM theorem.

notion from algebraic topology is used. The ideas of this approach come from Horvath [11] and Lassonde [14].

For a comprehensive account on Helly-type results, the reader is referred to the survey papers of Danzer, Grünbaum, Klee [4] and Eckhoff [7].

2. Selection-extension property

By *polytope* we understand a simplicial CW-complex in the sense of Whitehead. We recall the definitions. A *triangulation* is a collection $\mathcal{T}$ of geometric closed simplexes such that every face of a simplex in $\mathcal{T}$ is itself a simplex in $\mathcal{T}$, and the intersection of any two simplexes in $\mathcal{T}$ is a face of both of them. We denote by $\mathcal{T}^{(n)}$ the collection of all simplexes of $\mathcal{T}$ of dimension less than or equal to n.

A *polytope* is a topological space P together with a triangulation $\mathcal{T}$ such that $P = \bigcup\{\, \sigma \mid \sigma \in \mathcal{T}\,\}$, and P is supplied with the CW-topology, i.e. $U \subseteq P$ is open if and only if for each $\sigma \in \mathcal{T}$, $U \cap \sigma$ is open in the Euclidean topology of σ. We denote by $P^{(n)}$ the n-skeleton of P, i.e. the subpolytope of P with triangulation $\mathcal{T}^{(n)}$. A *subpolytope* of P is a polytope $Q \subseteq P$ with a triangulation $\mathcal{S} \subseteq \mathcal{T}$.

The following theorem, though more general than required for the applications considered in this paper, is worth stating explicitly for future use. It strengthens a result of Horvath [11, Theorem 1] in several aspects; the proof is similar.

Theorem 1. *Let X be a topological space, P a polytope with triangulation $\mathcal{T}$, $Q \subseteq P$ a subpolytope with triangulation $\mathcal{S} \subseteq \mathcal{T}$, and $\{\, C_\sigma \mid \sigma \in \mathcal{T}\,\}$ a family of subsets of X such that:*

(a) $C_\sigma \subseteq C_\tau$ for every $\sigma, \tau \in \mathcal{T}$ with $\sigma \subseteq \tau$,

(b) C_σ is $(\dim \sigma - 1)$-connected for every $\sigma \in \mathcal{T} \setminus \mathcal{S}$.

Then any continuous map $f : Q \to X$ satisfying $f(\sigma) \subseteq C_\sigma$ for all $\sigma \in \mathcal{S}$ can be extended to a continuous map $g : P \to X$ satisfying $g(\sigma) \subseteq C_\sigma$ for all $\sigma \in \mathcal{T}$.

Proof. Since for any vertex $x \in \mathcal{T}^{(0)}$ the set $C_{\{x\}}$ is not empty because of (b), the map $f_0 : Q \cup P^{(0)} \to X$ given by $f_0(x) = f(x)$ for $x \in Q$, and $f_0(x) =$ any point in $C_{\{x\}}$ for $x \in P^{(0)} \setminus Q$ is a continuous extension of f such that $f_0(\sigma) \subseteq C_\sigma$ for all $\sigma \in \mathcal{S} \cup \mathcal{T}^{(0)}$. Starting from f_0, we define inductively a sequence of continuous extensions $f_n : Q \cup P^{(n)} \to X$ satisfying $f_n(\sigma) \subseteq C_\sigma$ for all $\sigma \in \mathcal{S} \cup \mathcal{T}^{(n)}$. The result then follows by taking $g : P \to X$ such that $g(x) = f_n(x)$ whenever $x \in Q \cup P^{(n)}$.

Suppose f_n is constructed. The required extension of f_n is obtained by pasting together a family of maps $f^\sigma_{n+1} : \sigma \to C_\sigma$, $\sigma \in \mathcal{S} \cup \mathcal{T}^{(n+1)}$, defined as follows. For $\sigma \in \mathcal{S} \cup \mathcal{T}^{(n)}$, simply let $f^\sigma_{n+1} = f_{n|\sigma}$. Otherwise, write $\partial\sigma = \sigma_1 \cup \ldots \cup \sigma_k$, where each σ_i belongs to $\mathcal{T}^{(n)}$; for each $i = 1, \ldots, k$, we have $f_n(\sigma_i) \subseteq C_{\sigma_i}$ by construction and $C_{\sigma_i} \subseteq C_\sigma$ by (a), so that $f_n(\partial\sigma) \subseteq C_\sigma$: let $f^\sigma_{n+1} : \sigma \to C_\sigma$ be a continuous extension of $f_{n|\partial\sigma} : \partial\sigma \to C_\sigma$ (which exists because of (b)). Since for any different members σ, τ in $\mathcal{S} \cup \mathcal{T}^{(n+1)}$ we have $f^\sigma_{n+1|\sigma\cap\tau} = f_{n|\sigma\cap\tau} = f^\tau_{n+1|\sigma\cap\tau}$, there is a map $f_{n+1} : Q \cup P^{(n+1)} \to X$ which is an extension of each f^σ_{n+1}. Obviously, such a map is continuous for the CW-topology, extends f_n, and satisfies $f_{n+1}(\sigma) \subseteq C_\sigma$ for all $\sigma \in \mathcal{S} \cup \mathcal{T}^{(n+1)}$. □

As an immediate special case of Theorem 1, we mention the following well-known property (see for example Eilenberg [8, p. 241]):

Corollary. *A topological space X is k-connected for every $k \leq n$ if and only if for any polytope P and any subpolytope $Q \subseteq P$, any continuous $f : Q \to X$ extends continuously over $Q \cup P^{(n+1)}$.*

Proof. The "if part" is obvious, and the "only if part" follows from Theorem 1 applied to the polytope $Q \cup P^{(n+1)}$ and the sets C_σ given by $C_\sigma = X$ for every $\sigma \in \mathcal{S} \cup \mathcal{T}^{(n+1)}$. □

3. KKM-type intersection theorems

We denote by $\mathcal{F}(\Delta_n)$ the set of all faces of Δ_n, and for any $\sigma \in \mathcal{F}(\Delta_n)$ we denote by $\sigma^{(0)}$ the set of all vertices of σ. The Knaster-Kuratowski-Mazurkiewicz theorem can be formulated as follows: *If a family $\{ A_i \mid i \in \Delta_n^{(0)} \}$ of subsets of Δ_n is such that all the sets are closed or all are open, and each face σ of Δ_n is contained in the corresponding union $\bigcup \{ A_i \mid i \in \sigma^{(0)} \}$, then there is a point common to all the sets.* (The original statement deals only with closed sets; a proof that open sets may be used equivalently is given in Lassonde [14].)

By combining Theorem 1 with the Knaster-Kuratowski-Mazurkiewicz theorem we get our basic topological intersection theorem:

Theorem 2. *In a topological space, let $\{ A_i \mid i \in \Delta_n^{(0)} \}$ be a family of sets, all closed or all open, and let $\{ C_\sigma \mid \sigma \in \mathcal{F}(\Delta_n) \}$ be an associated family of sets such that:*

(a) $C_\sigma \subseteq C_\tau$ for every $\sigma, \tau \in \mathcal{F}(\Delta_n)$ with $\sigma \subseteq \tau$,

(b) C_σ is $(\dim \sigma - 1)$-connected for every $\sigma \in \mathcal{F}(\Delta_n)$.

Then the following equivalent assertions hold:

(1) If each C_σ is contained in the corresponding union $\bigcup \{ A_i \mid i \in \sigma^{(0)} \}$, then C_{Δ_n} contains a point of the intersection $\bigcap \{ A_i \mid i \in \Delta_n^{(0)} \}$;

(2) If C_{Δ_n} is contained in the union $\bigcup \{ A_i \mid i \in \Delta_n^{(0)} \}$, then some C_σ contains a point of the corresponding intersection $\bigcap \{ A_i \mid i \in \sigma^{(0)} \}$.

Proof. Denote by X the underlying topological space, and assume that the condition of assertion (1) is satisfied. By Theorem 1, a continuous $g : \Delta_n \to X$ exists such that for each face σ of Δ_n we have $g(\sigma) \subseteq C_\sigma$, which in view of the assumption implies $g(\sigma) \subseteq \bigcup \{ A_i \cap C_{\Delta_n} \mid i \in \sigma^{(0)} \}$. Thus, each face σ of Δ_n is contained in the corresponding union $\bigcup \{ g^{-1}(A_i \cap C_{\Delta_n}) \mid i \in \sigma^{(0)} \}$. We derive from the theorem of Knaster-Kuratowski-Mazurkiewicz that the family $\{ g^{-1}(A_i \cap C_{\Delta_n}) \mid i \in \Delta_n^{(0)} \}$ has a nonempty intersection. Consequently, the intersection $\bigcap \{ A_i \cap C_{\Delta_n} \mid i \in \Delta_n^{(0)} \}$ is not empty, proving assertion (1).

As for assertion (2), it suffices to observe that it is the contraposition form of assertion (1) applied to the family $\{ X \setminus A_i \mid i \in \Delta_n^{(0)} \}$. □

Ky Fan's extension of Knaster-Kuratowski-Mazurkiewicz's theorem [9] and Ky Fan's matching theorem [10] are obtained respectively from assertion (1) and assertion (2) of the following:

Corollary. *In a convex subset of a vector space supplied with the finite topology, let $\{ A_i \mid i \in I \}$ be a finite family of sets, all closed or all open, and let $\{ x_i \mid i \in I \}$ be a family of points, indexed by the same set I. Then the following (equivalent) assertions hold:*

(1) If for every nonempty subset $J \subseteq I$ *the convex hull of* $\{ x_j \mid j \in J \}$ *is contained in the corresponding union* $\bigcup \{ A_j \mid j \in J \}$*, then the convex hull of* $\{ x_i \mid i \in I \}$ *contains a point of the intersection* $\bigcap \{ A_i \mid i \in I \}$*.*

(2) If the convex hull of $\{ x_i \mid i \in I \}$ *is contained in the union* $\bigcup \{ A_i \mid i \in I \}$*, then for some nonempty subset* $J \subseteq I$ *the convex hull of* $\{ x_j \mid j \in J \}$ *contains a point of the corresponding intersection* $\bigcap \{ A_j \mid j \in J \}$*.*

Proof. We may assume that the set of indices I is the set of vertices of Δ_n. The result then follows from Theorem 2 with C_σ being, for every $\sigma \in \mathcal{F}(\Delta_n)$, the convex hull of $\{ x_j \mid j \in \sigma^{(0)} \}$ (such sets are evidently contractible when supplied with the Euclidean topology). □

4. Klee-type intersection theorems

Our main result below provides topological generalizations of the familiar theorem of Klee [12]: *If* n *closed convex sets in a topological vector space are such that their union is convex and the intersection of every* $n-1$ *of them is nonempty, then all the sets have a point in common.*

Theorem 3. *Let* n *sets in a topological space be given so that all the sets are closed or all are open, and either of the following properties is satisfied:*

(1) For each $k \leq n$ *the union of every* k *of the sets is* $(k-2)$*-connected; or*

(2) The union of the n *sets is* $(n-2)$*-connected and for each* $k \leq n-1$ *the intersection of every* k *of the sets is* $(n-k-2)$*-connected.*

Then all the sets have a point in common.

Proof. Let $\{ A_i \mid i \in \Delta_{n-1}^{(0)} \}$ be a family of n sets satisfying the conditions of the theorem. We define an associated family of sets $\{ C_\sigma \mid \sigma \in \mathcal{F}(\Delta_{n-1}) \}$ satisfying the requirements of Theorem 2 as follows. In case (1), we let $C_\sigma = \bigcup \{ A_i \mid i \in \sigma^{(0)} \}$ so that assertion (1) of Theorem 2 applies. In case (2), we let $C_\sigma = \bigcup \{ A_i \mid i \in \sigma^{(0)} \}$ for $\sigma = \Delta_{n-1}$ and $C_\sigma = \bigcap \{ A_i \mid i \notin \sigma^{(0)} \}$ for $\sigma \neq \Delta_{n-1}$, so that assertion (2) of Theorem 2 gives the result. □

The following immediate consequence of Theorem 3 is worth mentioning:

Corollary. *A family of* n *closed convex sets in a topological vector space has a nonempty intersection if (and only if) the union of the* n *sets is* $(n-2)$*-connected and the intersection of every* $n-1$ *of them is nonempty.*

This result, which is a slight modification of Klee's theorem, implies the theorem of Brouwer stating that *the* n*-sphere* S^n *is not* n*-connected.* Indeed, consider the n-dimensional faces of Δ_{n+1}: they form a family of $n+2$ closed convex sets in $\mathbb{R}^{n+2}$, every intersection of $n+1$ of them is nonempty, but the whole intersection is empty. Hence their union, which is $\partial\Delta_{n+1}$, is not n-connected, so S^n is not n-connected either. Since this statement is equivalent to Brouwer's fixed point theorem, which is itself equivalent to the theorem of Knaster-Kuratowski-Mazurkiewicz, we infer that the above corollary is equivalent to all these results.

Remark. Case (2) of Theorem 3 is closely related to results of Dugundji for metric spaces (see [5, Theorem 4.1] for the open version, and [6, Theorem 5.2] for the closed version). Dugundji's assumptions are: *The singular* $(n-2)$*-homology group of the union of the* n *sets is trivial or a torsion group, and for each* $k \leq n-1$ *the intersection of every* k *of the sets is* p*-connected for every* $p \leq n-k-2$. In case (2) of Theorem 3 the hypothesis on the union is stronger but the hypotheses on the intersections are weaker.

5. Helly-type intersection theorems

We now turn to discussing Helly's theorem which asserts that *a finite family of convex sets in* $\mathbb{R}^n$ *has a nonempty intersection if (and only if) the intersection of every* $n+1$ *members of the family is nonempty.* As is well known, Helly's theorem can easily be proved by combining Klee's theorem with Carathéodory's theorem (see for example Berge [1, p. 173]).

Recall that a subset C of a vector space is said to be *starshaped* if there is a point $c \in C$ such that for every $x \in C$ the segment $[c, x]$ lies in C. Clearly, if the intersection of convex sets is nonempty, then their union is starshaped. Hence, the following result of Breen [2] is a generalization of Helly's theorem for closed sets: *A finite family of closed convex sets in* $\mathbb{R}^n$ *has a nonempty intersection if (and only if) the union of every* $n+1$ *or fewer members of the family is starshaped.* The two theorems of this section extend Breen's result in two different directions.

In our first theorem, starshapedness of the union is weakened to n-connectedness, and open sets are considered as well as closed sets. Like the above Klee-type results, this extension of Helly's theorem is equivalent to Brouwer's fixed point theorem.

Theorem 4. *Let a finite family of convex sets in* $\mathbb{R}^n$ *be given so that all the sets are closed or all are open, and for each* $k \leq n+1$ *the union of every* k *members of the family is* $(k-2)$*-connected. Then all the members of the family have a point in common.*

Proof. It follows from Theorem 3, case (1), that the intersection of every $n+1$ members of the family is nonempty, whence the conclusion by Helly's theorem. □

In our second theorem, convexity of the sets is weakened to starshapedness. The proof combines Theorem 3 with a Krasnosel'skiĭ-type theorem of Kołodziejczyk [13] stating that *if every* $n+1$ *members of a finite family of closed sets in* $\mathbb{R}^n$ *have a starshaped union, then the union of all the members of the family is starshaped.* Theorem 5 extends another result of Breen [3], which is the special case $n = 2$.

Theorem 5. *If every* $n+1$ *or fewer members of a finite family of closed sets in* $\mathbb{R}^n$ *have a starshaped union, then all the members of the family have a point in common.*

Proof. By the above-mentionned theorem of Kołodziejczyk, every union of members of the family is starshaped, so the conclusion follows from Theorem 3, case (1). □

References

1. C. Berge, *Espaces topologiques, fonctions multivoques*, Dunod, Paris, 1966. MR **21:**4401
2. M. Breen, *Starshaped unions and nonempty intersections of convex sets in* $\mathbb{R}^d$, Proc. Amer. Math. Soc. **108** (1990), *no.* 3, 817–820. MR **90e:**52007
3. M. Breen, *An intersection property for starshaped sets in the plane*, Geom. Dedicata **37** (1991), *no.* 3, 317–326. MR **92e:**52008
4. L. Danzer, B. Grünbaum, and V. Klee, *Helly's theorem and its relatives*, Convexity, Sympos. Pure Math., Vol. 7, Amer. Math. Soc., Providence, RI, 1962, pp. 101–180. MR **28:**524
5. J. Dugundji, *A duality property of nerves*, Fund. Math. **59** (1966), 213–219. MR **34:**5071
6. J. Dugundji, *Maps into nerves of closed coverings*, Ann. Scuola Norm. Sup. Pisa. Sci. Fis. Mat. **21** (1967), 121–136. MR **36:**7112

7. J. Eckhoff, *Helly, Radon, and Carathéodory type theorems*, Handbook of convex geometry (P. M. Gruber and J. M. Wills, eds.), Elsevier Science Publishers, 1993, pp. 389–447. MR **94k:**52010
8. S. Eilenberg, *Cohomology and continuous mappings*, Ann. Math. **41** (1940), 231–251. MR **1:**222b
9. K. Fan, *A generalization of Tychonoff's fixed point theorem*, Math. Ann. **142** (1961), 305–310. MR **24:**A1120
10. K. Fan, *Some properties of convex sets related to fixed point theorems*, Math. Ann. **266** (1984), 519–537. MR **85i:**47060
11. C. Horvath, *Contractibility and generalized convexity*, J. Math. Anal. Appl. **156** (1991), *no.* 2, 341–357. MR **92f:**52002
12. V. L. Klee, *On certain intersection properties of convex sets*, Canad. J. Math. **3** (1951), 272–275. MR **13:**153m
13. K. Kołodziejczyk, *On starshapedness of the union of closed sets in* $\mathbb{R}^n$, Colloq. Math. **53** (1987), 193–197. MR **89b:**52017
14. M. Lassonde, *Sur le principe KKM*, C.R. Acad. Sci. Paris Sér. I Math. **310** (1990), 573-576. MR **91h:**47061

Département de Mathématiques, Université de Perpignan, 66860 Perpignan Cedex, France

E-mail address: horvath@univ-perp.fr

Département de Mathématiques, Université des Antilles et de la Guyane, 97159 Pointe-à-Pitre Cedex, Guadeloupe, France

E-mail address: lassonde@univ-ag.fr