PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 4, April 1997, Pages 1197–1202
S 0002-9939(97)03627-7

HOLOMORPHIC HELICES IN A COMPLEX SPACE FORM

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(Communicated by Christopher B. Croke)

ABSTRACT. In a complex space form M we shall investigate a smooth curve γ which is generated by a holomorphic Killing vector field X on M.

INTRODUCTION

In Riemannian Geometry it is interesting to investigate "simple curves" in a certain sense. From this point of view some geometers studied Submanifold Theory (for example, see [1], [2]). In this paper, we consider "simple curves" in a complex space form.

Let M be an n-dimensional Kähler manifold with complex structure J and Riemannian metric $\langle \, , \rangle$. For a helix γ on M of order d ($\leq 2n$) with the associated Frenet frame $\{V_1, \ldots, V_d\}$, we define its *complex torsions* by $\tau_{ij}(t) = \langle V_i(t), JV_j(t)\rangle$ for $1 \leq i < j \leq d$. In the study of helices in a Kähler manifold their complex torsions play an important role. We shall call γ a *holomorphic helix* if all the complex torsions are constant. Ohnita and the first-named author proved in [3] that a smooth curve γ on a complex space form is a holomorphic helix if and only if it is generated by a holomorphic Killing vector field X. This is the complex version of the well-known fact that a smooth curve on a real space form is a helix if and only if it is generated by a Killing vector field. Study of holomorphic helices is one of the most interesting objects in differential geometry in a complex space form.

The main purpose of this paper is to study the moduli of holomorphic helices of order 3 in complex space forms. A helix of order 1 is nothing but a geodesic, and a helix of order 2 is called a *circle*. They are necessarily holomorphic helices. But in the class of helices of order d (≥ 3) we can find many helices which are not holomorphic helices.

We show that the moduli of all holomorphic helices of order 3 on an n-dimensional complex space form is parametrized by three real numbers or two real numbers according as $n \geq 3$ or $n = 2$ (see Theorem 5). Moreover, we investigate the moduli of all holomorphic helices in a 2-dimensional complex space form (see Theorems 4, 5).

The authors would like to express their hearty thanks to Professor K. Ogiue for his valuable suggestions during the preparation of this paper. Also, the first-named author is grateful to Professor Y. Ohnita for his kindness.

Received by the editors February 16, 1995 and, in revised form, August 30, 1995.
1991 *Mathematics Subject Classification.* Primary 53C35, 53C20.
Key words and phrases. Holomorphic helix, complex torsion, complex space form.

1. Complex torsions of holomorphic helices

We shall start by recalling the definition of helices. A smooth curve $\gamma = \gamma(t)$ parametrized by its arc length t is called a *helix of proper order* d if there exist an orthonormal frame $\{V_1 = \dot\gamma, \dots, V_d\}$ along γ and positive constants $k_1, \dots, k_{d-1}$ which satisfy the system of ordinary differential equations

$$\nabla_t V_j(t) = -k_{j-1}V_{j-1}(t) + k_j V_{j+1}(t), \qquad j = 1, \dots, d, \tag{1.1}$$

where $V_0 = V_{d+1} = 0$ and ∇_t denotes the covariant differentiation along γ. The constants k_j ($1 \le j \le d-1$) and the orthonormal frame $\{V_1, \dots, V_d\}$ are called the *curvatures* and the *Frenet frame* of γ, respectively. A curve is called a *helix of order* d if it is a helix of proper order r ($\le d$). For a helix of order d which is of proper order r ($\le d$), we use the convention in (1.1) that $k_j = 0$ ($r \le j \le d-1$) and $V_j = 0$ ($r+1 \le j \le d$). Note that every helix is a real analytic curve on a Kähler manifold M.

As a matter of course complex torsions of helices satisfy $|\tau_{ij}(t)| \le 1$ by their definitions. We first show that complex torsions and curvatures of holomorphic helices have relations. Differentiating all the complex torsions, we find by use of the equation (1.1) that

$$\frac{d}{dt}\tau_{ij}(t) = -k_{i-1}\tau_{i-1,j}(t) + k_i\tau_{i+1,j}(t) - k_{j-1}\tau_{i,j-1}(t) + k_j\tau_{i,j+1}(t),$$

where $\tau_{kl} = 0$ when $k = l$ or $k = 0$ or l is greater than the proper order. We hence get the following.

Proposition 1. *The complex torsions of a holomorphic helix of odd proper order d on a Kähler manifold satisfy the following relations*:

$$\begin{aligned}
\tau_{i,i+2k} &= 0 \text{ for } i = 1, 2, \dots, d-2k, \text{ where } k = 1, 2, \dots, (d-1)/2,\\
k_1\tau_{2d} &= k_{d-1}\tau_{1,d-1},\\
k_1\tau_{2j} + k_j\tau_{1,j+1} &= k_{j-1}\tau_{1,j-1} \text{ for } j = 3, 5, \dots, d-2,\\
k_{i-1}\tau_{i-1,d} + k_{d-1}\tau_{i,d-1} &= k_i\tau_{i+1,d} \text{ for } i = 3, 5, \dots, d-2,\\
k_{i-1}\tau_{i-1,j} + k_{j-1}\tau_{i,j-1} &= k_i\tau_{i+1,j} + k_j\tau_{i,j+1}\\
&\qquad \text{for } i = 2, 3, \dots, d-3,\ j = i+2, i+4, \dots, d-1.
\end{aligned}$$

Proposition 2. *The complex torsions of a holomorphic helix of even proper order d on a Kähler manifold satisfy the following relations*:

$$\begin{aligned}
\tau_{i,i+2k} &= 0 \text{ for } i = 1, 2, \dots, d-2k, \text{ where } k = 1, 2, \dots, (d-2)/2,\\
k_1\tau_{2d} &= k_{d-1}\tau_{1,d-1},\\
k_1\tau_{2j} + k_j\tau_{1,j+1} &= k_{j-1}\tau_{1,j-1} \text{ for } j = 3, 5, \dots, d-1,\\
k_{i-1}\tau_{i-1,d} + k_{d-1}\tau_{i,d-1} &= k_i\tau_{i+1,d} \text{ for } i = 2, 4, \dots, d-2,\\
k_{i-1}\tau_{i-1,j} + k_{j-1}\tau_{i,j-1} &= k_i\tau_{i+1,j} + k_j\tau_{i,j+1}\\
&\qquad \text{for } i = 2, 3, \dots, d-3,\ j = i+2, i+4, \dots, d-1.
\end{aligned}$$

Conversely, if the Frenet frame of a helix γ in a Kähler manifold satisfies the above relations at a point, then all nth derivatives of its complex torsions vanish at this point. Since γ is real analytic, we find that it is a holomorphic helix. We therefore have

Proposition 3. *For orthornormal vectors $v_1, \dots, v_d$ at a point p of a Kähler manifold M, we set $\tau_{ij} = \langle v_i, Jv_j\rangle$ $(1 \le i < j \le d)$. If positive constants $k_1, \dots, k_{d-1}$ and the vectors $v_1, \dots, v_d$ satisfy the relations in Proposition 1 or 2, there exists a unique holomorphic helix with curvatures $k_1, \dots, k_{d-1}$ satisfying that the initial value of its Frenet frame is $(v_1, \dots, v_d)$.*

The following is easily verified.

Proposition 4. *The complex torsions τ_{ij} of a holomorphic helix of proper order d on a Kähler manifold M satisfy $\sum_{j=1}^{i-1} \tau_{ji}^2 + \sum_{j=i+1}^{d} \tau_{ij}^2 \le 1$ for every i.*

Proof. Since $\tau_{i,i+2k} = 0$, we find that $\{V_{2l-1}, JV_{2l-1} | l = 1, 2, \dots, \}$ and $\{V_{2l}, JV_{2l} | l = 1, 2, \dots\}$ form orthornormal frames. When i is even, the left-hand-side of the inequality is the norm of the projection of $V_i(0)$ onto the complex linear subspace spanned by $\{V_{2l-1}(0), JV_{2l-1}(0) | l = 1, 2, \dots\}$. We hence get the inequality. For odd i we have a similar argument. $\square$

Here we treat holomorphic helices of order 3. We need to choose orthonormal vectors $v_1, v_2, v_3 \in T_pM$ which satisfy

$$k_1\langle v_2, Jv_3\rangle = k_2\langle v_1, Jv_2\rangle, \langle v_1, Jv_3\rangle = 0.$$

Identifying T_pM with C^n, we set v_1, v_2 and v_3 as

$$\begin{aligned} v_1 &= (1, 0, \dots, 0), \\ v_2 &= (-i\tau, \sqrt{1-\tau^2}, 0, \dots, 0), \\ v_3 &= (0, -i\rho/\sqrt{1-\tau^2}, \sqrt{1-\tau^2-\rho^2}/\sqrt{1-\tau^2}, 0, \dots, 0) \end{aligned}$$

for positive constants τ and ρ with $\tau^2 + \rho^2 \le 1$. Then they are orthonormal and satisfy $\langle v_1, Jv_2\rangle = \tau$, $\langle v_2, Jv_3\rangle = \rho$, $\langle v_1, Jv_3\rangle = 0$. We therefore have

Theorem 1. *Let M be a Kähler manifold of dimension greater than 2. Then the following hold*:

(1) *Every holomorphic helix of order 3 satisfies*

$$k_1\tau_{23} = k_2\tau_{12},\ \tau_{13} = 0,\ |\tau_{12}| \le k_1/\sqrt{k_1^2 + k_2^2}.$$

(2) *Conversely, if nonnegative constants k_1, k_2 and a constant τ satisfy $|\tau| \le k_1/\sqrt{k_1^2+k_2^2}$, then there exists a holomorphic helix of order 3 on M with the first curvature k_1 and the second curvature k_2, and with the first complex torsion $\tau_{12} = \tau$.*

(3) *If $|\tau| > k_1/\sqrt{k_1^2+k_2^2}$, we have no such a holomorphic helix of order 3 on M.*

Theorem 2. *Let M be a 2-dimensional Kähler manifold. Then the following hold*:

(1) *The complex torsions of each holomorphic helix of proper order 3 in M are*

$$\tau_{12} = k_1/\sqrt{k_1^2+k_2^2},\ \tau_{13} = 0,\ \tau_{23} = k_2/\sqrt{k_1^2+k_2^2} \tag{1.2}$$

or

$$\tau_{12} = -k_1/\sqrt{k_1^2+k_2^2},\ \tau_{13} = 0,\ \tau_{23} = -k_2/\sqrt{k_1^2+k_2^2}, \tag{1.3}$$

where its curvatures are k_1 and k_2.

(2) *Conversely for given positive constants k_1 and k_2, there exists a holomorphic helix of proper order 3 with curvatures k_1 and k_2, and with complex torsions defined by (1.2) or (1.3).*

Such a description as above for holomorphic helices of order 4 is much more complicated. We restrict ourselves here to holomorphic helices on a 2-dimensional Kähler manifold M. For given constants τ and ρ with $\tau^2+\rho^2=1$, we choose vectors

$$v_1=(1,0),\ v_2=(-i\tau,\rho),\ v_3=(0,-i),\ v_4=\mp(i\rho,\tau)$$

in $T_pM\simeq C^2$. Then they are orthonormal and satisfy

$$\langle v_1,Jv_2\rangle=\tau,\ \langle v_2,Jv_3\rangle=\rho,\ \langle v_1,Jv_4\rangle=\pm\rho$$
$$\langle v_1,Jv_3\rangle=\langle v_2,Jv_4\rangle=0,\ \langle v_3,Jv_4\rangle=\pm\tau.$$

On the other hand, Proposition 2 shows that a helix is a holomorphic helix if and only if

$$\tau_{13}(0)=\tau_{24}(0)=0,\ k_1\tau_{23}(0)+k_3\tau_{14}=k_2\tau_{12}(0),$$
$$k_1\tau_{14}(0)+k_3\tau_{23}(0)=k_2\tau_{34}(0).$$

We therefore have

Theorem 3. *Let M be a 2-dimensional Kähler manifold. Then the following hold*:

(1) *The complex torsions of each holomorphic helix of proper order 4 with curvatures k_1,k_2 and k_3 on M satisfy one of the following*:

$$\tau_{12}=\tau_{34}=\tau,\ \tau_{23}=\tau_{14}=k_2\tau/(k_1+k_3),\ \tau_{13}=\tau_{24}=0, \tag{1.4}$$

where $\tau=\pm(k_1+k_3)/\sqrt{k_2^2+(k_1+k_3)^2}$,

$$\tau_{12}=-\tau_{34}=\tau,\ \tau_{23}=-\tau_{14}=k_2\tau/(k_1-k_3),\ \tau_{13}=\tau_{24}=0, \tag{1.5}$$

when $k_1\neq k_3$, where $\tau=\pm(k_1-k_3)/\sqrt{k_2^2+(k_1-k_3)^2}$, or

$$\tau_{12}=\tau_{34}=\tau_{13}=\tau_{24}=0,\ \tau_{23}=-\tau_{14}=\pm1, \tag{1.5$'$}$$

when $k_1=k_3$.

(2) *Conversely, for given positive constants k_1,k_2 and k_3, there exist holomorphic helices of proper order 4 in M with curvatures k_1,k_2 and k_3, and with complex torsions defined by* (1.4), (1.5) *or* (1.4), (1.5$'$).

2. Moduli of holomorphic helices on a complex space form

Let $M_n(c)$ be an n-dimensional complete simply connected complex space form of constant holomorphic sectional curvature c. It is well-known that an arbitrary complex space form is locally complex analytically isometric to a complex projective space, a complex hyperbolic space or a complex Euclidean space according as the holomorphic sectional curvature c is positive, negative or zero. We have the following holomorphic congruence theorem for helices in $M_n(c)$.

Proposition 5 ([3])**.** *Let γ and σ be two helices of orders p and q in a complex space form $M_n(c)$, respectively. Let $\{k_1,\dots,k_{p-1}\}$ (resp. $\{\lambda_1,\dots,\lambda_{q-1}\}$) be the curvatures of γ (resp. σ), and let $\tau^{\gamma}_{ij}(t)$ (resp. $\tau^{\sigma}_{kl}(t)$) be the complex torsions of γ (resp. σ). Then there exists a holomorphic isometry φ of $M_n(c)$ satisfying $\gamma=\varphi\circ\sigma$ if and only if $p=q$, $k_i=\lambda_i$ $(1\leq i\leq p-1)$ and $\tau^{\gamma}_{ij}(0)=\tau^{\sigma}_{ij}(0)$ $(1\leq i<j\leq p)$.*

In this section we denote by $Hh^d(M_n(c))$ the set of the equivalence classes of all holomorphic helices of order d $(\leq 2n)$ in $M_n(c)$ with respect to holomorphic isometries of $M_n(c)$. By Proposition 5 the set $Hh^d(M_n(c))$ is naturally regarded as a subset of $[0,\infty)^{d-1}\times[-1,1]^{d(d-1)/2}\subset R^{(d+2)(d-1)/2}$. Needless to say, every helix which lies on a totally real totally geodesic submanifold $M^n(c/4)$ of $M_n(c)$ is a holomorphic helix (whose complex torsions are zero in $M_n(c)$), so that the set of the equivalence classes of all holomorphic helices of proper order d ($\leq n$) with respect to holomorphic isometries of $M_n(c)$, which is a subset of $Hh^d(M_n(c))$, is not empty.

Theorem 4. *For given positive constants k_1, k_2 and k_3, there exist four equivalence classes of holomorphic helices of proper order 4 with curvatures k_1, k_2 and k_3 with respect to holomorphic isometries of $M_2(c)$. In addition, these four equivalence classes are given by* (1.4), (1.5) *or* (1.4), (1.5′).

We give some examples here of holomorphic helices of proper order 4 in a 2-dimensional complex projective space $CP_2(c)$. Let $\pi\colon S^{2n+1}(1)(\subset C^{n+1}) \to CP^n(4)$ denote the Hopf fibration.

Example 1. For any k satisfying $0 < k < \sqrt{2}$, we put

$$\begin{aligned}
A &= \sqrt{(4-k^2-\sqrt{(2-k^2)(8-k^2)})/(2(8-k^2))},\\
B &= 2/\sqrt{8-k^2},\\
C &= \sqrt{(4-k^2+\sqrt{(2-k^2)(8-k^2)})/(2(8-k^2))},\\
\alpha &= \left(\sqrt{2-k^2}+\sqrt{8-k^2}\right)/\sqrt{2},\\
\beta &= \sqrt{2-k^2}/\sqrt{2},\\
\delta &= \left(\sqrt{2-k^2}-\sqrt{8-k^2}\right)/\sqrt{2}.
\end{aligned}$$

Let $\tilde{\gamma}$ be a curve in C^3 defined by $\tilde{\gamma}(t) = (Ae^{i\alpha t}, Be^{i\beta t}, Ce^{i\delta t})$. Then $\tilde{\gamma}$ is a horizontal curve on $S^5(1)$ parametrized by arc length t. Moreover $\pi(\tilde{\gamma})$ is a holomorphic helix of order 4 in $CP_2(4)$ with curvatures $k_1 = k$, $k_2 = \sqrt{(18-9k^2)/2}$ and $k_3 = k$, and with complex torsions $\tau_{12} = \tau_{13} = \tau_{24} = \tau_{34} = 0$, $\tau_{14} = 1$ and $\tau_{23} = -1$. The curve $\pi(\tilde{\gamma})$ satisfies (1.5′).

Example 2. Let $\tilde{\gamma}$ be a curve in C^3 defined by

$$\tilde{\gamma}(t) = \left(\left(1/\sqrt{3}\right)e^{it}, \left(1/\sqrt{14}\right)e^{2it}, \left(5/\sqrt{42}\right)e^{-4it/5}\right).$$

Then $\pi(\tilde{\gamma})$ is a holomorphic helix of order 4 in $CP_2(4)$ with curvatures $k_1 = 3\sqrt{2}/5$, $k_2 = 11\sqrt{2}/10$ and $k_3 = 1/\sqrt{2}$, and with complex torsions $\tau_{12} = \tau_{14} = \tau_{23} = \tau_{34} = -1/\sqrt{2}$ and $\tau_{13} = \tau_{24} = 0$. The curve $\pi(\tilde{\gamma})$ satisfies (1.4).

Finally we shall investigate the moduli spaces $Hh^d(M_n(c))$ ($d = 1, 2, 3$). The moduli space $Hh^1(M_n(c))$ clearly consists of one point. As an immediate consequence of Theorem 1, Theorem 2 and Proposition 5 we can establish the following.

Theorem 5. (1) *The moduli space $Hh^2(M_n(c))$ is homeomorphic to a cone in R^2 or a half line according as $n \geq 2$ or $n = 1$. More precisely, $Hh^2(M_n(c))$ is $[0,\infty)\times[-1,1]/\sim$ or $[0,\infty)$ according as $n \geq 2$ or $n = 1$, where the equivalence relation $\sim$ means that $(0,\tau) \sim (0,\rho)$ if $\tau, \rho \in [-1,1]$.*

(2) *The moduli space* $Hh^3(M_n(c))$ *is connected and*

$$Hh^3(M_n(c)) = \begin{cases} \{(k_1, k_2, \tau) \in [0,\infty) \times [0,\infty) \times [-1,1] | \tau^2 \leq k_1^2/(k_1^2+k_2^2)\}/\sim, & n \geq 3, \\ ([0,\infty) \times \{0\} \times [-1,1] \\ \qquad \cup \{ \left(k_1, k_2, \pm k_1/\sqrt{k_1^2+k_2^2}\right) | k_1 > 0, k_2 > 0\})/\sim, & n = 2, \end{cases}$$

where the equivalence relation $\sim$ *means that* $(0,k,\tau) \sim (0,l,\rho)$ *if* $k, l \in [0,\infty)$ *and* $\tau, \rho \in [-1,1]$.

Remark. Let γ be a holomorphic helix of proper order 3 with curvatures k_1 and k_2, and with the first complex torsion $\tau_{12} = \tau$ in a complex space form $M_n(c)$. Then γ lies on a totally real totally geodesic submanifold $M^3(c/4)$ of $M_n(c)$ if and only if $\tau = 0$, and γ lies on a holomorphic totally geodesic submanifold $M_2(c)$ of $M_n(c)$ if and only if $\tau = \pm k_1/\sqrt{k_1^2+k_2^2}$.

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