HOLOMORPHIC HELICES IN A COMPLEX SPACE FORM

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Abstract. In a complex space form $M$ we shall investigate a smooth curve $\gamma$ which is generated by a holomorphic Killing vector field $X$ on $M$.

Introduction

In Riemannian Geometry it is interesting to investigate “simple curves” in a certain sense. From this point of view some geometers studied Submanifold Theory (for example, see [1], [2]). In this paper, we consider “simple curves” in a complex space form.

Let $M$ be an $n$-dimensional Kähler manifold with complex structure $J$ and Riemannian metric $\langle \cdot, \cdot \rangle$. For a helix $\gamma$ on $M$ of order $d$ ($\leq 2n$) with the associated Frenet frame $\{V_1, \ldots, V_d\}$, we define its complex torsions by $\tau_{ij}(t) = \langle V_i(t), JV_j(t) \rangle$ for $1 \leq i < j \leq d$. In the study of helices in a Kähler manifold their complex torsions play an important role. We shall call $\gamma$ a holomorphic helix if all the complex torsions are constant. Ohnita and the first-named author proved in [3] that a smooth curve $\gamma$ on a complex space form is a holomorphic helix if and only if it is generated by a holomorphic Killing vector field $X$. This is the complex version of the well-known fact that a smooth curve on a real space form is a helix if and only if it is generated by a Killing vector field. Study of holomorphic helices is one of the most interesting objects in differential geometry in a complex space form.

The main purpose of this paper is to study the moduli of holomorphic helices of order 3 in complex space forms. A helix of order 1 is nothing but a geodesic, and a helix of order 2 is called a circle. They are necessarily holomorphic helices. But in the class of helices of order $d$ ($\geq 3$) we can find many helices which are not holomorphic helices.

We show that the moduli of all holomorphic helices of order 3 on an $n$-dimensional complex space form is parametrized by three real numbers or two real numbers according as $n \geq 3$ or $n = 2$ (see Theorem 5). Moreover, we investigate the moduli of all holomorphic helices in a 2-dimensional complex space form (see Theorems 4, 5).

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1. Complex Torsions of Holomorphic Helices

We shall start by recalling the definition of helices. A smooth curve $\gamma = \gamma(t)$ parametrized by its arc length $t$ is called a helix of proper order $d$ if there exist an orthonormal frame $\{V_1 = \dot{\gamma}, \ldots, V_d\}$ along $\gamma$ and positive constants $k_1, \ldots, k_{d-1}$ which satisfy the system of ordinary differential equations
\begin{equation}
\nabla_t V_j(t) = -k_{j-1} V_{j-1}(t) + k_j V_{j+1}(t), \quad j = 1, \ldots, d,
\end{equation}
where $V_0 = V_{d+1} = 0$ and $\nabla_t$ denotes the covariant differentiation along $\gamma$. The constants $k_j$ ($1 \leq j \leq d-1$) and the orthonormal frame $\{V_1, \ldots, V_d\}$ are called the curvatures and the Frenet frame of $\gamma$, respectively. A curve is called a helix of order $d$ if it is a helix of proper order $r \leq d)$. For a helix of order $d$ which is of proper order $r \leq d)$, we use the convention in (1.1) that $k_j = 0$ ($r \leq j \leq d-1$) and $V_j = 0$ ($r + 1 \leq j \leq d$). Note that every helix is a real analytic curve on a Kähler manifold $M$.

As a matter of course complex torsions of helices satisfy $|\tau_j(t)| \leq 1$ by their definitions. We first show that complex torsions and curvatures of holomorphic helices have relations. Differentiating all the complex torsions, we find by use of the equation (1.1) that
\[
\frac{d}{dt} \tau_{ij}(t) = -k_{i-1} \tau_{i-1,j}(t) + k_i \tau_{i+1,j}(t) - k_{j-1} \tau_{i,j-1}(t) + k_j \tau_{i,j+1}(t),
\]
where $\tau_{kl} = 0$ when $k = l$ or $k = 0$ or $l$ is greater than the proper order. We hence get the following.

**Proposition 1.** The complex torsions of a holomorphic helix of odd proper order $d$ on a Kähler manifold satisfy the following relations:
\[
\tau_{i,i+2k} = 0 \text{ for } i = 1, 2, \ldots, d-2k, \text{ where } k = 1, 2, \ldots, (d-1)/2,
\]
\[
k_1 \tau_{2d} = k_d \tau_{1,d-1},
\]
\[
k_1 \tau_{2j} + k_j \tau_{1,j+1} = k_{j-1} \tau_{1,j-1} \text{ for } j = 3, 5, \ldots, d-2,
\]
\[
k_{i-1} \tau_{i-1,d} + k_{d-1} \tau_{i,d-1} = k_i \tau_{i+1,d} \text{ for } i = 3, 5, \ldots, d-2,
\]
\[
k_{i-1} \tau_{i-1,j} + k_{j-1} \tau_{i,j-1} = k_i \tau_{i+1,j} + k_j \tau_{i,j+1} \text{ for } i = 2, 3, \ldots, d-3, \text{ } j = i+2, i+4, \ldots, d-1.
\]

**Proposition 2.** The complex torsions of a holomorphic helix of even proper order $d$ on a Kähler manifold satisfy the following relations:
\[
\tau_{i,i+2k} = 0 \text{ for } i = 1, 2, \ldots, d-2k, \text{ where } k = 1, 2, \ldots, (d-2)/2,
\]
\[
k_1 \tau_{2d} = k_d \tau_{1,d-1},
\]
\[
k_1 \tau_{2j} + k_j \tau_{1,j+1} = k_{j-1} \tau_{1,j-1} \text{ for } j = 3, 5, \ldots, d-1,
\]
\[
k_{i-1} \tau_{i-1,d} + k_{d-1} \tau_{i,d-1} = k_i \tau_{i+1,d} \text{ for } i = 2, 4, \ldots, d-2,
\]
\[
k_{i-1} \tau_{i-1,j} + k_{j-1} \tau_{i,j-1} = k_i \tau_{i+1,j} + k_j \tau_{i,j+1} \text{ for } i = 2, 3, \ldots, d-3, \text{ } j = i+2, i+4, \ldots, d-1.
\]

Conversely, if the Frenet frame of a helix $\gamma$ in a Kähler manifold satisfies the above relations at a point, then all $n$th derivatives of its complex torsions vanish at this point. Since $\gamma$ is real analytic, we find that it is a holomorphic helix. We therefore have
Proposition 3. For orthonormal vectors \(v_1, \ldots, v_d\) at a point \(p\) of a Kähler manifold \(M\), we set \(\tau_{ij} = \langle v_i, Jv_j \rangle\) (1 \(\leq i < j \leq d\)). If positive constants \(k_1, \ldots, k_{d-1}\) and the vectors \(v_1, \ldots, v_d\) satisfy the relations in Proposition 1 or 2, there exists a unique holomorphic helix with curvatures \(k_1, \ldots, k_{d-1}\) satisfying that the initial value of its Frenet frame is \((v_1, \ldots, v_d)\).

The following is easily verified.

Proposition 4. The complex torsions \(\tau_{ij}\) of a holomorphic helix of proper order \(d\) on a Kähler manifold \(M\) satisfy \(\sum_{j=1}^{i-1} \tau_{ji}^2 + \sum_{j=i+1}^{d} \tau_{ij}^2 \leq 1\) for every \(i\).

Proof. Since \(\tau_{i,i+2k} = 0\), we find that \(\{V_{2l-1}, JV_{2l-1}|l = 1, 2, \ldots, \}\) and \(\{V_{2l}, JV_{2l}|l = 1, 2, \ldots, \}\) form orthonormal frames. When \(i\) is even, the left-hand-side of the inequality is the norm of the projection of \(V_i(0)\) onto the complex linear subspace spanned by \(\{V_{2l-1}(0), JV_{2l-1}(0)|l = 1, 2, \ldots, \}\). We hence get the inequality. For odd \(i\) we have a similar argument. \(\square\)

Here we treat holomorphic helices of order 3. We need to choose orthonormal vectors \(v_1, v_2, v_3 \in T_pM\) which satisfy

\[k_1(v_2, Jv_3) = k_2(v_1, Jv_2), \langle v_1, Jv_3 \rangle = 0.\]

Identifying \(T_pM\) with \(C^n\), we set \(v_1, v_2\) and \(v_3\) as

\[v_1 = (1, 0, \ldots, 0),\]
\[v_2 = (-i\tau, \sqrt{1 - \tau^2}, 0, \ldots, 0),\]
\[v_3 = (0, -i\rho/\sqrt{1 - \tau^2}, \sqrt{1 - \tau^2 - \rho^2/\sqrt{1 - \tau^2}}, 0, \ldots, 0)\]

for positive constants \(\tau\) and \(\rho\) with \(\tau^2 + \rho^2 \leq 1\). Then they are orthonormal and satisfy \(\langle v_1, Jv_2 \rangle = \tau, \langle v_2, Jv_3 \rangle = \rho, \langle v_1, Jv_3 \rangle = 0\). We therefore have

Theorem 1. Let \(M\) be a Kähler manifold of dimension greater than 2. Then the following hold:

1. Every holomorphic helix of order 3 satisfies
\[k_1\tau_{23} = k_2\tau_{12}, \ \tau_{13} = 0, \ |\tau_{12}| \leq k_1/\sqrt{k_1^2 + k_2^2}.
\]

2. Conversely, if nonnegative constants \(k_1, k_2\) and a constant \(\tau\) satisfy \(|\tau| \leq k_1/\sqrt{k_1^2 + k_2^2}\), then there exists a holomorphic helix of order 3 on \(M\) with the first curvature \(k_1\) and the second curvature \(k_2\), and with the first complex torsion \(\tau_{12} = \tau\).

3. If \(|\tau| > k_1/\sqrt{k_1^2 + k_2^2}\), we have no such a holomorphic helix of order 3 on \(M\).

Theorem 2. Let \(M\) be a 2-dimensional Kähler manifold. Then the following hold:

1. The complex torsions of each holomorphic helix of proper order 3 in \(M\) are

\[(1.2) \quad \tau_{12} = k_1/\sqrt{k_1^2 + k_2^2}, \ \tau_{13} = 0, \ \tau_{23} = k_2/\sqrt{k_1^2 + k_2^2},
\]

or

\[(1.3) \quad \tau_{12} = -k_1/\sqrt{k_1^2 + k_2^2}, \ \tau_{13} = 0, \ \tau_{23} = -k_2/\sqrt{k_1^2 + k_2^2},
\]

where its curvatures are \(k_1\) and \(k_2\).

2. Conversely for given positive constants \(k_1\) and \(k_2\), there exists a holomorphic helix of proper order 3 with curvatures \(k_1\) and \(k_2\), and with complex torsions defined by \((1.2)\) or \((1.3)\).
Such a description as above for holomorphic helices of order 4 is much more complicated. We restrict ourselves here to holomorphic helices on a 2-dimensional Kähler manifold \( M \). For given constants \( \tau \) and \( \rho \) with \( \tau^2 + \rho^2 = 1 \), we choose vectors
\[
v_1 = (1, 0), \quad v_2 = (-i\tau, \rho), \quad v_3 = (0, -i), \quad v_4 = \mp(i\rho, \tau)
\]
in \( T_p M \cong \mathbb{C}^2 \). Then they are orthonormal and satisfy
\[
\langle v_1, Jv_2 \rangle = \tau, \quad \langle v_2, Jv_3 \rangle = \rho, \quad \langle v_1, Jv_4 \rangle = \pm \rho \\
\langle v_1, Jv_3 \rangle = \langle v_2, Jv_4 \rangle = 0, \quad \langle v_3, Jv_4 \rangle = \pm \tau.
\]
On the other hand, Proposition 2 shows that a helix is a holomorphic helix if and only if
\[
\tau_{13}(0) = \tau_{24}(0) = 0, \quad k_1 \tau_{23}(0) + k_3 \tau_{14} = k_2 \tau_{12}(0), \\
k_1 \tau_{14}(0) + k_3 \tau_{25}(0) = k_2 \tau_{34}(0).
\]
We therefore have

**Theorem 3.** Let \( M \) be a 2-dimensional Kähler manifold. Then the following hold:

1. The complex torsions of each holomorphic helix of proper order 4 with curvatures \( k_1, k_2 \) and \( k_3 \) on \( M \) satisfy one of the following:

   \[
   \tau_{12} = \tau_{34} = \tau, \quad \tau_{23} = \tau_{14} = k_2 \tau/(k_1 + k_3), \quad \tau_{13} = \tau_{24} = 0,
   \]
   where \( \tau = \pm(k_1 + k_3)/\sqrt{k_2^2 + (k_1 + k_3)^2} \),

   \[
   \tau_{12} = -\tau_{34} = \tau, \quad \tau_{23} = -\tau_{14} = k_2 \tau/(k_1 - k_3), \quad \tau_{13} = \tau_{24} = 0,
   \]
   when \( k_1 \neq k_3 \), where \( \tau = \pm(k_1 - k_3)/\sqrt{k_2^2 + (k_1 - k_3)^2} \), or

   \[
   \tau_{12} = \tau_{34} = \tau_{13} = \tau_{24} = 0, \quad \tau_{23} = -\tau_{14} = \pm 1,
   \]
   when \( k_1 = k_3 \).

2. Conversely, for given positive constants \( k_1, k_2 \) and \( k_3 \), there exist holomorphic helices of proper order 4 in \( M \) with curvatures \( k_1, k_2 \) and \( k_3 \), and with complex torsions defined by (1.4), (1.5) or (1.4), (1.5').

2. **Moduli of holomorphic helices on a complex space form**

Let \( M_n(c) \) be an \( n \)-dimensional complete simply connected complex space form of constant holomorphic sectional curvature \( c \). It is well-known that an arbitrary complex space form is locally complex analytically isometric to a complex projective space, a complex hyperbolic space or a complex Euclidean space according as the holomorphic sectional curvature \( c \) is positive, negative or zero. We have the following holomorphic congruence theorem for helices in \( M_n(c) \).

**Proposition 5** ([3]). Let \( \gamma \) and \( \sigma \) be two helices of orders \( p \) and \( q \) in a complex space form \( M_n(c) \), respectively. Let \( \{k_1, \ldots, k_{p-1}\} \) (resp. \( \{\lambda_1, \ldots, \lambda_{q-1}\} \)) be the curvatures of \( \gamma \) (resp. \( \sigma \)), and let \( \tau_{ij}^\gamma(t) \) (resp. \( \tau_{ij}^\sigma(t) \)) be the complex torsions of \( \gamma \) (resp. \( \sigma \)). Then there exists a holomorphic isometry \( \varphi \) of \( M_n(c) \) satisfying \( \gamma = \varphi \circ \sigma \) if and only if \( p = q \), \( k_i = \lambda_i \) \((1 \leq i \leq p-1)\) and \( \tau_{ij}^\gamma(0) = \tau_{ij}^\sigma(0) \) \((1 \leq i < j \leq p)\).

In this section we denote by \( Hh^d(M_n(c)) \) the set of the equivalence classes of all holomorphic helices of order \( d \) \((\leq 2n)\) in \( M_n(c) \) with respect to holomorphic isometries of \( M_n(c) \). By Proposition 5 the set \( Hh^d(M_n(c)) \) is naturally regarded as a subset of \([0, \infty)^{d-1} \times [-1, 1]^{d(d-1)/2} \subset R^{(d+2)(d-1)/2} \). Needless to say, every
holomorphic helix which lies on a totally real totally geodesic submanifold $M^n(c/4)$ of $M_n(c)$ is a holomorphic helix (whose complex torsions are zero in $M_n(c)$), so that the set of the equivalence classes of all holomorphic helices of proper order $d (\leq n)$ with respect to holomorphic isometries of $M_n(c)$, which is a subset of $Hh^d(M_n(c))$, is not empty.

**Theorem 4.** For given positive constants $k_1$, $k_2$ and $k_3$, there exist four equivalence classes of holomorphic helices of proper order 4 with curvatures $k_1$, $k_2$ and $k_3$ with respect to holomorphic isometries of $M_2(c)$. In addition, these four equivalence classes are given by (1.4), (1.5) or (1.4), (1.5').

We give some examples here of holomorphic helices of proper order 4 in a 2-dimensional complex projective space $CP_2(c)$. Let $\pi: S^{2n+1}(\subset C^{n+1}) \rightarrow CP^n(4)$ denote the Hopf fibration.

**Example 1.** For any $k$ satisfying $0 < k < \sqrt{2}$, we put
\[
A = \sqrt{(4 - k^2 - \sqrt{(2 - k^2)(8 - k^2)}/2(8 - k^2)),
B = 2/\sqrt{8 - k^2},
C = \sqrt{(4 - k^2 + \sqrt{(2 - k^2)(8 - k^2)}/2(8 - k^2)),
\alpha = (\sqrt{2 - k^2 + \sqrt{8 - k^2}})/\sqrt{2},
\beta = \sqrt{2 - k^2}/\sqrt{2},
\delta = (\sqrt{2 - k^2 - \sqrt{8 - k^2}})/\sqrt{2}.
\]
Let $\tilde{\gamma}$ be a curve in $C^3$ defined by $\tilde{\gamma}(t) = (Ae^{\text{rot}}, Be^{i\beta t}, Ce^{i\alpha t})$. Then $\tilde{\gamma}$ is a horizontal curve on $S^5(1)$ parametrized by arc length $t$. Moreover $\pi(\tilde{\gamma})$ is a holomorphic helix of order 4 in $CP_2(4)$ with curvatures $k_1 = k$, $k_2 = \sqrt{(18 - 9k^2)/2}$ and $k_3 = k$, and with complex torsions $\tau_{12} = \tau_{13} = \tau_{24} = \tau_{34} = 0$, $\tau_{14} = 1$ and $\tau_{23} = -1$. The curve $\pi(\tilde{\gamma})$ satisfies (1.5').

**Example 2.** Let $\tilde{\gamma}$ be a curve in $C^3$ defined by
\[
\tilde{\gamma}(t) = \left(1/\sqrt{3}\right)e^{it}, \left(1/\sqrt{14}\right)e^{2it}, \left(5/\sqrt{42}\right)e^{-4it/5}\right).
\]
Then $\pi(\tilde{\gamma})$ is a holomorphic helix of order 4 in $CP_2(4)$ with curvatures $k_1 = 3\sqrt{2}/5$, $k_2 = 11\sqrt{2}/10$ and $k_3 = 1/\sqrt{2}$, and with complex torsions $\tau_{12} = \tau_{14} = \tau_{23} = \tau_{34} = -1/\sqrt{2}$ and $\tau_{13} = \tau_{24} = 0$. The curve $\pi(\tilde{\gamma})$ satisfies (1.4).

Finally we shall investigate the moduli spaces $Hh^d(M_n(c))$ $(d = 1, 2, 3)$. The moduli space $Hh^1(M_n(c))$ clearly consists of one point. As an immediate consequence of Theorem 1, Theorem 2 and Proposition 5 we can establish the following.

**Theorem 5.** (1) The moduli space $Hh^2(M_n(c))$ is homeomorphic to a cone in $R^2$ or a half line according as $n \geq 2$ or $n = 1$. More precisely, $Hh^2(M_n(c))$ is $[0, \infty) \times [-1, 1] / \sim$ or $[0, \infty)$ according as $n \geq 2$ or $n = 1$, where the equivalence relation $\sim$ means that $(0, \tau) \sim (0, \rho)$ if $\tau, \rho \in [-1, 1]$. 

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(2) The moduli space $Hh^3(M_n(c))$ is connected and
$$Hh^3(M_n(c)) = \begin{cases} 
\{(k_1, k_2, \tau) \in [0, \infty) \times [0, \infty) \times [-1, 1] | \tau^2 \leq k_1^2/(k_1^2 + k_2^2) \} / \sim, & n \geq 3, \\
\{(0, \infty) \times \{0\} \times [-1, 1] \\
\cup \{(k_1, k_2, \pm k_1/\sqrt{k_1^2 + k_2^2}) | k_1 > 0, k_2 > 0 \} / \sim, & n = 2,
\end{cases}$$
where the equivalence relation $\sim$ means that $(0, k, \tau) \sim (0, l, \rho)$ if $k, l \in [0, \infty)$ and $\tau, \rho \in [-1, 1]$.

Remark. Let $\gamma$ be a holomorphic helix of proper order 3 with curvatures $k_1$ and $k_2$, and with the first complex torsion $\tau_{12} = \tau$ in a complex space form $M_n(c)$. Then $\gamma$ lies on a totally real totally geodesic submanifold $M^3(c/4)$ of $M_n(c)$ if and only if $\tau = 0$, and $\gamma$ lies on a holomorphic totally geodesic submanifold $M_2(c)$ of $M_n(c)$ if and only if $\tau = \pm k_1/\sqrt{k_1^2 + k_2^2}$.

References


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