HYPERSURFACES IN A SPHERE WITH CONSTANT MEAN CURVATURE

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ABSTRACT. Let $M^n$ be a closed hypersurface of constant mean curvature immersed in the unit sphere $S^{n+1}$. Denote by $S$ the square of the length of its second fundamental form. If $S < 2\sqrt{n-1}$, $M$ is a small hypersphere in $S^{n+1}$. We also characterize all $M^n$ with $S = 2\sqrt{n-1}$.

1. INTRODUCTION

Let $M^n$ be a closed submanifold with parallel mean curvature vector field immersed in the unit sphere $S^{n+p}$. Denote by $H$ the length of the mean curvature vector field and by $S$ the square of the length of the second fundamental form of $M^n$. It is important to characterize those $M$ immersed as $n$-spheres in $S^{n+p}$ by $H$ and $S$.

When $M$ is minimal, J. Simons [9] obtained a pinching constant $n/(2 - 1/p)$ of $S$ and Chern-do Carmo-Kobayashi [3] showed that it is sharp and characterized all $M$ with $S = n/(2 - 1/p)$. M. Okumura [6, 7] first discussed the general case and gave a pinching constant of $S$, but it is not sharp. Recently the sharp ones were obtained by H. Alencar-M. do Carmo [1] for $p = 1$, W. Santos [8] for $p > 1$ and H. W. Xu [11] for $p \geq 1$ respectively. But all of them were expressed by the mean curvature $H$. S. T. Yau [12] obtained a pinching constant for $p > 1$ which depended only on $n$ and $p$. H. W. Xu [10] improved Yau's result, but far from sharpness.

In the present paper, we shall give a pinching constant for $p = 1$ which depends only on $n$ and show the sharpness of it. More precisely, we want to prove the following theorems:

Theorem A. Let $M^n$ be a hypersurface of constant mean curvature immersed in $S^{n+1}$ with constant length of the second fundamental form. Then:

1. If $S < 2\sqrt{n-1}$, $M^n$ is locally a piece of small hypersphere $S^n(r)$ of radius $r = \sqrt{n/(n + S)}$.
2. If $S = 2\sqrt{n-1}$, $M$ is locally a piece of either $S^n(r_0)$ or $S^1(r) \times S^{n-1}(s)$ where $r_0^2 = n/(n + 2\sqrt{n-1})$, $r^2 = 1/(\sqrt{n-1} + 1)$ and $s^2 = \sqrt{n-1}/(\sqrt{n-1} + 1)$.

Theorem A'. Let $M^n$ be a closed hypersurface of constant mean curvature immersed in $S^{n+1}$. Then:

1. If $S < 2\sqrt{n-1}$, $M^n$ is a small hypersphere $S^n(r)$ of radius $r = \sqrt{n/(n + S)}$. 

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(2) If \( S = 2\sqrt{n-1} \), \( M \) is either a small hypersphere \( S^n(r_0) \) or a \( H(r) \)-torus \( S^1(r) \times S^{n-1}(s) \), where \( r_0, r \) and \( s \) are taken as before.

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2. PROOF OF THE THEOREMS

Let \( M \) be a closed hypersurface immersed in the unit sphere \( S^{n+1} \). Take a local orthonormal coframe field \( \{\omega_i\}_{i=1}^n \) on \( M \). Then the second fundamental form can be expressed as \( \omega = (h_{ij})_{n \times n} \). The mean curvature \( H \) and the square of the length of the second fundamental form \( S \) are defined by
\[
H = \frac{1}{n} \sum_i h_{ii}, \quad S = \sum_{i,j} (h_{ij})^2.
\]
From now on, we shall always use \( i, j, k, \ldots \) for indices running from 1 to \( n \).

Denote the covariant differentials of \( \{h_{ij}\} \) by \( \{h_{ijk}\} \) and \( \{h_{ijkl}\} \). Then the Laplacian of \( h_{ij} \) is defined by \( \Delta h_{ij} = \sum_k h_{ijkk} \). It follows that
\[
\sum_{(i,j)} h_{ij} \Delta h_{ij} = nS + nHf - n^2H^2 - S^2,
\]
where \( f = \text{Tr} L^3 \) (cf. e.g. [2] or [7]).

M. Okumura [7] established the following lemma (see also [1] or [11]).

**Lemma.** Let \( \{a_i\}_{i=1}^n \) be a set of real numbers satisfying \( \sum (i) a_i = 0, \sum (i) a_i^2 = t^2 \), where \( t \geq 0 \). Then we have
\[
-\frac{n-2}{\sqrt{n(n-1)}} t^3 \leq \sum (i) a_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}} t^3,
\]
and equalities hold if and only if at least \( (n-1) \) of the \( a_i \)'s are equal to one another.

Suppose that \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the principal curvatures of \( M \). Then we have
\[
nH = \sum (i) \lambda_i, \quad S = \sum (i) \lambda_i^2, \quad f = \sum (i) \lambda_i^3.
\]
Set \( S = -nH^2 \), \( \tilde{f} = f - 3HS + 2nH^3 \) and \( \tilde{\lambda}_i = \lambda_i - H \) \((1 \leq i \leq n)\). Then (3) changes into
\[
0 = \sum (i) \tilde{\lambda}_i, \quad \tilde{S} = \sum (i) \tilde{\lambda}_i^2, \quad \tilde{f} = \sum (i) \tilde{\lambda}_i^3.
\]

By applying Okumura’s Lemma to \( \tilde{f} \) in (4), we have
\[
\tilde{f} \geq -\frac{n-2}{\sqrt{n(n-1)}} \tilde{S}\sqrt{\tilde{S}} \iff f \geq 3HS - 2nH^3 - \frac{n-2}{\sqrt{n(n-1)}} \tilde{S}\sqrt{\tilde{S}}.
\]
Substituting this into (1), we have
\[
\sum_{(i,j)} h_{ij} \Delta h_{ij} \geq \tilde{S} \left\{ n - (\tilde{S} - nH^2) - (n-2)H \frac{n}{\sqrt{n-1}} \right\}.
\]

Consider the quadratic form \( Q(u, t) = u^2 - \frac{n-2}{\sqrt{n-1}} ut - t^2 \). By the orthogonal transformation
\[
\begin{align*}
\tilde{u} &= \frac{1}{\sqrt{2n}} \left\{ (1 + \sqrt{n-1})u + (1 - \sqrt{n-1})t \right\}, \\
\tilde{t} &= \frac{1}{\sqrt{2n}} \left\{ (\sqrt{n-1} - 1)u + (\sqrt{n-1} + 1)t \right\}.
\end{align*}
\]
Q(u, t) turns into \( Q(u, t) = \frac{1}{2\sqrt{n-1}}(\tilde{u}^2 - \tilde{t}^2) \), where \( \tilde{u}^2 + \tilde{t}^2 = u^2 + t^2 = S \).

Take \( t = \sqrt{S} \) and \( u = \sqrt{n}H \) in \( Q(u, t) \), and substitute it into (5). We can see

\[
(6) \quad \sum_{(i,j)} h_{ij} \Delta h_{ij} \geq \tilde{S} \left( n - \frac{n}{2\sqrt{n-1}}S + \frac{n}{\sqrt{n-1}}\tilde{u}^2 \right) \geq \tilde{S} \left( n - \frac{n}{2\sqrt{n-1}}S \right).
\]

Therefore we have

\[
(7) \quad \frac{1}{2} \Delta S = \sum_{(i,j,k)} h_{ijk}^2 + \sum_{(i,j)} h_{ij} \Delta h_{ij} \geq \tilde{S} \left( n - \frac{n}{2\sqrt{n-1}}S \right).
\]

**Theorem A.** Let \( M^n \) be a hypersurface of constant mean curvature immersed in \( S^{n+1} \) with constant length of the second fundamental form. Then:

1. If \( S < 2\sqrt{n-1} \), \( M \) is locally a piece of a small hypersphere \( S^n(r) \) in \( S^{n+1} \), where \( r = \sqrt{n/(n+S)} \).
2. If \( S = 2\sqrt{n-1} \), \( M \) is locally a piece of either \( S^n(r_0) \) or \( S^1(r) \times S^{n-1}(s) \), where \( r_0^2 = n/(n+2\sqrt{n-1}) \), \( r^2 = 1/(\sqrt{n-1}+1) \) and \( s^2 = \sqrt{n-1}/(\sqrt{n-1}+1) \).

**Proof.** Since \( S \) is constant, the left-hand side of (7) is zero. When \( S \leq 2\sqrt{n-1} \), we have

\[
(8) \quad \tilde{S} \left( n - \frac{n}{2\sqrt{n-1}}S \right) = 0, \quad h_{ijk} = 0, \quad 1 \leq i, j, k \leq n.
\]

If \( S < 2\sqrt{n-1} \), we have \( \tilde{S} = 0 \), which means that \( M \) is totally umbilical and hence is locally a piece of hypersphere \( S^n(r) \) where \( r = \sqrt{n/(n+S)} \).

Suppose \( S = 2\sqrt{n-1} \). Then all of the inequalities in (5)–(7) become equal ones. Okumura’s Lemma implies that at least \( n-1 \) of \( \lambda_i \)’s are equal to one another. When \( \lambda_1 = \lambda_2 = \cdots = \lambda_n \), \( M \) is totally umbilical and hence is locally a piece of hypersphere \( S^n(r) \) where \( r^2 = n/(n+2\sqrt{n-1}) \). When \( M \) is not totally umbilical, there are exactly \( n-1 \) of \( \lambda_i \)’s that are equal to one another. The same arguments as those developed by Chern-do Carmo-Kobayashi (see [3], p. 68) show that \( M \) is locally a piece of \( S^1(r) \times S^{n-1}(s) \) in \( S^{n+1} \). To determine the radii \( r \) and \( s \), we refer to the examples of K. Nomizu and B. Smyth [5], from which we have

\[ H = -\frac{1}{n} \left( \frac{s}{r} \right)^2 + \frac{n-1}{n} \left( \frac{r}{s} \right)^2, \quad S = \left( \frac{s}{r} \right)^2 + (n-1) \left( \frac{r}{s} \right)^2. \]

It is easy to see that

\[
\left( \frac{s}{r} \right)^2 + (n-1) \left( \frac{r}{s} \right)^2 \geq 2\sqrt{n-1}
\]

and equality holds if and only if \( \left( \frac{s}{r} \right)^2 = \sqrt{n-1} \). Therefore we have \( r^2 = \frac{1}{\sqrt{n-1}+1} \) and \( s^2 = \frac{\sqrt{n-1}}{\sqrt{n-1}+1} \).

When \( M \) is closed, the integral of the left-hand side of (7) on \( M \) is equal to zero, and so is that of the right-hand side. After the same deduction as in the proof of Theorem A, we can obtain the following:

**Theorem A’.** Suppose \( M \) is a closed hypersurface of constant mean curvature immersed in \( S^{n+1} \). Then:

1. If \( S < 2\sqrt{n-1} \), \( M \) is a small hypersphere \( S^n(r) \), where \( r = \sqrt{n/(n+S)} \).
2. If \( S = 2\sqrt{n-1} \), \( M \) is either a small hypersphere \( S^n(r_0) \) or \( S^1(r) \times S^{n-1}(s) \), where \( r_0, r \) and \( s \) are taken as in Theorem A.
We can show an application of Theorem A'. H. W. Xu [10] proved the following:

**Proposition (Xu).** Let \( M^n \) be an \( n \)-dimensional compact submanifold with parallel mean curvature vector field in \( S^{n+p} \) and \( p > 1 \). If

\[
S \leq \min \left\{ \frac{2n}{1 + \sqrt{n}}, \frac{n}{2 - (p - 1)^{-1}} \right\},
\]

and the Gauss mapping of \( M \) is relatively affine, then \( M^n \) is a standard hypersphere in a totally geodesic \( S^{n+1} \) of \( S^{n+p} \).

By Theorem A', we can remove the assumption that the Gauss mapping is relatively affine. Namely we can obtain the following

**Corollary.** Let \( M^n \) be an \( n \)-dimensional compact submanifold with parallel mean curvature vector field in \( S^{n+p} \) and \( p > 1 \). If

\[
S \leq \min \left\{ \frac{2n}{1 + \sqrt{n}}, \frac{n}{2 - (p - 1)^{-1}} \right\},
\]

then \( M^n \) is a standard hypersphere in a totally geodesic \( S^{n+1} \) of \( S^{n+p} \).

**Proof.** It is easy to check that \((\sqrt{n} + 1)/n > 1/\sqrt{n - 1}\). Therefore we have

\[
\sqrt{n - 1} > \frac{n}{\sqrt{n} + 1} \iff 2\sqrt{n - 1} > \frac{2n}{\sqrt{n} + 1} \geq S.
\]

\[\square\]

**REFERENCES**


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