PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 4, April 1997, Pages 1089–1094
S 0002-9939(97)03685-X

WILSON'S FUNCTIONAL EQUATION FOR VECTOR AND MATRIX FUNCTIONS

PAVLOS SINOPOULOS

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. We determine the general solution of the functional equation

$$f(x+y)+f(x-y)=A(y)f(x) \qquad (x,y\in G),$$

where G is a 2-divisible abelian group, f is a vector-valued function and A is a matrix-valued function. Using this result we solve the scalar equation

$$f(x+y)+f(x-y)=g_1(x)h_1(y)+g_2(x)h_2(y) \qquad (x,y\in G),$$

which contains as special cases, among others, the d'Alembert and Wilson equations and the parallelogram law.

1. INTRODUCTION

Let G be an abelian group divisible by 2 (that is, $2G=G$) and K an algebraically closed commutative field of characteristic different from 2 and 3. Throughout this paper we denote by $\mathcal{L},\mathcal{M},\mathcal{N}$ the sets of all solutions $f\colon G\to K$ of the functional equations $f(x+y)=f(x)+f(y)$, $f(x+y)=f(x)f(y)$ and $f(x+y)+f(x-y)=2f(x)+2f(y)$, respectively. Also, K_2 is the set of all 2×2 matrices over K, K^2 is the set of all 2×1 matrices (column vectors) over K, $\mathcal{F}$ is the set of all vector-valued functions with linearly independent components, I is the unit matrix of K_2 and 0 (the zero of K) is also used for the zeros of G, K^2 and K_2. If $a\in\mathcal{L}$, then a is called additive.

Clearly, if $M\in K_2$, $f\in\mathcal{F}$ and $Mf=0$, then $M=0$.

In this paper we determine the general solutions $f\colon G\to K^2$, $A\colon G\to K_2$ of the functional equation

$$f(x+y)+f(x-y)=A(y)f(x) \qquad (x,y\in G),$$

which can be viewed as a vector analogue of Wilson's functional equation (1.1) below (cf. also Eq. (3.4) of [4]).

The main result is given in Theorem 1. Using this result we solve in Theorem 2 the scalar equation

$$f(x+y)+f(x-y)=g_1(x)h_1(y)+g_2(x)h_2(y) \qquad (x,y\in G).$$

This last equation contains as special cases the Wilson equation

$$f(x+y)+f(x-y)=f(x)g(y), \tag{1.1}$$

Received by the editors August 4, 1995 and, in revised form, September 22, 1995.
1991 *Mathematics Subject Classification.* Primary 39B42, 39B52, 39B62.

introduced in [9], the equations

$$f(x+y)+f(x-y)=2f(x)+g(x)h(y), \tag{1.2}$$

$$f(x+y)+f(x-y)=f(x)g(y)+g(x)f(y), \tag{1.3}$$

$$f(x+y)+f(x-y)=f(x)f(y)+g(x)g(y), \tag{1.4}$$

which were solved in [1], [3], and the parallelogram law

$$f(x+y)+f(x-y)=2f(x)+2f(y), \tag{1.5}$$

which characterizes the diagonalization of biadditive functions ([1], [2]). More general equations were solved e.g. in [4], but, as remarked in [3], to extract the solutions of (1.1)–(1.5) from [4] is not simple.

2. The results

Theorem 1. *If $A\colon G\to K_2$, $f\colon G\to K^2$ with $f\in\mathcal{F}$ is a solution of the functional equation*

$$f(x+y)+f(x-y)=A(y)f(x) \qquad (x,y\in G), \tag{2.1}$$

then

$$\begin{cases} A(y)=C[E(y)+E(-y)]C^{-1}, \\ f(x)=C[E(x)\alpha+E(-x)\beta], \end{cases} \tag{2.2}$$

where $C\in K_2$ $(\det C\neq 0)$, $\alpha,\beta\in K^2$ and $E(x)$ has one of the forms

$$\begin{gathered} E(x)=\chi(x)\begin{Vmatrix}1 & \phi(x)\\ 0 & 1\end{Vmatrix}, \\ \begin{Vmatrix}1 & n(x)+\phi(x)\\ 0 & 1\end{Vmatrix}, \quad \begin{Vmatrix}\phi(x)+1 & c\phi(x)^3+3c\phi(x)^2+\phi_1(x)\\ 0 & \phi(x)+1\end{Vmatrix}, \\ \begin{Vmatrix}\chi_1(x)-[\chi_1(0)-1][\phi_1(x)+1] & 0\\ 0 & \chi_2(x)-[\chi_2(0)-1][\phi_2(x)+1]\end{Vmatrix}, \end{gathered} \tag{2.3}$$

with $c\in K$, $\phi,\phi_1,\phi_2\in\mathcal{L}$, $\chi,\chi_1,\chi_2\in\mathcal{M}$, $\chi(x)\not\equiv 0$, and $n\in\mathcal{N}$.

Conversely, any A, f as described above satisfy (2.1).

Note. If $f\notin\mathcal{F}$, then (2.1) reduces to a system of two Wilson equations and the solutions are obtained by application of Theorem (2.2) of [5].

Proof. Setting $y=u+v$ and $y=u-v$ in (2.1), adding the resulting equations and using (2.1) again we obtain

$$A(v)A(u)f(x)=[A(u+v)+A(u-v)]f(x).$$

Since $f\in\mathcal{F}$ this leads to the matrix d'Alembert equation

$$A(v)A(u)=A(u+v)+A(u-v). \tag{2.4}$$

Replacing y by $-y$ in (2.1) we see that $A(y)f(x)=A(-y)f(x)$ and so $A(y)=A(-y)$. Hence, interchanging u, v in (2.4), we have $A(u)A(v)=A(v)A(u)$. Since K is algebraically closed there exists an invertible matrix $C\in K_2$ such that the matrix function $B(x):=C^{-1}A(x)C$ has the form

$$B(x)=\begin{Vmatrix}b_1(x) & b_0(x)\\ 0 & b_2(x)\end{Vmatrix}$$

with $b_1(x)\equiv b_2(x)$ or $b_0(x)\equiv 0$ (see [8], [4]).

Let $g(x) := C^{-1}f(x)$. Substituting

$$A(x) = CB(x)C^{-1} \quad \text{and} \quad f(x) = Cg(x) \tag{2.5}$$

into (2.1) we get

$$g(x+y) + g(x-y) = B(y)g(x), \tag{2.6}$$

that is, with $g = \begin{Vmatrix} g_1 \\ g_2 \end{Vmatrix}$,

$$g_1(x+y) + g_1(x-y) = g_1(x)b_1(y) + g_2(x)b_0(y), \tag{2.6a}$$

$$g_2(x+y) + g_2(x-y) = g_2(x)b_2(y). \tag{2.6b}$$

Substituting also the first of (2.5) into (2.4) we have

$$B(u+v) + B(u-v) = B(u)B(v), \tag{2.7}$$

that is,

$$b_i(u+v) + b_i(u-v) = b_i(u)b_i(v), \qquad i = 1, 2, \tag{2.7a}$$

$$b_0(u+v) + b_0(u-v) = b_1(u)b_0(v) + b_0(u)b_2(v). \tag{2.7b}$$

We distinguish three cases.

Case 1. $b_1(x) \equiv b_2(x) \not\equiv 2$. Then $b_1(x_0) = b_2(x_0) \neq 2$ for some $x_0 \in G$. Setting $y = 0$ in (2.6) gives $B(0) = 2I$ (since $g \in \mathcal{F}$) and from (2.7) it follows that $B(2x) + 2I = B(x)^2$. So the matrix $B(2x_0) - 2I = B(x_0)^2 - 4I$ is invertible. This and all subsequent 2×2 matrices are upper triangular with equal diagonal elements and hence commute.

Now the solutions of the matrix equation (2.7) can be written in the form $B(x) = \Xi(x) + \Xi(-x)$ where $\Xi(x) = \chi(x) \begin{Vmatrix} 1 & \phi(x) \\ 0 & 1 \end{Vmatrix}$ with $\phi \in \mathcal{L}$, $\chi \in \mathcal{M}$ and $\Xi(x+y) = \Xi(x)\Xi(y)$ (see, e.g., the proof of Lemma 1 of [7]).

In order to solve the equation (2.6) we follow the method used in the proof of Theorem 2.2 of [5]. In (2.6) we replace x by $-x$, we subtract the resulting equation from (2.6) and, setting $h(x) = g(x) - g(-x)$, we obtain

$$h(x+y) + h(x-y) = B(y)h(x)$$

with $h(0) = 0$, $h(-x) = -h(x)$ and $h(2x) = B(x)h(x)$. Therefore

$$\begin{aligned} [2B(x+y) - B(x)B(y)]h(x) &= 2B(x+y)h(x) - B(y)h(2x) \\ &= 2[h(2x+y) - h(y)] - [h(2x+y) + h(2x-y)] \\ &= h(y+2x) + h(y-2x) - 2h(y) = B(2x)h(y) - 2h(y). \end{aligned}$$

Since $2B(x+y) - B(x)B(y) = [\Xi(x) - \Xi(-x)][\Xi(y) - \Xi(-y)]$ we have

$$[B(2x) - 2I]h(y) = [\Xi(x) - \Xi(-x)][\Xi(y) - \Xi(-y)]h(x).$$

Setting $x = x_0$ we obtain

$$g(y) - g(-y) = [\Xi(y) - \Xi(-y)]\gamma$$

where $\gamma \in K^2$. Adding this to $g(y) + g(-y) = [\Xi(y) + \Xi(-y)]g(0)$ (which follows from (2.6) with $x = 0$) we obtain

$$g(x) = \Xi(x)\alpha + \Xi(-x)\beta,$$

where $\alpha, \beta \in K^2$. So by (2.5) we obtain (2.2) with the first form in (2.3).

Case 2. $b_1(x) \equiv b_2(x) \equiv 2$. Then (2.7b) shows that $b_0 \in \mathcal{N}$ and from (2.6b) (a Jensen equation) we have $g_2(x) = \phi_2(x) + \lambda$ with $\lambda \in K$ and $\phi_2 \in \mathcal{L}$. Now (2.6a) becomes

$$g_1(x+y) + g_1(x-y) = 2g_1(x) + [\phi_2(x) + \lambda]b_0(y). \tag{2.8}$$

If $\phi_2(x) \equiv 0$, then interchanging x, y in (2.8) and subtracting we find

$$g_1(x-y) - g_1(y-x) = [2g_1(x) - \lambda b_0(x)] - [2g_1(y) - \lambda b_0(y)],$$

a Pexider equation which gives $2g_1(x) - \lambda b_0(x) = 2\phi_0(x) + 2\mu$ ($\mu \in K, \phi_0 \in \mathcal{L}$). Hence

$$g(x) = \begin{Vmatrix} \frac{1}{2}\lambda b_0(x) + \phi_0(x) + \mu \\ \lambda \end{Vmatrix}, \qquad B(y) = \begin{Vmatrix} 2 & b_0(y) \\ 0 & 2 \end{Vmatrix}.$$

Using (2.5) we obtain (2.2) with the second form in (2.3).

If $\phi_2(x) \not\equiv 0$, then applying Lemma 4 of [1] to (2.8) (here we need the condition $\operatorname{char} K \neq 2, 3$) we obtain

$$g_1(x) = c\phi_2(x)^3 + 3c\lambda\phi_2(x)^2 + \phi_0(x) + \mu, \qquad b_0(y) = 6c\phi_2(y)^2,$$

where $c, \mu \in K$ and $\phi_0 \in \mathcal{L}$. So by (2.5) we obtain (2.2) with the third form in (2.3).

Case 3. $b_0(x) \equiv 0$. Then applying Theorem 2.2 of [5] to (2.6a) and (2.6b) we see that $g(x)$, $B(y)$ have one of the forms

$$g(x) = \begin{Vmatrix} \lambda\theta(x) + \mu\theta(-x) \\ \nu\eta(x) + \kappa\eta(-x) \end{Vmatrix}, \qquad B(y) = \begin{Vmatrix} \theta(y) + \theta(-y) & 0 \\ 0 & \eta(y) + \eta(-y) \end{Vmatrix},$$

$$g(x) = \begin{Vmatrix} a_0(x) + \lambda \\ \mu\theta(x) + \nu\theta(-x) \end{Vmatrix}, \qquad B(y) = \begin{Vmatrix} 2 & 0 \\ 0 & \theta(y) + \theta(-y) \end{Vmatrix},$$

$$g(x) = \begin{Vmatrix} \mu\theta(x) + \nu\theta(-x) \\ a_0(x) + \lambda \end{Vmatrix}, \qquad B(y) = \begin{Vmatrix} \theta(y) + \theta(-y) & 0 \\ 0 & 2 \end{Vmatrix},$$

$$g(x) = \begin{Vmatrix} a_0(x) + \lambda \\ \phi_0(x) + \mu \end{Vmatrix}, \qquad B(y) = \begin{Vmatrix} 2 & 0 \\ 0 & 2 \end{Vmatrix},$$

where $\theta, \eta \in \mathcal{M}$, $a_0, \phi_0 \in \mathcal{L}$ and $\kappa, \lambda, \mu, \nu \in K$. So by (2.5) we obtain (2.2) with the last form in (2.3).

To prove the converse observe that the substitution of (2.2) into (2.1) gives

$$\begin{aligned}&[E(x+y) + E(x-y) - [E(y) + E(-y)]E(x)]\alpha \\ &\qquad + [E(-x+y) + E(-x-y) - [E(y) + E(-y)]E(-x)]\beta = 0.\end{aligned}$$

On the other hand, it is easy to verify that the functions E in (2.3) satisfy the functional equation

$$E(x+y) + E(x-y) = [E(y) + E(-y)]E(x) \qquad (x, y \in G) \tag{2.9}$$

and this completes the proof of Theorem 1. □

In the following theorem the prime indicates transposition.

Theorem 2. *If $f\colon G \to K$, $g, h\colon G \to K^2$ is a solution of the functional equation*

$$f(x+y) + f(x-y) = h'(y)g(x) \qquad (x, y \in G), \tag{2.10}$$

then f has the form

$$f(x) = \lambda'[E(x)\alpha + E(-x)\beta], \tag{2.11}$$

where $\lambda, \alpha, \beta \in K^2$ and E is given by (2.3).

Conversely, for any f as described above there exist g, h which together with f constitute a solution of (2.10).

Notes. 1. Once f is known, it is easy to determine the complete list for the vector-valued functions g, h.

2. The right-hand side of (2.10), as a scalar product, can also be written $g'(x)h(y)$.

Proof. If $g \notin \mathcal{F}$ or $h \notin \mathcal{F}$, then (2.10) reduces to a Wilson equation. Applying Theorem 2.2 of [5] to this equation we see that the solutions are contained in (2.11). So in the following we assume that $g, h \in \mathcal{F}$.

Setting $y = u + v$ and $y = u - v$ in (2.10), adding the resulting equations and using again (2.10) we find

$$h'(v)[g(x+u) + g(x-u)] = [h'(u+v) + h'(u-v)]g(x).$$

Since $h \in \mathcal{F}$, using Lemma 14.1 of [2] we obtain

$$g(x+u) + g(x-u) = A(u)g(x),$$

with $A\colon G \to K_2$. Since $g \in \mathcal{F}$, applying Theorem 1 to this equation we obtain

$$g(x) = C[E(x)\alpha + E(-x)\beta)],$$

where, $\alpha, \beta \in K^2$, $C \in K_2$ and E is given by (2.3).

Now from (2.10) with $y = 0$ we have $f(x) = \frac{1}{2}h'(0)g(x)$ which leads to (2.11).

The converse is verified by substitution of (2.11) into (2.10) and using (2.9). □

Remark. The following result generalizes Theorem 1 and is proved in the same way. Here S is an arbitrary set.

Theorem 3. *If $A\colon G \to K_2$, $f\colon G \times S \to K^2$ with $f \in \mathcal{F}$ is a solution of the functional equation*

$$f(x+y, s) + f(x-y, s) = A(y)f(x, s) \quad (x, y \in G, s \in S), \tag{2.12}$$

then

$$\begin{cases} A(y) = C[E(y, s_0) + E(-y, s_0)]C^{-1}, \\ f(x, s) = C[E(x, s)\alpha(s) + E(-x, s)\beta(s)], \end{cases} \tag{2.13}$$

where $s_0 \in S$, $C \in K_2$ $(\det C \neq 0)$, $\alpha, \beta\colon S \to K^2$ are arbitrary functions and $E(x, s)$ has one of the forms

$$\begin{gathered} E(x,s) = \chi(x) \begin{Vmatrix} 1 & \phi(x) \\ 0 & 1 \end{Vmatrix}, \\ \begin{Vmatrix} 1 & n(x) + \phi_1(x,s) \\ 0 & 1 \end{Vmatrix}, \quad \begin{Vmatrix} \phi(x) + 1 & c\phi(x)^3 + 3c\phi(x)^2 + \phi_1(x,s) \\ 0 & \phi(x) + 1 \end{Vmatrix}, \\ \begin{Vmatrix} \chi_1(x) - [\chi_1(0) - 1][\phi_1(x,s) + 1] & 0 \\ 0 & \chi_2(x) - [\chi_2(0) - 1][\phi_2(x,s) + 1] \end{Vmatrix}, \end{gathered} \tag{2.14}$$

with $c \in K$, $\phi \in \mathcal{L}$, $\chi, \chi_1, \chi_2 \in \mathcal{M}$, $\chi(x) \not\equiv 0$, $n \in \mathcal{N}$ and $\phi_1, \phi_2\colon G \times S \to K$ additive in the first variable.

Conversely, any A, f as described above satisfy (2.12).

(Clearly, for E given by (2.14), the sum $E(y,s)+E(-y,s)$ is independent of the second variable, that is, $E(y,s)+E(-y,s)=E(y,t)+E(-y,t)$ for all $y \in G$ and $s,t \in S$. So A in the first of (2.13) is independent of the choice of s_0.)

Theorem 3 can be used, e.g., in the solution of the functional equation

$$f(x+t,y-t)+f(x-t,y+t)=g_1(x,y)h_1(t)+g_2(x,y)h_2(t)$$

which generalizes (2.10) (and (3.2) of [6]).

Acknowledgment

I wish to thank the referee for his valuable remarks which helped in improving this paper.

References

1. J. Aczél, J. K. Chung and C. T. Ng, *Symmetric second differences in product form on groups*, Topics in mathematical analysis, (edited by Th. M. Rassias), World Scientific Publ., 1989, pp. 1-22. MR **92g:**39007
2. J. Aczél and J. Dhombres, *Functional equations in several variables*, Cambridge Univ. Press, 1989. MR **90h:**39001
3. J. K. Chung, Pl. Kannappan and C. T. Ng, *On two trigonometric functional equations*, Math. Rep. Toyama Univ. **11** (1988), 153–165. MR **89j:**39010
4. A. L. Rukhin, *The solution of the functional equation of d'Alembert's type for commutative groups*, Intern. J. Math. Sci. **5** (1982), 315–335. MR **84g:**39006
5. P. Sinopoulos, *Generalized sine equations*, I, Aeq. Mathematicae **48** (1994), 171–193. MR **95i:**39020
6. P. Sinopoulos, *Generalized sine equations*, II, Aeq. Mathematicae **49** (1995), 122–152. MR **96b:**39020
7. P. Sinopoulos, *A functional equation in three variables for five unknown functions*, Submitted.
8. D. A. Suprunenco and R. I. Tyshkevich, *Commutative matrices*, Academic Press, 1968.
9. W. H. Wilson, *On certain related functional equations*, Bull. Amer. Math. Soc. **26** (1919–20), 300–312.

18 Vergovitsas Street, GR-11475 Athens, Greece