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ON THE VON NEUMANN-JORDAN CONSTANT FOR BANACH SPACES

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Abstract. Let $C_{\mathrm{NJ}}(E)$ be the von Neumann-Jordan constant for a Banach space E. It is known that $1 \le C_{\mathrm{NJ}}(E) \le 2$ for any Banach space E; and E is a Hilbert space if and only if $C_{\mathrm{NJ}}(E) = 1$. We show that: (i) If E is uniformly convex, $C_{\mathrm{NJ}}(E)$ is less than two; and conversely the condition $C_{\mathrm{NJ}}(E) < 2$ implies that E admits an equivalent uniformly convex norm. Hence, denoting by $\widetilde{C}_{\mathrm{NJ}}(E)$ the infimum of all von Neumann-Jordan constants for equivalent norms of E, E is super-reflexive if and only if $\widetilde{C}_{\mathrm{NJ}}(E) < 2$. (ii) If $\widetilde{C}_{\mathrm{NJ}}(E) = 2^{2/p-1}$, $1 < p \le 2$ (the same value as that of L_p-space), E is of Rademacher type r and cotype r' for any r with $1 \le r < p$, where $1/r + 1/r' = 1$; the converse holds if E is a Banach lattice and l_p is finitely representable in E or E'.

Introduction and preliminaries

In connection with the famous work [8] of Jordan and von Neumann concerning inner products, the von Neumann-Jordan (NJ-) constant $C_{\mathrm{NJ}}(E)$ for a Banach space E was introduced by Clarkson [3] as the smallest constant C for which

$$\frac{1}{C} \le \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \le C \tag{1}$$

holds for all $x, y \in E$ with $(x, y) \ne (0, 0)$. As is easily seen, if C is best possible in the right-hand side inequality of (1), then so is $1/C$ in the left. Despite its fundamental nature very little is known about the NJ-constant; a summary is given in Theorem A below. In what follows, let $1 \le p,\ q, r, t \le \infty$ unless otherwise stated, and let p', q', r', t' be their conjugate numbers, respectively.

A. Theorem. (i) *$1 \le C_{\mathrm{NJ}}(E) \le 2$ for all Banach spaces E; and $C_{\mathrm{NJ}}(E) = 1$ if and only if E is a Hilbert space (Jordan and von Neumann [8]).*

(ii) *$C_{\mathrm{NJ}}(L_p) = 2^{2/t-1}$, where $t = \min\{p, p'\}$ (Clarkson [3]; see also [5]).*

(iii) *For $L_p(L_q)$, L_q-valued L_p-space (on arbitrary measure spaces), $C_{\mathrm{NJ}}(L_p(L_q)) = 2^{2/t-1}$, where $t = \min\{p, q, p', q'\}$; and for the Sobolev space $W_p^k(\Omega)$, $C_{\mathrm{NJ}}(W_p^k(\Omega)) = 2^{2/t-1}$, where $t = \min\{p, p'\}$ (Kato and Miyazaki [10]).*

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(iv) *For* $E = C_c(K)$, *resp.* $C_b(K)$ (*the spaces of continuous functions on a locally compact Hausdorff space* K *which have compact support, resp. are bounded*), $C_{\mathrm{NJ}}(E) = 2$ (*Kato and Miyazaki* [9]).

The following facts are readily seen:

B. Proposition. (i) $C_{\mathrm{NJ}}(E) = 2^{2/t-1}$, $1 \leq t \leq 2$, *if and only if*

$$\|A\colon l_2^2(E) \to l_2^2(E)\| = 2^{1/t},$$

where $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ *and* $l_2^2(E)$ *denotes the* E*-valued* l_2^2*-space*; *and hence*

(ii) $C_{\mathrm{NJ}}(E') = C_{\mathrm{NJ}}(E)$, *where* E' *is the dual space of* E. (*This was observed for* L_p *in Clarkson* [3].)

In this paper we first show that a uniformly convex Banach space E is nearly characterized by the condition $C_{\mathrm{NJ}}(E) < 2$; more precisely the uniform convexity for E implies $C_{\mathrm{NJ}}(E) < 2$, while conversely $C_{\mathrm{NJ}}(E) < 2$ assures the existence of an equivalent uniformly convex norm in E. Hence, letting $\widetilde{C}_{\mathrm{NJ}}(E)$ be the infimum of all NJ-constants for equivalent norms of E, E is super-reflexive if and only if $\widetilde{C}_{\mathrm{NJ}}(E) < 2$. Secondly we show that if $\widetilde{C}_{\mathrm{NJ}}(E) = 2^{2/p-1}$, $1 < p \leq 2$ (the same value as that of L_p and of $L_{p'}$ as well), then E is of type r and cotype r' for any r with $1 \leq r < p$; and the converse holds if E is a Banach lattice and l_p is finitely representable in E or E'. Thus, for a Banach lattice E the number p satisfying $\widetilde{C}_{\mathrm{NJ}}(E) = 2^{2/p-1}$ is coincident with $\min\{p(E), q(E)'\}$, where $p(E) = \sup\{p; E \text{ is of type } p\}$, $q(E) = \inf\{q; E \text{ is of cotype } q\}$ and $1/q(E) + 1/q(E)' = 1$.

Let us recall some definitions. A Banach space E is called *strictly convex* if $\|(x+y)/2\| < 1$ whenever $\|x\| = \|y\| = 1$, $x \neq y$. E is said to be *uniformly convex* provided for each ε $(0 < \varepsilon < 2)$ there exists a $\delta > 0$ such that $\|(x+y)/2\| < 1 - \delta$ whenever $\|x - y\| \geq \varepsilon$, $\|x\| = \|y\| = 1$. E is called $(2, \varepsilon)$*-convex*, $\varepsilon > 0$, if $\min\{\|x+y\|, \|x-y\|\} \leq 2(1-\varepsilon)$ whenever $\|x\| = \|y\| = 1$.

A Banach space F is said to be *finitely representable in* E provided for any $\lambda > 1$ and each finite-dimensional subspace F_1 of F, there is an isomorphism T of F_1 into E for which

$$\lambda^{-1}\|x\| \leq \|Tx\| \leq \lambda\|x\| \quad \text{for all } x \in F_1.$$

E is said to be *super-reflexive* ([7]; cf. [1], [12]) if no non-reflexive Banach space is finitely representable in E.

Super-reflexive spaces are characterized as those uniformly convexifiable:

C. Theorem (Enflo [4]; cf. [1], [15])**.** *A Banach space* E *is super-reflexive if and only if* E *admits an equivalent uniformly convex norm.*

A Banach space E is said to be *of* (*Rademacher*) *type* p $(1 \leq p \leq 2)$, resp. *of cotype* q $(2 \leq q \leq \infty)$, if there exists some $M > 0$ such that

$$\int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\| dt \leq M \left\{ \sum_{j=1}^n \|x_j\|^p \right\}^{1/p},$$

resp.

$$\left\{ \sum_{j=1}^n \|x_j\|^q \right\}^{1/q} \leq M \int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\| dt,$$

for all finite systems $\{x_j\}$ in E, where $r_j(t)$ are the Rademacher functions, i.e., $r_j(t) = \operatorname{sgn}(\sin 2^j \pi t)$.

D. Theorem (Maurey and Pisier [14]; also [16], esp. Theorem 3.11, or [12],v. 3)**.** *Let E be a Banach space of infinite dimension. Let*

$$p(E) := \sup\{p;\ E \textit{ is of type } p\},$$

resp.

$$q(E) := \inf\{q;\ E \textit{ is of cotype } q\}.$$

Then

$$p(E) = \min\{p;\ l_p \textit{ is finitely representable in } E\},$$

resp.

$$q(E) = \max\{q;\ l_q \textit{ is finitely representable in } E\}.$$

(*Note that $p(E) \le 2 \le q(E)$.*)

A Banach lattice E is called *p-convex*, resp. *p-concave*, $1 \le p < \infty$, if there exists a constant M such that

$$\left\| \left\{ \sum_{j=1}^{n} |x_j|^p \right\}^{1/p} \right\| \le M \left\{ \sum_{j=1}^{n} \|x_j\|^p \right\}^{1/p}, \tag{2}$$

resp.

$$\left\{ \sum_{j=1}^{n} \|x_j\|^p \right\}^{1/p} \le M \left\| \left\{ \sum_{j=1}^{n} |x_j|^p \right\}^{1/p} \right\|, \tag{3}$$

for all finite systems $\{x_j\}$ in E. The smallest possible value of M in (2), resp. (3), is called the *p-convexity*, resp. *p-concavity*, *constant* for E.

Now, we begin with the following proposition which will give effective examples later.

1. Proposition. *Let $1 < p \le 2$ and $\lambda > 1$. Let $X_{p,\lambda}$ be the space $l_{p'}$ equipped with the norm $\|x\|_{p,\lambda} := \max\{\|x\|_{p'},\ \lambda\|x\|_\infty\}$, where $1/p + 1/p' = 1$. Then*:

(i) $C_{\mathrm{NJ}}(X_{p,\lambda}) = \begin{cases} \lambda^2 2^{2/p-1} & \textit{if } 1 < \lambda \le 2^{1/p'}, \\ 2 & \textit{if } \lambda \ge 2^{1/p'}, \end{cases}$

(ii) *$X_{2,\lambda}$ is isomorphic to a Hilbert space and*

$$C_{\mathrm{NJ}}(X_{2,\lambda}) = \min\{\lambda^2, 2\},$$

(iii) *$X_{p,\lambda}$ is not strictly convex for any $\lambda > 1$.*

Proof. (i) Since

$$\|x\|_{p'} \le \|x\|_{p,\lambda} \le \lambda \|x\|_{p'} \quad \text{for all } x \in X_{p,\lambda}, \tag{4}$$

we have

$$\begin{aligned} \|x+y\|_{p,\lambda}^2 + \|x-y\|_{p,\lambda}^2 &\le \lambda^2(\|x+y\|_{p'}^2 + \|x-y\|_{p'}^2) \\ &\le \lambda^2 2^{2/p}(\|x\|_{p'}^2 + \|y\|_{p'}^2) \\ &\le \lambda^2 2^{2/p}(\|x\|_{p,\lambda}^2 + \|y\|_{p,\lambda}^2), \end{aligned}$$

where the second inequality follows from the fact $C_{\mathrm{NJ}}(l_{p'}) = 2^{2/p-1}$ (Theorem A (ii)) and Proposition B (i). This implies $C_{\mathrm{NJ}}(X_{p,\lambda}) \le \lambda^2 2^{2/p-1}$. Let $1 < \lambda \le 2^{1/p'}$. Put $x = (1/\lambda, 1/\lambda, 0, \dots)$, $y = (1/\lambda, -1/\lambda, 0, \dots) \in X_{p,\lambda}$. Then, since $\|x\|_{p'} = \|y\|_{p'} = \lambda^{-1}2^{1/p'} \ge 1$ and $\|x\|_\infty = \|y\|_\infty = \lambda^{-1}$, we have

$$\|x\|_{p,\lambda} = \|y\|_{p,\lambda} = \lambda^{-1}2^{1/p'},$$

and clearly

$$\|x+y\|_{p,\lambda} = \|x-y\|_{p,\lambda} = 2. \tag{5}$$

Hence,

$$\frac{\|x+y\|_{p,\lambda}^2 + \|x-y\|_{p,\lambda}^2}{2(\|x\|_{p,\lambda}^2 + \|y\|_{p,\lambda}^2)} = \lambda^2 2^{2/p-1}, \tag{6}$$

which implies $C_{\mathrm{NJ}}(X_{p,\lambda}) = \lambda^2 2^{2/p-1}$. Let $\lambda \ge 2^{1/p'}$, and let x and y be as above. Then we have $\|x\|_{p,\lambda} = \|y\|_{p,\lambda} = 1$ and (5). Therefore the left side of (6), in this case, takes the value two, or $C_{\mathrm{NJ}}(X_{p,\lambda}) = 2$.

(ii) is clear by (4) and (i).

(iii) Take α as $(1/\lambda)^{p'} + \alpha^{p'} \le 1$ and $0 < \alpha \le 1/\lambda$. Put $x = (1/\lambda, 0, 0, \dots)$ and $y = (1/\lambda, \alpha, 0, \dots)$. Then $\|x\|_{p,\lambda} = \|y\|_{p,\lambda} = 1$, whereas $\|x+y\|_{p,\lambda} = 2$, so X is not strictly convex. This completes the proof. □

2. Theorem. *Let E be uniformly convex. Then $C_{\mathrm{NJ}}(E) < 2$.*

Proof. Let E be uniformly convex. Let ε be any positive number with $0 < \varepsilon < 2^{1/2}$. Then there exists a $\delta > 0$ such that $\|x\| \le 1$, $\|y\| \le 1$, and $\|x-y\| \ge \varepsilon$ imply $\|(x+y)/2\|^2 \le (1-\delta)[(\|x\|^2 + \|y\|^2)/2]$ (see [1], p. 190, or [11], p. 360). Let x and y be any elements in E with $\|x\|^2 + \|y\|^2 = 1$. We first assume that $\|x-y\| \ge \varepsilon$. Then

$$\left\|\frac{x+y}{2}\right\|^2 + \left\|\frac{x-y}{2}\right\|^2 \le (1-\delta)\frac{\|x\|^2+\|y\|^2}{2} + \frac{\|x\|^2+\|y\|^2}{2} \le 1 - \frac{\delta}{2},$$

and hence

$$\|x+y\|^2 + \|x-y\|^2 \le 2(2-\delta). \tag{7}$$

Next, let $\|x-y\| \le \varepsilon$. Then

$$\|x+y\|^2 + \|x-y\|^2 \le 2(\|x\|^2+\|y\|^2) + \varepsilon^2 \le 2(1+\varepsilon^2/2). \tag{8}$$

Consequently, by (7) and (8) we have

$$\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2+\|y\|^2)} \le 1 + \max\{1-\delta, \varepsilon^2/2\},$$

or $C_{\mathrm{NJ}}(E) < 2$. □

The converse of Theorem 2 is not true:

3. Theorem. *For any $\varepsilon > 0$ there exists a Banach space E, isomorphic to a Hilbert space, with $C_{\mathrm{NJ}}(E) < 1 + \varepsilon$ which is not strictly convex.*

In fact, for the spaces $X_{2,\lambda}$ $(\lambda > 1)$, we have $C_{\mathrm{NJ}}(X_{2,\lambda}) \to 1$ as $\lambda \to 1$, whereas $X_{2,\lambda}$ is not strictly convex.

Although the condition $C_{\mathrm{NJ}}(E) < 2$ does not even imply strict convexity for E, it assures the existence of an equivalent norm on E for which E becomes uniformly convex (cf. Theorem C).

4. Theorem. *Let $C_{\mathrm{NJ}}(E) < 2$. Then E is super-reflexive; the converse is not true.*

Proof. Assume $C := C_{\mathrm{NJ}}(E) < 2$. We see that E is $(2,\varepsilon)$-convex with some ε, which implies the super-reflexivity for E (James [6]; see also [1], [12]). Let x and y be any elements in E with $\|x\| = \|y\| = 1$. Then

$$\begin{aligned}\min_{\varepsilon_i=\pm1} \|\varepsilon_1 x + \varepsilon_2 y\| &\le \left\{\frac{1}{2}(\|x+y\|^2 + \|x-y\|^2)\right\}^{1/2} \\ &\le C^{1/2}(\|x\|^2 + \|y\|^2)^{1/2} \\ &= (2C)^{1/2} = 2(1-\varepsilon),\end{aligned}$$

where $\varepsilon = 1 - (C/2)^{1/2}$. For the latter assertion consider the space $X_{2,\sqrt{2}}$. Indeed, $X_{2,\sqrt{2}}$ is isomorphic to a Hilbert space and hence super-reflexive, whereas $C_{\mathrm{NJ}}(X_{2,\sqrt{2}}) = 2$ by Proposition 1. (l_1^n and l_∞^n are also such examples.) □

5. Definition. Let $\widetilde{C}_{\mathrm{NJ}}(E)$ denote the infimum of all von Neumann-Jordan constants for equivalent norms of a Banach space E.

Owing to Theorems 2 and 4, we obtain the following theorem.

6. Theorem. *A Banach space E is super-reflexive if and only if $\widetilde{C}_{\mathrm{NJ}}(E) < 2$.*

So far we have treated Banach spaces whose NJ-constant is less than two or can be so taken by renorming. Let us now consider the spaces with NJ-constant $2^{2/t-1}$, $1 < t \le 2$, which is the same as that of L_p-space if $t = \min\{p, p'\}$. We first note the following facts; the proof is easy and is omitted.

7. Proposition. (i) *Let F be a Banach space which is finitely representable in E. Then $C_{\mathrm{NJ}}(F) \le C_{\mathrm{NJ}}(E)$.*

(ii) *If l_p is finitely representable in E, $C_{\mathrm{NJ}}(E) \ge 2^{2/t-1}$, where $t = \min\{p, p'\}$.*

8. Proposition. *Let $C_{\mathrm{NJ}}(E) \le 2^{2/p-1}$, $1 < p \le 2$. Then E is of type r and cotype r' for any r with $1 \le r < p$. The converse is not true.*

Proof. Assume $C_{\mathrm{NJ}}(E) \le 2^{2/p-1}$. Suppose that E is not of type r for some $r < p$. Then l_r is finitely representable in E by Theorem D. Hence, $C_{\mathrm{NJ}}(E) \ge 2^{2/r-1} > 2^{2/p-1}$ by the preceding proposition, which is a contradiction. By Proposition B, E' is also of type r for any $r < p$, which implies that E is of cotype r' for any such r (cf. [13], Proposition 1.e.17; or [1], p. 309).

The latter part is a consequence of a result by Pisier and Xu [17] which asserts that for any $q > 2$ there exists a non-reflexive Banach space E ($C_{\mathrm{NJ}}(E) = 2$) such that E is of type 2 and cotype q. □

9. *Remark.* Propositions 7 and 8 remain valid if $C_{\mathrm{NJ}}(E)$ is replaced by $\widetilde{C}_{\mathrm{NJ}}(E)$ (note that "finite representability" and "type, cotype" are isomorphic properties).

For a Banach lattice the situation in Proposition 8 or Remark 9 is preferable, and we obtain the following result.

10. Theorem. *Let E be a Banach lattice and let $1 < p \le 2$. Then the following assertions are equivalent*:

(i) $\widetilde{C}_{\mathrm{NJ}}(E) \le 2^{2/p-1}$,

(ii) *E is of type r and cotype r' for any $r < p$,*

(iii) *E and E' are of type r for any $r < p$,*
(iv) *E and E' are of cotype r' for any $r < p$,*
(v) *E is r-convex and r'-concave for any $r < p$.*

For the proof of Theorem 10 we need the following two lemmas.

11. Lemma ([13], Corollary 1.f.9)**.** *Let $1 < r < \infty$. Let E be a Banach lattice of type r, resp. cotype r. Then E is $(r - \varepsilon)$-convex, resp. $(r + \varepsilon)$-concave, for any $\varepsilon > 0$.*

The following lemma is an easy consequence of Theorem 1.d.1 of [13] (see also the proof of Theorem 1.f.1 of [13]).

12. Lemma (cf. [13], Theorem 1.d.1)**.** *Let $1 \le p \le 2$. Let E be a Banach lattice. Then, for any x and y in E,*

$$(|x+y|^{p'} + |x-y|^{p'})^{1/p'} \le 2^{1/p'}(|x|^p + |y|^p)^{1/p}.$$

Proof of Theorem 10. We have already seen that the implications (i)⇒(iii)⇒(ii) are valid in the proof of Proposition 8 and Remark 9. The implication (ii)⇒(iv) is clear, since if E is of type r, then E' is of cotype r' (cf. [13], Proposition 1.e.17; or [1], p. 309).

(iv)⇒(v): Assume that E and E' are of cotype r' for any $r < p$. Then by Lemma 11, E and E' are r'-concave for any $r < p$, and the latter implies r-convexity of E (see [13], Proposition 1.d.4).

(v)⇒(i): Let E be r-convex and r'-concave for any $r < p$. Here we can take both r-convex and r'-concave constants to be one by a suitable renorming ([13], Proposition 1.d.8). We denote such an equivalent norm by $\|\cdot\|_1$. Then by Lemma 12 we have

$$\begin{aligned}(\|x+y\|_1^2 + \|x-y\|_1^2)^{1/2} &\le 2^{1/2-1/r'}(\|x+y\|_1^{r'} + \|x-y\|_1^{r'})^{1/r'}\\ &\le 2^{1/2-1/r'}\|(|x+y|^{r'} + |x-y|^{r'})^{1/r'}\|_1\\ &\le 2^{1/2}\|(|x|^r + |y|^r)^{1/r}\|_1\\ &\le 2^{1/2}(\|x\|_1^r + \|y\|_1^r)^{1/r}\\ &\le 2^{1/r}(\|x\|_1^2 + \|y\|_1^2)^{1/2},\end{aligned}$$

which implies $C_{\mathrm{NJ}}((E, \|\cdot\|_1)) \le 2^{2/r-1}$. Consequently $\widetilde{C}_{\mathrm{NJ}}(E) \le 2^{2/p-1}$. This completes the proof. □

13. *Remark.* In Theorem 10 the assertion (i) may be replaced by
(i′) $\widetilde{C}_{\mathrm{NJ}}(E) = 2^{2/p-1}$
if l_p is finitely representable in E or E' (cf. Remark 9).

By Theorem 10 we obtain the following result.

14. Corollary. *Let E be a Banach lattice, and let $\widetilde{C}_{\mathrm{NJ}}(E) = 2^{2/p_0-1}$, $1 \le p_0 \le 2$. Then*

$$\begin{aligned} p_0 &= \sup\{p: \ E \text{ is of type } p \text{ and cotype } p'\} \\ &= \sup\{p: \ E \text{ and } E' \text{ are of type } p\} \\ &= \sup\{p: \ E \text{ and } E' \text{ are of cotype } p'\} \\ &= \sup\{p: \ E \text{ is } p\text{-convex and } p'\text{-concave}\} \\ &= \sup\{p: \ E \text{ and } E' \text{ are } p\text{-convex}\} \\ &= \sup\{p: \ E \text{ and } E' \text{ are } p'\text{-concave}\}. \end{aligned}$$

The proof for the case $p_0 = 1$ goes in the same way as in the proof of Theorem 10. (Note Lemma 1.f.3 of [13] for the last three identities.)

15. *Remark.* Let E be a Banach lattice, and let $p(E)$, $q(E)$ be as in Theorem D. Then the number p_0 in Corollary 14 is also represented as

$$p_0 = \min\{p(E),\ q(E)'\} = 2\{1 + \log_2 \widetilde{C}_{\mathrm{NJ}}(E)\}^{-1},$$

where $1/q(E) + 1/q(E)' = 1$.

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