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ON FUNCTIONS ARISING AS POTENTIALS ON SPACES OF HOMOGENEOUS TYPE

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ABSTRACT. On a space of homogeneous type we consider functions F in L^p, $1 < p < \infty$, which are potentials of order α of L^p functions. We show that these functions belong to the class of smooth functions $C^{p,\alpha}$ of Calderón-Scott. This result has applications to tangential convergence.

1. Introduction

This note is motivated by the results in [CDS] on boundary tangential convergence on spaces of homogeneous type. In that paper the authors showed that $C^{p,\alpha}$ smoothness of functions defined on a space of homogeneous type suffices for obtaining boundary tangential convergence of "convolutions" with approximate identities, $1 < p \leq \frac{1}{\alpha}$, thus extending the results in [NRS]. It is shown in [CDS] that in the case of stratified nilpotent Lie groups the spaces of potentials, as defined in [F], are continuously embedded in $C^{p,\alpha}$ spaces.

Our purpose is to show that in a general space of homogeneous type L^p functions that are potentials of order α of functions in L^p are included in the classes $C^{p,\alpha}$, and that this can be proved by adapting a classical argument for sharp functions [S, p. 158].

2. Definitions and statement of the results

In this paper (X, δ, μ) is a space of homogeneous type as defined in [CW] or [MS] such that $\mu(X) = \infty$ and $\mu(\{x\}) = 0$ for all x in X. As shown in [MS] we can assume without loss of generality that (X, δ, μ) is a normal space of order γ, $0 < \gamma \leq 1$. This means that the quasidistance δ and the measure μ have the following two properties:

1. For all $r > 0$ and all x in X the balls $B_r(x) = \{y \in X \mid \delta(x, y) < r\}$ satisfy

$$c_1 r \leq \mu(B_r(x)) \leq c_2 r \tag{2.1}$$

with positive constants c_1 and c_2 independent of x and r.

2. There exists a number γ, $0 < \gamma \leq 1$, called the order of (X, δ, μ) such that for all x, x' and y in X we have

$$|\delta(x, y) - \delta(x', y)| \leq M\delta^{\gamma}(x, x')\{\delta(x, y) + \delta(x', y)\}^{1-\gamma} \tag{2.2}$$

with a positive constant M independent of x, x' and y.

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We shall need the following known Lemma [GV, Lemma II.3].

Lemma. *Let $0 < \alpha < 1$. Then for all x, x' and y in X such that $\delta(x,y) > 2\kappa\delta(x,x')$*

$$|\delta^{\alpha-1}(x,y) - \delta^{\alpha-1}(x',y)| \leq K\delta^{\gamma}(x,x')\delta^{\alpha-\gamma-1}(x,y) \tag{2.3}$$

where κ is the constant of the "triangle" inequality of δ, and K is a constant independent of x, x' and y.

The (Riesz) potential of order α, $0 < \alpha < \gamma$, of a function f in L^p with $1 < p < \frac{1}{\alpha}$ is defined as in [GV] by

$$I_\alpha f(x) = \int_X \frac{f(y)}{\delta^{1-\alpha}(x,y)} d\mu(y), \tag{2.4}$$

and for $\frac{1}{\alpha} \leq p < \infty$ by

$$\widetilde{I}_\alpha f(x) = \int_X f(y) \left[\frac{1}{\delta^{1-\alpha}(x,y)} - \frac{\psi_u(y)}{\delta^{1-\alpha}(u,y)}\right] d\mu(y) \tag{2.5}$$

where u is a fixed point in X, $\psi_u(y)$ is the characteristic function of the complement of the ball $B_1(u)$ when $p = \frac{1}{\alpha}$, and, for $\frac{1}{\alpha} < p$, $\psi_u(y)$ is identically 1 on X. It can be seen that $I_\alpha f(x)$ and $\widetilde{I}_\alpha f(x)$ converge absolutely for almost every x in X [GV].

The spaces $C^{p,\alpha}$ of smooth functions of Calderón-Scott [CS] were introduced on spaces of homogeneous type in [CDS]. For $f \in L^p$, $1 < p < \infty$, we consider the sharp functions

$$f^{\#}_\alpha(x) = \sup_{x \in B} \frac{1}{\mu(B)^{1+\alpha}} \int_B |f(y) - m_B(f)|\, d\mu(y)$$

where $m_B(f) = \frac{1}{\mu(B)} \int_B f(y)\, d\mu(y)$. The space $C^{p,\alpha}$ is the set of all functions $f \in L^p$ with $f^{\#}_\alpha \in L^p$ equipped with the norm $\|f\|_{C^{p,\alpha}} = \|f^{\#}_\alpha\|_p + \|f\|_p$, where $\|\ \|_p$ denotes the L^p-norm. The letter c will denote a constant, not necessarily the same in different occurrences.

We now state the results.

Theorem 2.1. *Let $0 < \alpha < \gamma$, $1 < p < \frac{1}{\alpha}$ and $F \in L^p$ with $F = I_\alpha(f)$ and $f \in L^p$. Then $F \in C^{p,\alpha}$ and $\|F\|_{C^{p,\alpha}} \leq c(\|F\|_p + \|f\|_p)$, with a constant c independent of F and f.*

Theorem 2.2. *Let $0 < \alpha < \gamma$, $\frac{1}{\alpha} \leq p < \infty$, and $F \in L^p$ with $F = \widetilde{I}_\alpha(f) + C_F$ where $f \in L^p$ and C_F is a constant. Then $F \in C^{p,\alpha}$ and $\|F\|_{C^{p,\alpha}} \leq c(\|F\|_p + \|f\|_p)$, with c independent of F and f.*

3. Proof

Proof of Theorem 2.1. Fix $x \in X$; we will estimate $F^{\#}_\alpha(x)$. Let $B = B_r(x_0)$ be a ball containing x and $\widetilde{B} = B_{2\kappa r}(x_0)$, where κ is the constant of the "triangle" inequality of δ, and, finally, let χ be the characteristic function of $\widetilde{B}$. Set $F = F_1 + F_2$ with $F_1 = I_\alpha(f\chi)$ and $F_2 = I_\alpha(f(1-\chi))$. To estimate $(F_1)^{\#}_\alpha(x)$, let $1 < s < p$ and $\frac{1}{t} = \frac{1}{s} - \alpha$. Hölder's inequality with exponent t followed by the Hardy-Sobolev inequality (see [GV], [S]) give

$$\frac{1}{\mu(B)^{1+\alpha}}\int_B |F_1|\,d\mu \le \frac{1}{\mu(B)^{\alpha+\frac{1}{t}}}\left(\int_B |I_\alpha(f\chi)|^t\,d\mu\right)^{\frac{1}{t}}$$
$$\le \left(\frac{1}{\mu(B)}\int_{\widetilde{B}} |f|^s\,d\mu\right)^{\frac{1}{s}} \le c\left(\frac{1}{\mu(\widetilde{B})}\int_{\widetilde{B}} |f|^s\,d\mu\right)^{\frac{1}{s}}$$
$$\le c\,M_s(f)(x),$$

where $M_s(f)=[M(|f|^s)]^{\frac{1}{s}}$, and M is the Hardy-Littlewood maximal function. By a known property of the sharp function we have

$$(F_1)^{\#}_{\alpha}(x) \le 2cM_s(f)(x). \tag{3.1}$$

Now we will estimate $(F_2)^{\#}_{\alpha}(x)$. Observe that

$$F_2(z)-F_2(x_0)=\int_{(\widetilde{B})^c} f(y)\left[\frac{1}{\delta^{1-\alpha}(z,y)}-\frac{1}{\delta^{1-\alpha}(x_0,y)}\right]d\mu(y);$$

then by (2.3)

(3.2)

$$\int_B |F_2(x)-F_2(x_0)|\,d\mu(z) \le K\int_B \delta^{\gamma}(z,x_0)\left(\int_{(\widetilde{B})^c}\frac{|f(y)|}{\delta^{1-\alpha+\gamma}(x_0,y)}\,d\mu(y)\right)d\mu(z).$$

To estimate the inner integral we write

$$\int_{(\widetilde{B})^c}\frac{|f(y)|}{\delta^{1-\alpha+\gamma}(x_0,y)}\,d\mu(y) \le \sum_{k=1}^{\infty}\int_{2^{k-1}r\le\delta(x_0,y)<2^k r}\frac{|f(y)|}{(2^{k-1}r)^{1-\alpha+\gamma}}\,d\mu(y)$$
$$\le 2^{1-\alpha+\gamma}r^{\alpha-\gamma}\sum_{k=1}^{\infty}\frac{1}{2^{(\gamma-\alpha)k}}\frac{1}{2^k r}\int_{B_{2^k r}(x_0)}|f(y)|\,d\mu(y) \le cr^{\alpha-\gamma}Mf(x).$$

Using this estimate in (3.2) and a known property of the sharp function we get

$$(F_2)^{\#}_{\alpha}(x) \le 2cMf(x). \tag{3.3}$$

Since $Mf(x)\le M_s f(x)$, (3.1) and (3.3), and the fact that the sharp operator is subadditive we have

$$F^{\#}_{\alpha}(x) \le cM_s(f)(x).$$

Finally, using the strong type $\frac{p}{s}$ of M we have

$$\|F^{\#}_{\alpha}\|_p \le c\|f\|_p.$$

This concludes the proof of Theorem 2.1.

Proof of Theorem 2.2. The proof of Theorem 2.2 is similar to that of Theorem 2.1. Set

$$F-C_F=\widetilde{I}_\alpha(f\chi)+\widetilde{I}_\alpha(f(1-\chi))=F_1+F_2.$$

To estimate $(F_1)^{\#}_{\alpha}(x)$, let $c_B=-\int_X (f\chi)\frac{\psi_u(y)}{\delta^{1-\alpha}(u,y)}\,d\mu(y)$ and $1<s<\frac{1}{\alpha}$, $\frac{1}{t}=\frac{1}{s}-\alpha$. Then by the same argument used in the proof of Theorem 2.1 we get

$$\frac{1}{\mu(B)^{1+\alpha}}\int_B |F_1-c_B|\,d\mu \le cM_s(f)(x).$$

To estimate $(F_2)^{\#}_{\alpha}(x)$, note that

$$F_2(z) - F_2(x_0) = \int_{(\widetilde{B})^c} f(y) \left[\frac{1}{\delta^{1-\alpha}(z,y)} - \frac{1}{\delta^{1-\alpha}(x_0,y)} \right] d\mu(y),$$

and therefore as in Theorem 2.1 we have $(F_2)^{\#}_{\alpha}(x) \leq 2cMf(x)$. The rest of the proof is the same as that of Theorem 2.1.

References

[CDS] P. Cifuentes, J. R. Dorronsoro, and J. Sueiro, *Boundary tangential convergence on spaces of homogeneous type*, Trans. Amer. Math. Soc. **332** (1992), 331–350. MR **92j:**42019

[CS] A. P. Calderón and R. Scott, *Sobolev type inequalities for* $p > 0$, Studia Math. **62** (1978), 75–92. MR **58:**7057

[CW] R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977), 569–645. MR **56:**6264

[F] G. B. Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Arkiv f. Math. **13** (1975), 161–207. MR **58:**13215

[GV] A. E. Gatto and S. Vági, *Fractional integrals on spaces of homogeneous type*, in Analysis and Partial Differential Equations, Cora Sadosky, editor, Marcel Dekker, New York and Basel, 1990. MR **91e:**42032

[MS] R. Macías and C. Segovia, *Lipschitz functions on spaces of homogeneous type*, Adv. Math. **33** (1979), 257–270. MR **81c:**32017a

[NRS] A. Nagel, W. Rudin, and J. Shapiro, *Tangential boundary behavior of functions in Dirichlet type spaces*, Ann. of Math. **116** (1982), 331–360. MR **84a:**31002

[S] E. M. Stein, *Harmonic analysis, real-variable methods, orthogonality, and oscillatory integrals*, Princeton Univ. Press, Princeton, NJ, 1993. MR **95c:**42002

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