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HOPFIAN AND CO-HOPFIAN G-CW-COMPLEXES

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ABSTRACT. We determine conditions for a G-CW-complex to be a Hopfian or a co-Hopfian object in the G-homotopy category of G-path-connected G-CW-complexes with base points.

1. INTRODUCTION

The notion of a Hopfian and a co-Hopfian object of a category is fairly well known. An object X of a category $\mathcal{C}$ is called Hopfian (respectively, co-Hopfian) if every self-epimorphism (respectively, self-monomorphism) $f : X \longrightarrow X$ is an equivalence; this notion plainly makes sense in any category, since epimorphisms and monomorphisms are categorically defined. It is interesting to recognize Hopfian and co-Hopfian objects in a specific category. Several results are known in this direction, [1], [5], [6], [7], [8], [9], [10], [11]. In [5] and [9], the authors studied Hopfian and co-Hopfian objects of $\mathcal{H}$, the homotopy category of pointed path-connected CW-complexes.

Let G be a discrete group and $G\mathcal{H}$ denote the G-homotopy category of G-path-connected G-CW-complexes with base points (base points are G-fixed). In this paper we determine conditions for an object of $G\mathcal{H}$ to be a Hopfian or co-Hopfian object of $G\mathcal{H}$. Our results extend the results of [9] to the category $G\mathcal{H}$. Since G is discrete, every object of $G\mathcal{H}$ is also an object of $\mathcal{H}$. We provide examples to show that the Hopficity and the co-Hopficity of an object X of $G\mathcal{H}$ are independent of the Hopficity and the co-Hopficity of X when considered as an object of $\mathcal{H}$.

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2. HOPFIAN OBJECTS

Let O_G denote the category of canonical orbits. More precisely, objects of O_G are homogeneous spaces G/H, H a subgroup of G, and a morphism $\hat{g} : G/H \longrightarrow G/K$ of O_G is given by a subconjugacy relation $g^{-1}H\ g \subset K$ (cf. [2]).

An abelian O_G-group is a contravariant functor from the category O_G to the category $\mathcal{A}b$ of abelian groups. Such objects, along with obvious morphisms (natural transformations) between them form an abelian category $\mathcal{C}_G$. We shall denote the zero object in the abelian category $\mathcal{C}_G$ by $\underline{0} : O_G \longrightarrow \mathcal{A}b$, $G/H \mapsto 0$, the trivial group.

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If X is an object of $G\mathcal{H}$, then for every $i \geq 0$, we have an abelian O_G-group $\underline{H}_i X : O_G \longrightarrow \mathcal{A}b$, defined by $\underline{H}_i X(G/H) = H_i(X^H)$, the i-th integral homology group of X^H, X^H being the H-fixed point set of X, for every object G/H of O_G, and $\underline{H}_i X(\hat{g}) = H_i(g) : H_i(X^K) \longrightarrow H_i(X^H)$ for every morphism $\hat{g} : G/H \longrightarrow G/K$ of O_G, where $g : X^K \longrightarrow X^H$ is induced by the action of G on X. Similarly, we have O_G-groups $\underline{\pi}_i X$, $\underline{\pi}_1 X$ need not be abelian. A morphism $f : X \longrightarrow Y$ of $G\mathcal{H}$ induces a natural transformation $f_* : \underline{H}_n X \longrightarrow \underline{H}_n Y$, where $f_*(G/H) = H_n(f^H) : H_n(X^H) \longrightarrow H_n(Y^H)$, $n \geq 0$.

We have the following easy lemma.

Lemma 2.1. *A morphism $\eta : T \longrightarrow S$ in $\mathcal{C}_G$ is an epimorphism (respectively, monomorphism) in $\mathcal{C}_G$ if and only if $\eta(G/H) : T(G/H) \longrightarrow S(G/H)$ is an epimorphism (respectively, monomorphism) in $\mathcal{A}b$ for every object G/H of O_G.* □

Remark 2.2. If $\mathcal{C}'_G$ is the category of O_G-groups, then a morphism $\eta : T \longrightarrow S$ in $\mathcal{C}'_G$ is a monomorphism if and only if $\eta(G/H)$ is a monomorphism in the category $\mathcal{G}$ of groups. If a morphism $\eta : T \longrightarrow S$ satisfies that $\eta(G/H)$ is onto for every object G/H of O_G, then η is an epimorphism in $\mathcal{C}'_G$.

It follows immediately from the above discussion that :

Proposition 2.3. *If an object T in $\mathcal{C}_G$ satisfies the condition that $T(G/H)$ is Hopfian (respectively, co-Hopfian) in $\mathcal{A}b$ for every object G/H of O_G, then T is a Hopfian (respectively, co-Hopfian) object in $\mathcal{C}_G$.* □

Since the Hopfian and co-Hopfian objects in $\mathcal{A}b$ are by now well studied (cf. [1]), the above result gives an idea about the Hopfian and co-Hopfian objects in $\mathcal{C}_G$.

Definition 2.4. A morphism $f : X \longrightarrow Y$ in $G\mathcal{H}$ is a *weak G-homology equivalence* if $f_* : \underline{H}_n X \longrightarrow \underline{H}_n Y$ is an isomorphism for every $n \geq 0$.

Note that if a morphism $f : X \longrightarrow Y$ of $G\mathcal{H}$ is such that $f_*(G/H) : \pi_n(X^H) \longrightarrow \pi_n(Y^H)$ is an isomorphism for every $n \geq 0$, then f is a G-homotopy equivalence [3].

Proposition 2.5. *Let $f : X \longrightarrow Y$ be an epimorphism in $G\mathcal{H}$. Then $f_* : \underline{H}_k X \longrightarrow \underline{H}_k Y$ is an epimorphism in $\mathcal{C}_G$ for all $k \geq 0$.*

Proof. We may without loss of generality assume (by replacing Y by the equivariant mapping cylinder of f) that f is an inclusion. Then consider the maps $\pi : Y \longrightarrow Y/X$ and $c : Y \longrightarrow Y/X$, where π is the quotient and c is the constant G-map. Then $\pi \circ f = c \circ f$. Since f is an epimophism, it follows that π is G-homotopic to c. Now for every $H \subset G$, it follows from the exact homology sequence

$$\cdots \longrightarrow H_k(X^H) \longrightarrow H_k(Y^H) \longrightarrow H_k((Y/X)^H) = H_k(Y^H/X^H) \longrightarrow \cdots$$

that $f_*^H : H_k(X^H) \longrightarrow H_k(Y^H)$ is an epimorphisn in $\mathcal{A}b$ for every $k \geq 1$. The result follows from Lemma 2.1. □

Remark 2.6. Note that for any morphism $f : X \longrightarrow Y$ in $G\mathcal{H}$, $f_* : \underline{H}_0 X \longrightarrow \underline{H}_0 Y$ is an isomorphism. This follows from the fact that $H_0(X^H)$ is generated by the homology class of the base point $x^0 \in X^G$ in $H_0(X^H)$, and f being a morphism in $G\mathcal{H}$, $(f^H)_*$ maps the generator of $H_0(X^H)$ onto the generator of $H_0(Y^H)$.

Theorem 2.7. *Let $f : X \longrightarrow X$ be a self-epimorphism in $G\mathcal{H}$. If $\underline{H}_n X$, $n \geq 1$ are Hopfian objects in $\mathcal{C}_G$, then f is a weak G-homology equivalence.*

Proof. By the Remark 2.6 $f_* : \underline{H}_0X \longrightarrow \underline{H}_0X$ is an isomorphism. Since f is an epimorphism, Proposition 2.5 implies that $f_* : \underline{H}_nX \longrightarrow \underline{H}_nX$ is an epimorphism for every $n \geq 1$. The result now follows as $\underline{H}_nX$, $n \geq 1$ are Hopfian objects in $\mathcal{C}_G$. □

Recall from [4] the following definition.

Definition 2.8. A G-space X is *nilpotent* if each $\underline{\pi}_nX$, $n \geq 1$ is nilpotent as an O_G-module over $\underline{\pi}_1X$, that is, there are O_G-submodules

$$\{\underline{0}\} = \underline{\pi}_{n,0}X \subset \underline{\pi}_{n,1}X \subset \cdots \subset \underline{\pi}_{n,r_n}X = \underline{\pi}_nX$$

such that the subquotients $A_{n,j} = \underline{\pi}_{n,j+1}X/\underline{\pi}_{n,j}X$ are abelian with trivial $\underline{\pi}_1X$-action.

This is equivalent to saying each X^H is nilpotent in the usual sense with a uniform bound on the order of nilpotence in each dimension (of course, this last condition is vacuous if G is finite).

Corollary 2.9. *Let an object X of $G\mathcal{H}$ be nilpotent as a G-space, and $\underline{H}_nX$, $n \geq 1$ are Hopfian in $\mathcal{C}_G$, then X is Hopfian in $G\mathcal{H}$.*

Proof. Let $f : X \longrightarrow X$ be a self-epimorphism in $G\mathcal{H}$. Then by Theorem 2.7 f is a weak G-homology equivalence. But X^H being nilpotent for every $H \subset G$, it follows that $f^H : X^H \longrightarrow X^H$ is a homotopy equivalence and hence $f : X \longrightarrow X$ is a G-homotopy equivalence. This completes the proof. □

It may be noted that Corollary 1.1 of [9] follows from Corollary 2.9 by taking G to be the trivial group.

Let $\lambda : O_G \longrightarrow \mathcal{G}$ be an O_G-group, and $K(\lambda, 1)$ denote the equivariant Eilenberg-Mac Lane complex of the type $(\lambda, 1)$ [4]. It may be remarked that for any O_G-group $\lambda : O_G \longrightarrow \mathcal{A}b$, $K(\lambda, n)$ is the classifying space for the Bredon cohomology with coefficient λ [2].

Proposition 2.10. *For any object X of $G\mathcal{H}$ and O_G-group $\lambda : O_G \longrightarrow \mathcal{G}$ there is an adjunction equivalence $[X, K(\lambda, 1)]_G \leftrightarrow Hom(\underline{\pi}_1X, \lambda)$.*

Proof. If $f : X \longrightarrow K(\lambda, 1)$ represents an element of $[X, K(\lambda, 1)]_G$, then the corresponding natural transformation in $Hom(\underline{\pi}_1X, \lambda)$ is given by $f_* : \underline{\pi}_1X \longrightarrow \lambda$ (note that $\underline{\pi}_1K(\lambda, 1) = \lambda$). Conversely, a natural transformation $T : \underline{\pi}_1X \longrightarrow \lambda$ induces a G-map $T_* : K(\underline{\pi}_1X, 1) \longrightarrow K(\lambda, 1)$ (cf. [4]). Note that X can be regarded as a G-subcomplex of $K(\underline{\pi}_1X, 1)$, for we may obtain $K(\underline{\pi}_1X, 1)$ from X by attaching suitable equivariant cells to X to kill the higher homotopy groups of the fixed point sets of X. The class represented by T_*/X in $[X, K(\lambda, 1)]_G$ is then the element which corresponds to T. □

It follows immediately from Proposition 2.10 that :

Proposition 2.11. *If $f : X \longrightarrow Y$ is an epimorphism in $G\mathcal{H}$, then $f_* : \underline{\pi}_1X \longrightarrow \underline{\pi}_1Y$ is an epimorphism in $\mathcal{C}'_G$.* □

Corollary 2.12. *If $\lambda : O_G \longrightarrow \mathcal{G}$ is Hopfian in $\mathcal{C}'_G$, then $K(\lambda, 1)$ is a Hopfian object in $G\mathcal{H}$.* □

Corollary 2.13. *If X is G-$(n-1)$-connected, $n > 1$, (that is, each X^H is $(n-1)$-connected) and $f : X \longrightarrow X$ is an epimorphism in $G\mathcal{H}$, then $f_* : \underline{\pi}_nX \longrightarrow \underline{\pi}_nX$ is an epimorphism.*

Proof. Note that since X is G-$(n-1)$-connected, the natural transformation $\underline{\pi}_n X \longrightarrow \underline{H}_n X$ given by the Hurewicz homomorphism is actually an isomorphism. The result now follows from Proposition 2.5. □

Thus in view of Remark 2.2, it follows that if $\lambda : O_G \longrightarrow \mathcal{G}$ is such that $\lambda(G/H)$ is a Hopfian group for every $H \subset G$, then $K(\lambda, 1)$ is a Hopfian object in $G\mathcal{H}$. In fact, if $\lambda : O_G \longrightarrow \mathcal{A}b$ is a Hopfian object in $\mathcal{C}_G$, then $K(\lambda, n)$ is Hopfian for every integer $n > 1$. To see this, we first need to prove the following result.

Proposition 2.14. *If X is G-$(n-1)$-connected, $n > 1$, then there is an adjunction equivalence $[X, K(\lambda, n)]_G \leftrightarrow Hom(\underline{\pi}_n X, \lambda)$, for any $\lambda : O_G \longrightarrow \mathcal{A}b$.*

Proof. Recall from [2] that there exists a spectral sequence whose E_2 term is $E_2^{p,q} = Ext^p(\underline{H}_q X, \lambda) \Rightarrow H_G^{p+q}(X; \lambda)$, here $H_G^{p+q}(X; \lambda)$ is the Bredon cohomology group of X with coefficient λ. There is an edge homomorphism $H_G^n(X; \lambda) \longrightarrow Hom(\underline{H}_n X; \lambda)$ of the above spectral sequence, which is an isomorphism if $\underline{H}_q X$ is projective for $q < n$. Now, since X is G-$(n-1)$-connected, $\underline{H}_q X = \underline{0}$ for $0 < q < n$ and $\underline{H}_n X \cong \underline{\pi}_n X$, where $\underline{0} : O_G \longrightarrow \mathcal{A}b$ is the zero object in the category $\mathcal{C}_G$. Moreover, since X is G-path-connected, $\underline{H}_0 X(G/H) = \mathbb{Z}\langle x^0 \rangle$, where x^0 is the base point and $\langle x^0 \rangle$ is the homology class of x^0 and $\underline{H}_0 X(\hat{g}) = id$. The result now follows from the fact that if B is projective in $\mathcal{A}b$, then $\underline{B}$ is projective in $\mathcal{C}_G$, where $\underline{B}$ is defined by $\underline{B}(G/H) = B$ for every $H \subset G$ and $\underline{B}(\hat{g}) = id$, for every morphism $\hat{g} : G/H \longrightarrow G/K$ of O_G. □

Corollary 2.15. *If $\lambda : O_G \longrightarrow \mathcal{A}b$ is Hopfian in $\mathcal{C}_G$, then $K(\lambda, n)$ is Hopfian in $G\mathcal{H}$.* □

Example 2.16. Let X be a G-connected finite G-CW-complex (that is, X has a finite number of equivariant cells) which has one G-fixed 0-cell and no 1-cell. Since $H_i(X^H)$ is finitely generated abelian for every $H \subset G$, by Lemma 2.1 $\underline{H}_i X$ is Hopfian. Moreover it is clear that X is G-simply-connected and hence nilpotent. Thus by Corollary 2.9, X is Hopfian in $G\mathcal{H}$.

Example 2.17. Consider the real $4k$-dimensional Euclidean space $\mathbb{R}^{4k}$ as the quaternion k-space $\mathbb{H}^k$. Let τ be a quaternion of norm one and order p, an odd prime. We can take $\tau = e^{2\pi i/p}$. Define an action of $\mathbb{Z}_p$ on $\mathbb{R}^{4k} \cong \mathbb{H}^k$ by

$$\tau\ (a_1, a_2, \cdots, a_k) = (\tau a_1 \tau^{-1}, \tau a_2 \tau^{-1}, \cdots, \tau a_k \tau^{-1}),$$

for any k-tuple of quaternions $(a_1, a_2, \cdots, a_k)$. Since this action is norm preserving, there results a $\mathbb{Z}_p$-action on the $(4k-1)$-sphere S^{4k-1}. The fixed point sets are S^{4k-1} and $(S^{4k-1})^{\mathbb{Z}_p}$. To determine $(S^{4k-1})^{\mathbb{Z}_p}$ we proceed as follows. Let $(a_1, a_2, \cdots, a_k) \in (S^{4k-1})^{\mathbb{Z}_p}$, where $a_r = a_r^1 + a_r^2 i + a_r^3 j + a_r^4 k = A_r^1 + A_r^2 j$, and $A_r^1 = a_r^1 + a_r^2 i, A_r^2 = a_r^3 + a_r^4 i$, $r = 1, 2, \cdots, k$. We must have $\tau a_r = a_r \tau$. Now $\tau a_r = \tau A_r^1 + \tau A_r^2 j$, whereas $a_r \tau = (A_r^1 + A_r^2 j)\tau = A_r^1 \tau + A_r^2 \overline{\tau} j$. Thus we must have $\tau A_r^2 = \overline{\tau} A_r^2$ or $(\tau - \overline{\tau}) A_r^2 = 0$. Therefore, $A_r^2 = 0$, as $\tau \neq \overline{\tau}$. Therefore $(a_1, a_2, \cdots, a_k) \in \mathbb{C}^k$, with $||(a_1, a_2, \cdots, a_k)|| = 1$. Thus $(S^{4k-1})^{\mathbb{Z}_p} = S^{2k-1}$. Now S^{4k-1} is a smooth compact $\mathbb{Z}_p$-manifold, it admits a structure of a finite $\mathbb{Z}_p$-CW-complex which is $\mathbb{Z}_p$-path-connected and has a base point. Moreover, note that the fixed point sets S^{4k-1} and S^{2k-1} being simply-connected, are nilpotent. It is now easy to check that all the conditions of Corollary 2.9 are satisfied and hence it is a Hopfian object in $G\mathcal{H}$ where $G = \mathbb{Z}_p$.

3. Co-Hopfian objects

In this section we obtain conditions for an object X of $G\mathcal{H}$ to be a co-Hopfian object. The following proposition is a straightforward consequence of Proposition 2.10.

Proposition 3.1. *$f : S \longrightarrow T$ is a monomorphism in $\mathcal{C}'_G$ if and only if the induced map $f_* : K(S,1) \longrightarrow K(T,1)$ is a monomorphism in $G\mathcal{H}$.* □

Corollary 3.2. *For an object $\lambda : O_G \longrightarrow \mathcal{G}$ in $\mathcal{C}'_G$, $K(\lambda, 1)$ is co-Hopfian if and only if λ is co-Hopfian.* □

As before, we may obtain from Proposition 2.14 that :

Corollary 3.3. *For $\lambda : O_G \longrightarrow \mathcal{A}b$ in $\mathcal{C}_G$ and $n > 1$, $K(\lambda, n)$ is co-Hopfian in $G\mathcal{H}$ if and only if λ is co-Hopfian in $\mathcal{C}_G$.* □

Definition 3.4. For an object X of $G\mathcal{H}$, we say $\underline{\pi}_i X$ is *finitely generated* if $\underline{\pi}_i X(G/H) = \pi_i(X^H)$ is finitely generated for every $H \subset G$. X will be called G-homotopically finite type if $\underline{\pi}_i X$ is finitely generated for all $i \geq 2$.

Theorem 3.5. *Suppose an object X of $G\mathcal{H}$ is G-homotopically finite type and such that $\pi_1(X^H)$ is a co-Hopfian group and the inclusion $X^H \subset X$ is a monomorphism in $\mathcal{H}$ for every $H \subset G$. Then X is a co-Hopfian object in $G\mathcal{H}$.*

Proof. Let $f : X \longrightarrow X$ be a self-monomorphism in $G\mathcal{H}$. We show that under the given hypothesis $f^H : X^H \longrightarrow X^H$ is a monomorphism in $\mathcal{H}$ for every $H \subset G$. Since $\pi_i(X^H)$ is finitely generated for all $i \geq 2$ and $\pi_1(X^H)$ is co-Hopfian, it will follow from Theorem 7 and Corollary 2 of [5] that $f^H : X^H \longrightarrow X^H$ is a homotopy equivalence. Hence f is a G-homotopy equivalence.

First we show that $f = f^{\{e\}} : X = X^{\{e\}} \longrightarrow X^{\{e\}} = X$ is a monomorphism in $\mathcal{H}$. We assume that the base point $x^0 \in X^G$ is a G-fixed 0-cell in X. Let $\alpha,\ \beta : Y \longrightarrow X$ be morphisms in $\mathcal{H}$ such that $f \circ \alpha \simeq f \circ \beta$. Let $F : Y \times I \longrightarrow X$ be the homotopy $f \circ \alpha \simeq f \circ \beta$. Consider $Y \times G$ as a G-space, where the action of G is given by $g(y, h) = (y, gh)$, for all $g \in G,\ h \in G,\ y \in Y$. Clearly, the above action is free. Let y^0 be the base point of Y, which is a 0-cell of Y. Define $\overline{\alpha} : Y \times G \longrightarrow X$ by $\overline{\alpha}(y, e) = \alpha(y)$ and $\overline{\alpha}(y, g) = g\alpha(y)$. Then $\overline{\alpha}$ is a G-map. Note that $\overline{\alpha}(y^0, g) = x^0$ for all $g \in G$. Let Y_G be the space obtained from $Y \times G$ by identifying all points $(y^0, g),\ g \in G$. Then Y_G is a G-complex having a natural base point, which is a G-fixed 0-cell and is clearly an object of $G\mathcal{H}$. The map $\overline{\alpha}$ induces a G-map $\tilde{\alpha} : Y_G \longrightarrow X$ which is base point preserving. Similarly, we have $\tilde{\beta} : Y_G \longrightarrow X$. The homotopy $F : Y \times I \longrightarrow X$ gives rise to a G-homotopy $\overline{F} : Y \times G \times I \longrightarrow X$, between $f \circ \overline{\alpha}$ and $f \circ \overline{\beta}$, by setting $\overline{F}(y, e, t) = F(y, t)$ and $\overline{F}(y, g, t) = gF(y, t)$ for all $g \in G,\ t \in I$. Since the homotopy F is base point preserving, $\overline{F}$ induces a G-homotopy $\tilde{F} : Y_G \times I \longrightarrow X$, such that $\tilde{F}$ is a G-homotopy between $f \circ \tilde{\alpha}$ and $f \circ \tilde{\beta}$. Since f is a monomorphism in $G\mathcal{H}$, $\tilde{\alpha}$ is G-homotopic to $\tilde{\beta}$. Let $\tilde{F}_1 : Y_G \times I \longrightarrow X$ be a G-homotopy between them. Let $i : Y \longrightarrow Y_G$ be the imbedding $y \mapsto [y, e]$. Let $F_1 : Y \times I \longrightarrow X$ be the composition of $i \times id : Y \times I \longrightarrow Y_G \times I$ and $\tilde{F}_1$. Then it is easy to see that $F_1 : \alpha \simeq \beta$. Thus $f^{\{e\}} : X^{\{e\}} \longrightarrow X^{\{e\}}$ is a monomorphism in $\mathcal{H}$.

Next, let $H \subset G$. Let $\alpha,\ \beta : Y \longrightarrow X^H$ be any two morphisms in $\mathcal{H}$ such that $f^H \circ \alpha \simeq f^H \circ \beta$. Let i denote the inclusion $X^H \subset X$. Then, $i \circ f^H \circ \alpha \simeq i \circ f^H \circ \beta$. This implies $f \circ i \circ \alpha \simeq f \circ i \circ \beta$, since f being a G-map $i \circ f^H = f \circ i$. Since $f^{\{e\}}$

is a monomorphism in $\mathcal{H}$, we conclude $i \circ \alpha \simeq i \circ \beta$. Now since $i : X^H \subset X$ is a monomorphism, it follows that $\alpha \simeq \beta$. Therefore, f^H is a monomorphism in $\mathcal{H}$. This completes the proof of the theorem. □

As an immediate corollary we get

Corollary 3.6. *Suppose X is an object of $G\mathcal{H}$ such that the action of G is semifree and $X^G = \{x^0\}$, x^0 is the G-fixed 0-cell. Moreover, suppose that $\pi_i(X)$ is finitely generated for $i \geq 2$ and $\pi_1(X)$ is a co-Hopfian group. Then X is a co-Hopfian object in $G\mathcal{H}$.* □

Example 3.7. Let $n \geq 2$ and $X = S^n \vee S^n$. Then X has a $\mathbb{Z}_2$-CW-complex structure as described below. It has one 0-cell of the type $\mathbb{Z}_2/\mathbb{Z}_2$ and one equivariant n-cell of the type $\mathbb{Z}_2/\{e\}$, where e denotes the identity element of $\mathbb{Z}_2$. This action is given by "switching coordinates", regarding the wedge as a subspace of the Cartesian product $S^n \times S^n$. Since X is a 1-connected finite complex, $\pi_q(X)$ is finitely generated. Moreover, $\pi_1(X) = \{0\}$. Hence it follows from Corollary 3.6 that X is co-Hopfian in $G\mathcal{H}$ where $G = \mathbb{Z}_2$.

4. $G\mathcal{H}$ versus $\mathcal{H}$

Recall that if G is discrete and X a G-CW-complex, then X is in a canonical way a CW-complex (cf. [3], p. 102). Thus if X is an object of $G\mathcal{H}$, then X can also be regarded as an object of $\mathcal{H}$. We shall show by the following examples that an object X of $G\mathcal{H}$ can be Hopfian (respectively, co-Hopfian) in $G\mathcal{H}$ without being Hopfian (respectively, co-Hopfian) in $\mathcal{H}$, and vice versa.

Example 4.1. Let $G = \mathbb{Z}_2$. Define an O_G-group $\lambda : O_G \longrightarrow \mathcal{A}b$ as follows. $\lambda(G/G) = \mathbb{Z}, \lambda(G/\{e\}) = \{0\}$, the trivial group, and $\lambda(G/\{e\} \longrightarrow G/G) : \mathbb{Z} \longrightarrow \{0\}$ is the obvious homomorphism. Let $X = K(\lambda, 1)$. Then X is co-Hopfian in $\mathcal{H}$, but not co-Hopfian in $G\mathcal{H}$.

To see this, note that $X = X^{\{e\}} = K(\lambda(G/\{e\}), 1)$. Hence X is contractible. Therefore X is co-Hopfian in $\mathcal{H}$. Next, note that λ is not co-Hopfian in $\mathcal{C}_G$. For, $\eta : \lambda \longrightarrow \lambda$ defined by $\eta(G/G) : x \longmapsto 2x$, $\eta(G/\{e\}) = id_{\{0\}}$ is a monomorphism, but not an isomorphism in $\mathcal{C}_G$ by Lemma 2.1. It follows from Corollary 3.2 that X is not co-Hopfian in $G\mathcal{H}$.

Let $G = \mathbb{Z}$, and H_n denote the subgroup $2^n\mathbb{Z}$, $n \geq 0$. If H is a subgroup of G, $H \neq H_n$ for all n, then $H = k\mathbb{Z}$, where $k = 2^{n_i}\ell$, ℓ odd , ℓ is not 1 or -1 and $n_i \geq 0$. Clearly $k\mathbb{Z} \subset H_{n_i}$ and there is no subconjugacy relation of the type $H_m \subset k\mathbb{Z}$. Also note that $H_{n+1} \subset H_n$ for all n. We define an O_G-group $\lambda : O_G \longrightarrow \mathcal{A}b$ as follows. Let Q^∞ denote the direct sum $\bigoplus_i Qe_i$ of countable copies of Q with basis $\{e_1, e_2, \cdots, e_n, \cdots\}$. Thus Q^∞ is a vector space over Q. Clearly, Q^∞ is neither Hopfian nor co-Hopfian in $\mathcal{A}b$. Let $Q^n = \bigoplus_{i=1}^n Qe_i$. Note that every group homomorphism $Q^n \longrightarrow Q^n$ is actually a Q-linear homomorphism $Q^n \longrightarrow Q^n$. Then it is easy to see that Q^n is both Hopfian and co-Hopfian in $\mathcal{A}b$. Set $\lambda(G/\{e\}) = Q^\infty, \lambda(G/H_n) = Q^n$ for all $n \geq 0$. If $H = k\mathbb{Z}, k = 2^{n_i}\ell$, ℓ odd and not equal to 1 or -1, $n_i \geq 0$, then we set $\lambda(G/k\mathbb{Z}) = Q^{n_i}$. Here, $Q^0 = \{0\}$, the trivial group. For every subconjugacy relation $H_{n+1} \subset H_n$, let

$$\lambda(G/H_{n+1} \longrightarrow G/H_n) : Q^n \longrightarrow Q^{n+1}$$

be the standard inclusion. For $k\mathbb{Z} \subset H_{n_i}, k = 2^{n_i}\ell$, ℓ odd and not equal to 1 or -1, $n_i \geq 0$, let $\lambda(G/k\mathbb{Z} \longrightarrow G/H_{n_i}) : Q^{n_i} \longrightarrow Q^{n_i}$ be the identity. Again, for the inclusions $\{e\} \subset H_n$ and $\{e\} \subset k\mathbb{Z}$, $k = 2^{n_i}\ell$, ℓ odd and not equal to 1 or -1, $n_i \geq 0$, we set $\lambda(G/\{e\} \longrightarrow G/H_n) : Q^n \longrightarrow Q^\infty$ and $\lambda(G/\{e\} \longrightarrow G/k\mathbb{Z}) : Q^{n_i} \longrightarrow Q^\infty$ to be the obvious inclusions. Then it is easy to see that λ is a contravariant functor from O_G to $\mathcal{A}b$.

Example 4.2. Let λ be as above and $X = K(\lambda, 1)$. Then X is co-Hopfian in $G\mathcal{H}$, but not co-Hopfian in $\mathcal{H}$.

Since $X = X^{\{e\}} = K(\lambda(G/\{e\}), 1)$ and Q^∞ is not co-Hopfian, it follows that X is not co-Hopfian in $\mathcal{H}$. To show that X is co-Hopfian in $G\mathcal{H}$, by Corollary 3.2, it is enough to show that λ is co-Hopfian. Let $\eta : \lambda \longrightarrow \lambda$ be a monomorphism. Then, by Lemma 2.1, $\eta(G/H) : \lambda(G/H) \longrightarrow \lambda(G/H)$ is a monomorphism for every subgroup H of G. By construction of λ, it is clear that $\eta(G/H)$ is an isomorphism for every subgroup $H \neq \{e\}$. We shall show that $\eta(G/\{e\})$ is also an isomorphism. Let $x \in Q^\infty$. Then we can find n such that $x \in Q^n$. Since $\eta(G/H_n)$ is an isomorphism, x lies in the image of $\eta(G/H_n)$. By naturality of η we have

$$\eta(G/\{e\})\lambda(G/\{e\} \longrightarrow G/H_n) = \lambda(G/\{e\} \longrightarrow G/H_n)\eta(G/H_n).$$

It follows from Lemma 2.1 that η is an isomorphism. Thus λ is co-Hopfian.

Example 4.3. Let $G = \mathbb{Z}$, and $\lambda : O_G \longrightarrow \mathcal{A}b$ be as in Example 4.2. Let $X = K(\lambda, 1)$. Then X is Hopfian in $G\mathcal{H}$, but not Hopfian in $\mathcal{H}$.

By an argument similar to the previous case, one can show that every epimorphism $\eta : \lambda \longrightarrow \lambda$ is an isomorphism. Thus λ is Hopfian. It follows from Corollary 2.12 that X is Hopfian in $G\mathcal{H}$. To show that X is not Hopfian in $\mathcal{H}$, it is enough to produce a self-epimorphism of X which is not an equivalence. Since Q^∞ is not Hopfian, we have an epimorphism $f : Q^\infty \longrightarrow Q^\infty$ which is not an isomorphism. Let $F : X = K(Q^\infty, 1) \longrightarrow K(Q^\infty, 1) = X$ be the map induced by f. Clearly, F is not an equivalence as $\pi_1(F) = f$ is not an isomorphism. We claim that F is an epimorphism. Let $\alpha, \beta : X \longrightarrow Y$ be base point preserving maps such that $\alpha \circ F \simeq \beta \circ F$. Since $f : Q^\infty \longrightarrow Q^\infty$ is surjective, there exists a homomorphism $s : Q^\infty \longrightarrow Q^\infty$ such that $f \circ s = id$. Let $S : X \longrightarrow X$ be the map induced by s. Then $F \circ S \simeq id_X$. This implies $\alpha \simeq \beta$. Thus F is an epimorphism. Hence X is not Hopfian in $\mathcal{H}$.

Example 4.4. Let $G = \mathbb{Z}_2$, and $\lambda : O_G \longrightarrow \mathcal{A}b$ be the O_G-group defined as follows: $\lambda(G/G) = Q^\infty, \lambda(G/\{e\}) = \{0\}$, and $\lambda(G/\{e\} \longrightarrow G/G) : Q^\infty \longrightarrow \{0\}$ is the obvious homomorphism. Then $X = K(\lambda, 1)$ is Hopfian in $\mathcal{H}$, but not Hopfian in $G\mathcal{H}$.

Clearly X is Hopfian in $\mathcal{H}$, as X is contractible. To see that X is not Hopfian in $G\mathcal{H}$, it is enough to find an epimorphism in $G\mathcal{H}$ which is not an equivalence. Let $\alpha : \lambda \longrightarrow \lambda$ be the natural transformation defined as follows: $\alpha(G/\{e\}) = id_{\{0\}}$ and $\alpha(G/G) : Q^\infty \longrightarrow Q^\infty$ any epimorphism which is not an isomorphism. Let $\beta : Q^\infty \longrightarrow Q^\infty$ be a homomorphism such that $\alpha(G/G) \circ \beta = id$. This defines a right inverse $\underline{\beta} : \lambda \longrightarrow \lambda$ of α, where $\underline{\beta}(G/G) = \beta$ and $\underline{\beta}(G/\{e\}) = id_{\{0\}}$. Let $T, S : X \longrightarrow X$ be the G-maps induced by α and $\underline{\beta}$ respectively. Then $T \circ S$ is G-homotopic to id_X. Clearly, T is not a G-equivalence, as $\pi_1(T^G) = \alpha(G/G)$ is not an isomorphism. Now proceeding as in Example 4.3 one shows that $T : X \longrightarrow X$ is an epimorphism in $G\mathcal{H}$.

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