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A NOTE ON THE ZERO-SEQUENCES OF SOLUTIONS OF $f'' + Af = 0$

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Abstract. We give a sufficient condition for complex sequences to be zero-sequences of solutions of $f'' + Af = 0$ where A is transcendental entire and of finite order.

1. Introduction

We consider the zero distribution of solutions of the complex differential equation

$$f'' + Af = 0 \tag{1}$$

where A is a transcendental entire function of finite order. It is well known that all nontrivial solutions f of (1) are of infinite order and the only possible deficient value of f is 0 (see [BL], [L]). For a complete introduction to the oscillation theory of complex differential equations we refer to [L]. It is natural to ask which zero-sequences with finite exponent of convergence can occur for a solution f. In [B] Bank gave the following necessary condition:

Theorem. *Let z_n be an infinite sequence of distinct nonzero complex numbers with $z_n \to \infty$ and having a finite exponent of convergence. Let p denote the genus of z_n and set*

$$\lambda_k := \sum_{m \neq k} \left(\frac{z_k}{z_m}\right)^p (z_m - z_k)^{-1}.$$

Then, if z_n is the zero-sequence of a solution of an equation (1) *where A is an entire function of finite order, then there must exist a real number $b > 0$ and a positive integer k_0 such that*

$$|\lambda_k| \leq \exp\left(|z_k|^b\right) \tag{2}$$

for all $k \geq k_0$.

This shows that not every sequence $z_n \to \infty$ with finite exponent of convergence is the zero-sequence of a solution of an equation (1). In fact Bank constructed a sequence with convergence exponent zero which does not fulfill the requirements of the foregoing theorem. Conversely, it was shown in [B] that every sequence z_n with the rather restrictive property $|z_{n+1}| \geq K|z_n|$ for some $K > 1$ is the zero-sequence

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of a solution of (1). It is easy to see that in this case the exponent of convergence of z_n is zero. The purpose of this note is to give a more general sufficient condition for z_n to be a zero-sequence. This will be done in Theorem 1. Further we show in Theorem 2 that a zero-sequence which does not fulfill our sufficient condition must have a special property. We note that if the requirement that A is of finite order is dropped, then any two distinct sequences tending to infinity are the zero-sequences of linearly independent solutions of an equation (1). This was proved in [S]. Finally let us fix some notations. In the sequel z_n will always denote a sequence of distinct nonzero complex numbers with $|z_{n+1}| \geq |z_n|$ for all $n \in \mathbb{N}$ and $z_n \to \infty$. Further we will assume that the exponent of convergence γ of z_n is finite, i.e. that

$$\gamma := \inf \left\{ c > 0 \,\middle|\, \sum_{n=1}^{\infty} |z_n|^{-c} \text{ converges} \right\} < \infty.$$

The smallest nonnegative integer p such that $\sum |z_n|^{-(p+1)}$ converges is the genus of z_n. Further G will always denote the canonical product formed by z_n. The Weierstraß convergence factors are defined by $e_p(z) := \exp(\sum_{j=1}^{p} z^j/j)$ where we set $e_0 \equiv 1$.

2. Results

Theorem 1. *Let z_n be a sequence with finite exponent of convergence. Let p be its genus and set*

$$\mu_k := \prod_{m \neq k} \left(1 - \frac{z_k}{z_m}\right)^{-1} e_p(z_k/z_m)^{-1}.$$

If there exists a real number $b > 0$ and a positive integer k_0 such that

$$|\mu_k| \leq \exp\left(|z_k|^b\right) \tag{3}$$

for all $k \geq k_0$, then z_n is the zero-sequence of a solution of an equation (1) *with transcendental A of finite order.*

Proof. Let G be the canonical product formed by z_n. If a function Ge^g satisfies an equation (1), it is easy to see that G satisfies the differential equation

$$G'' + 2g'G' + ((g')^2 + g'' + A)G = 0. \tag{4}$$

Evaluation at z_k gives

$$g'(z_k) = -\frac{G''(z_k)}{2G'(z_k)}. \tag{5}$$

This was already pointed out in [B] and yields the necessary condition (2). Conversely, if a function g of finite order satisfies (5), then Ge^g satisfies (1) with

$$A := -(g')^2 - g'' - (G'' + 2g'G')/G. \tag{6}$$

Clearly A will then be of finite order. Once we have constructed g satisfying (5) we only need to show that g is of finite order and can be chosen such that A is transcendental. To solve the interpolation problem (5) we use a method known as *Mittag-Lefflerscher Anschmiegungssatz* in German literature (see [BS], p. 257, Satz 29). For convinience we set $\sigma_k := -G''(z_k)/2G'(z_k)$ and form the functions $\sigma_k/G(z) = c_k/(z-z_k)+P_k(z)$ with P_k holomorphic at z_k. Now construct a Mittag-Leffler series H which is holomorphic in $\mathbb{C}$ except for poles of first order at all z_k with singular part $c_k/(z-z_k)$. Then HG has the interpolation property: Clearly HG

is entire and Laurent expansion of H around z_k gives $H(z) = c_k/(z - z_k) + Q_k(z)$ with Q_k holomorphic at z_k. Thus

$$HG(z_k) = \lim_{z \to z_k} H(z)G(z) = \lim_{z \to z_k} \left(\frac{\sigma_k}{G(z)} - P_k(z) + Q_k(z) \right) G(z) = \sigma_k.$$

Since G is of finite order, we only need to show that it is possible to construct H such that it is of finite order. The standard construction leads to

$$H(z) = \sum_{k=1}^{\infty} \left(\frac{z}{z_k} \right)^{n_k} \frac{c_k}{z - z_k} \tag{7}$$

where the n_k are nonnegative integers chosen such that the series converges compactly (see [HC], pp. 113-115). In order to choose n_k suitably we need an estimation of $|c_k|$. We thus have to determine the residue of σ_k/G at z_k. A routine computation shows $c_k = -z_k \sigma_k \mu_k / e_p(1)$. On the other hand one obtains $\mu_k^{-1} = -z_k G'(z_k)/e_p(1)$ and thus

$$2c_k = \frac{-z_k^2 G''(z_k)\mu_k^2}{e_p(1)^2}. \tag{8}$$

Since G'' is of finite order and using the assumption of the theorem we thus find an estimation $|c_k| \leq C \exp\left(|z_k|^c\right)$ with suitable $c > 0$ and $C > 0$. We set $n_k := 2[|z_k|^c]$ where $[x]$ is the smallest integer that satisfies $[x] \geq x$. To prove the convergence of H it suffices to show that $\sum_{k=1}^{\infty} (z/z_k)^{n_k} c_k/z_k$ converges absolutely in $\mathbb{C}$ (see [HC]). From the estimation for $|c_k|$ we get for $|z| \geq e^{-1}$

$$\sum_{|z_k| > e|z|} \left| \frac{z}{z_k} \right|^{n_k} \left| \frac{c_k}{z_k} \right| \leq C \sum_{|z_k| > e|z|} \exp(-|z_k|^c). \tag{9}$$

Since the exponent of convergence of z_n is finite, there exists $n \in \mathbb{N}$ such that $\sum |z_k|^{-cn}$ converges. For sufficiently large k clearly $\exp(-|z_k|^c) < |z_k|^{-cn}$ and thus the series on the right in (9) converges. We will now show that H is of finite order. For this purpose we use a theorem of R. Nevanlinna ([N], p. 36) which states that for meromorphic f the order is the maximum of the exponent of convergence of its poles and the growth order of

$$I(r) := \frac{1}{r} \int_0^r \log^+ M(t, f)\, dt$$

where $M(t, f) = \sup_{|z|=t} |f(z)|$. Since the poles of H are the zeros of G, we only need to consider the growth of I. We set $K := C \sum_{k=1}^{\infty} \exp(-|z_k|^c)$. It follows for $|z| \geq 1/(e-1)$

$$|H(z)| \leq \sum_{|z_k| \leq e|z|} \left| \frac{z}{z_k} \right|^{n_k} \frac{|c_k|}{|z - z_k|} + K.$$

Thus

$$\begin{aligned}\log^+|H(z)| &\le \sum_{|z_k|\le e|z|}\left(n_k\log^+\left|\frac{z}{z_k}\right| + \log^+|c_k| + \log^+(|z-z_k|^{-1})\right)\\ &\quad + O(\log(|z|))\\ &\le [(e|z|)^c]n(e|z|,G)\left(2\log^+\left|\frac{z}{z_1}\right|+1\right) + n(e|z|,G)\log^+(C)\\ &\quad + \sum_{|z_k|\le e|z|}\log^+(|z-z_k|^{-1}) + O(\log(|z|)).\end{aligned}$$

Here n is the counting function for the zeros of G. Hence

$$\begin{aligned}I(r) &\le [(er)^c]n(er,G)\left(2\log^+\frac{r}{|z_1|}+1\right) + n(er,G)\log^+(C)\\ &\quad + \sum_{|z_k|\le er}\frac{1}{r}\int_{1/(e-1)}^{r}\log^+(|t-|z_k||^{-1})\,dt + O(\log(r))\\ &\le [(er)^c]n(er,G)\left(2\log^+\frac{r}{|z_1|}+1\right)\\ &\quad + n(er,G)\left(\log^+(C)+\frac{2}{r}\right) + O(\log(r)).\end{aligned} \tag{10}$$

Since G is of finite order, it follows that H is of finite order. To complete the proof we show that it is possible to choose H such that A is transcendental. Let g be any primitive of HG. Suppose A defined by (6) is a polynomial. An application of the Clunie lemma to (6) shows that g is a polynomial p_1. This means $H = p_1'/G$. We define $\tilde{H}$ by (7) with $\tilde{n_k} := 2[|z_k|^c]+1$. The same method as above shows that $\tilde{H}$ is also of finite order. Now $\tilde{H}$ cannot be of the form p_2/G with a polynomial p_2 since

$$\begin{aligned}\tilde{H}(z) - H(z) &= \sum_{k=1}^{\infty}\left(\frac{z}{z_k}-1\right)\left(\frac{z}{z_k}\right)^{n_k}\frac{c_k}{z-z_k}\\ &= \sum_{k=1}^{\infty}\frac{c_k}{z_k^{n_k+1}}z^{n_k}.\end{aligned}$$

This is an entire function and thus either not of the form $(p_1'-p_2)/G$ or identically zero. In the latter case clearly $c_k = 0$ for all $k\in\mathbb{N}$ and thus by the definition of c_k it follows $\sigma_k = 0$ for all $k\in\mathbb{N}$. In this case simply set $H\equiv 1$, i.e. we choose g as a primitive of G. □

It seems to us that condition (3) is not necessary. Nonetheless we can show that a zero-sequence which does not fulfill (3) must have a rather special property.

Theorem 2. *Let z_n be a zero-sequence of a solution of* (1) *such that the requirements of Theorem* 1 *are not fulfilled. Further let G be the canonical product formed by z_n. Then there exists a subsequence $z_{n_k}\to\infty$ such that for all $j\in\mathbb{N}, b>0$ there exists $k(j,b)\in\mathbb{N}$ with*

$$|G^{(j)}(z_{n_k})| \le \exp\left(-|z_{n_k}|^b\right) \tag{11}$$

for all $k \ge k(j,b)$.

Proof. We assume without loss of generality $b \in \mathbb{N}$. Since for all $b > 0$ and $k_0 \in \mathbb{N}$ there exists $k \geq k_0$ such that $|\mu_k| = e_p(1)/|z_k G'(z_k)| > \exp(|z_k|^b)$, we find for every $b \in \mathbb{N}$ a subsequence z_{b,n_l} of z_n with $|G'(z_{b,n_l})| \leq \exp(-|z_{b,n_l}|^b)$. We define z_{n_k} inductively by the usual *diagonal* method: Set $z_{n_1} := z_{1,n_l}$ where l is chosen such that $|z_{1,n_l}| \geq 1$. Now define z_{n_k} such that it is in the sequence z_{k,n_l} and $|z_{n_k}| > |z_{n_{k-1}}|$. It follows $|G'(z_{n_k})| \leq \exp(-|z_{n_k}|^k)$. By choosing $k(1,b) := [b]$ the assertion follows for $j = 1$. An application of the Clunie lemma to (6) and standard order considerations show $\rho(g) = \rho(A)$. Thus all coefficients in (4) are of finite order. Now from (4) we have

$$|G''(z_{n_k})| = |2g'(z_{n_k})G'(z_{n_k})| \leq |2g'(z_{n_k})| \exp(-|z_{n_k}|^k)$$

and thus for k large enough $|G''(z_{n_k})| \leq \exp(|z_{n_k}|^{\rho(g)+\varepsilon} - |z_{n_k}|^k)$. By enlarging k if needed we get $|G''(z_{n_k})| \leq \exp(-|z_{n_k}|^{k/2})$ and the assertion follows for $j = 2$. Differentiating (4) and an easy induction argument yield the statement. □

Remarks. a) Let γ be the exponent of convergence of z_n and b as in (3). Then (8) shows that the constant c can be chosen as any number bigger than $\max\{\gamma, b\}$. Thus it follows from (10) that H can be chosen such that $\rho(H) \leq \max\{2\gamma, b+\gamma\}+\varepsilon$ for fixed $\varepsilon > 0$. Since $\rho(G) = \gamma$, we get from (6) that $\rho(A) \leq \max\{2\gamma, b+\gamma\} + \varepsilon$.

b) The mentioned sufficient condition $|z_{n+1}| \geq K|z_n|$ in [B] is covered by (3) in the following sense: In this case the sequence μ_k is bounded as can be seen from equation (42) in [B] and thus (3) holds with any $b > 0$. Since the convergence exponent of z_n is zero, we get from our order estimation in a) that A can be chosen to be of arbitrarily small order. In [B] it was shown that A can be constructed with $\rho(A) = 0$.

c) Of course (3) implies the necessary condition (2). We want to remark that this can be verified from $\lambda_k = (z_k G''(z_k)\mu_k)/(2e_p(1)) + p/z_k$. This follows from $\lambda_k = \sigma_k + p/z_k$ (see [B], equation (12)) and $\mu_k = -e_p(1)/(z_k G'(z_k))$.

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