A NOTE ON THE ZERO-SEQUENCES
OF SOLUTIONS OF $f'' + Af = 0$

ANDREAS SAUER

(Communicated by Hal L. Smith)

Abstract. We give a sufficient condition for complex sequences to be zero-
sequences of solutions of $f'' + Af = 0$ where $A$ is transcendental entire and of
finite order.

1. Introduction

We consider the zero distribution of solutions of the complex differential equation

$$f'' + Af = 0 \tag{1}$$

where $A$ is a transcendental entire function of finite order. It is well known that
all nontrivial solutions $f$ of (1) are of infinite order and the only possible deficient
value of $f$ is 0 (see [BL], [L]). For a complete introduction to the oscillation theory
of complex differential equations we refer to [L]. It is natural to ask which zero-
sequences with finite exponent of convergence can occur for a solution $f$. In [B] Bank
gave the following necessary condition:

Theorem. Let $z_n$ be an infinite sequence of distinct nonzero complex numbers with
$z_n \to \infty$ and having a finite exponent of convergence. Let $p$ denote the genus of $z_n$
and set

$$\lambda_k := \sum_{m \neq k} \left( \frac{z_k}{z_m} \right)^p (z_m - z_k)^{-1}. \tag{2}$$

Then, if $z_n$ is the zero-sequence of a solution of an equation (1) where $A$ is an entire
function of finite order, then there must exist a real number $b > 0$ and a positive
integer $k_0$ such that

$$|\lambda_k| \leq \exp \left( |z_k|^b \right)$$

for all $k \geq k_0$.

This shows that not every sequence $z_n \to \infty$ with finite exponent of convergence
is the zero-sequence of a solution of an equation (1). In fact Bank constructed a
sequence with convergence exponent zero which does not fulfill the requirements of
the foregoing theorem. Conversely, it was shown in [B] that every sequence $z_n$ with
the rather restrictive property $|z_{n+1}| \geq K|z_n|$ for some $K > 1$ is the zero-sequence

Received by the editors October 11, 1995.
1991 Mathematics Subject Classification. Primary 34A20; Secondary 30D20.
This research was done during a visit at the University of Joensuu, Finland, financed by the
DFG (Deutsche Forschungsgemeinschaft).
of a solution of (1). It is easy to see that in this case the exponent of convergence of \( z_n \) is zero. The purpose of this note is to give a more general sufficient condition for \( z_n \) to be a zero-sequence. This will be done in Theorem 1. Further we show in Theorem 2 that a zero-sequence which does not fulfill our sufficient condition must have a special property. We note that if the requirement that \( A \) is of finite order is dropped, then any two distinct sequences tending to infinity are the zero-sequences of linearly independent solutions of an equation (1). This was proved in [S]. Finally let us fix some notations. In the sequel \( z_n \) will always denote a sequence of distinct nonzero complex numbers with \( |z_{n+1}| \geq |z_n| \) for all \( n \in \mathbb{N} \) and \( z_n \to \infty \). Further we will assume that the exponent of convergence \( \gamma \) of \( z_n \) is finite, i.e. that

\[
\gamma := \inf \left\{ c > 0 \mid \sum_{n=1}^{\infty} |z_n|^{-c} \text{ converges} \right\} < \infty.
\]

The smallest nonnegative integer \( p \) such that \( \sum |z_n|^{-(p+1)} \) converges is the genus of \( z_n \). Further \( G \) will always denote the canonical product formed by \( z_n \). The Weierstraß convergence factors are defined by \( e_p(z) := \exp(\sum_{j=1}^{p} z/j) \) where we set \( e_0 \equiv 1 \).

2. Results

**Theorem 1.** Let \( z_n \) be a sequence with finite exponent of convergence. Let \( p \) be its genus and set

\[
\mu_k := \prod_{m \neq k} \left( 1 - \frac{z_k}{z_m} \right)^{-1} e_p(z_k/z_m)^{-1}.
\]

If there exists a real number \( b > 0 \) and a positive integer \( k_0 \) such that

\[
|\mu_k| \leq \exp\left( |z_k|^b \right)
\]

for all \( k \geq k_0 \), then \( z_n \) is the zero-sequence of a solution of an equation (1) with transcendental \( A \) of finite order.

**Proof.** Let \( G \) be the canonical product formed by \( z_n \). If a function \( Ge^g \) satisfies an equation (1), it is easy to see that \( G \) satisfies the differential equation

\[
G'' + 2g'G' + ((g')^2 + g'' + A)G = 0.
\]

Evaluation at \( z_k \) gives

\[
g'(z_k) = -\frac{G''(z_k)}{2G'(z_k)}.
\]

This was already pointed out in [B] and yields the necessary condition (2). Conversely, if a function \( g \) of finite order satisfies (5), then \( Ge^g \) satisfies (1) with

\[
A := -(g')^2 - g'' - (G'' + 2g'G')/G.
\]

Clearly \( A \) will then be of finite order. Once we have constructed \( g \) satisfying (5) we only need to show that \( g \) is of finite order and can be chosen such that \( A \) is transcendental. To solve the interpolation problem (5) we use a method known as Mittag-Lefflerscher Anschmiegungssatz in German literature (see [BS], p. 257, Satz 29). For convenience we set \( \sigma_k := -G''(z_k)/2G'(z_k) \) and form the functions \( \sigma_k/G(z) = c_k/(z-z_k) + P_k(z) \) with \( P_k \) holomorphic at \( z_k \). Now construct a Mittag-Leffler series \( H \) which is holomorphic in \( \mathbb{C} \) except for poles of first order at all \( z_k \) with singular part \( c_k/(z-z_k) \). Then \( HG \) has the interpolation property: Clearly \( HG \)
is entire and Laurent expansion of $H$ around $z_k$ gives $H(z) = c_k/(z - z_k) + Q_k(z)$ with $Q_k$ holomorphic at $z_k$. Thus

$$HG(z_k) = \lim_{z \to z_k} H(z)G(z) = \lim_{z \to z_k} \left( \frac{\sigma_k}{G(z)} - P_k(z) + Q_k(z) \right)G(z) = \sigma_k.$$  

Since $G$ is of finite order, we only need to show that it is possible to construct $H$ such that it is of finite order. The standard construction leads to

$$H(z) = \sum_{k=1}^{\infty} \left( \frac{z}{z_k} \right)^{n_k} \frac{c_k}{z - z_k}$$

where the $n_k$ are nonnegative integers chosen such that the series converges compactly (see [HC], pp. 113-115). In order to choose $n_k$ suitably we need an estimation of $|c_k|$. We thus have to determine the residue of $\sigma_k/G$ at $z_k$. A routine computation shows $c_k = -z_k\sigma_k\mu_k/e_p(1)$. On the other hand one obtains $\mu_k^{-1} = -z_kG'(z_k)/e_p(1)$ and thus

$$2c_k = -z_k^2G''(z_k)\mu_k^2.$$  

Since $G''$ is of finite order and using the assumption of the theorem we thus find an estimation $|c_k| \leq C\exp\left(|z_k|^c\right)$ with suitable $c > 0$ and $C > 0$. We set $n_k := 2\|z_k|^c\|$ where $[x]$ is the smallest integer that satisfies $[x] \geq x$. To prove the convergence of $H$ it suffices to show that $\sum_{k=1}^{\infty} (z/z_k)^{n_k}c_k/z_k$ converges absolutely in $\mathbb{C}$ (see [HC]). From the estimation for $|c_k|$ we get for $|z| \geq e^{-1}$

$$\sum_{|z_k| > e|z|} \left| \frac{z}{z_k} \right|^{n_k} \frac{|c_k|}{|z_k|} \leq C \sum_{|z_k| > e|z|} \exp(-|z_k|^c).$$

Since the exponent of convergence of $z_n$ is finite, there exists $n \in \mathbb{N}$ such that $\sum |z_k|^{-cn}$ converges. For sufficiently large $k$ clearly $\exp(-|z_k|^c) < |z_k|^{-cn}$ and thus the series on the right in (9) converges. We will now show that $H$ is of finite order. For this purpose we use a theorem of R. Nevanlinna ([N], p. 36) which states that for meromorphic $f$ the order is the maximum of the exponent of convergence of its poles and the growth order of

$$I(r) := \frac{1}{\pi} \int_0^r \log^+ M(t,f) \, dt$$

where $M(t,f) = \sup_{|z| = t} |f(z)|$. Since the poles of $H$ are the zeros of $G$, we only need to consider the growth of $I$. We set $K := C \sum_{k=1}^{\infty} \exp(-|z_k|^c)$. It follows for $|z| \geq 1/(e - 1)$

$$|H(z)| \leq \sum_{|z_k| \leq |z|} \left| \frac{z}{z_k} \right|^{n_k} \frac{|c_k|}{|z - z_k|} + K.$$
Thus
\[
\log^+ |H(z)| \leq \sum_{|z_k| \leq |z|} \left( n_k \log^+ \left| \frac{z}{z_k} \right| + \log^+ |c_k| + \log^+ (|z - z_k|^{-1}) \right) + O(\log(|z|))
\]
\[
\leq [(e|z|)]^c n(e|z|, G) \left( 2 \log^+ \left| \frac{z}{z_1} \right| + 1 \right) + n(e|z|, G) \log^+(C)
\]
\[
+ \sum_{|z_k| \leq |z|} \log^+ (|z - z_k|^{-1}) + O(\log(|z|)).
\]

Here \( n \) is the counting function for the zeros of \( G \). Hence
\[
I(r) \leq [(er)^c] n(er, G) \left( 2 \log^+ \frac{r}{|z_1|} + 1 \right) + n(er, G) \log^+(C)
\]
\[
+ \sum_{|z_k| \leq er} \frac{1}{r} \int_{1/(e-1)}^r \log^+ (|t - |z_k||^{-1}) \, dt + O(\log(r))
\]
\[
\leq [(er)^c] n(er, G) \left( 2 \log^+ \frac{r}{|z_1|} + 1 \right) + n(er, G) \left( \log^+(C) + \frac{2}{r} \right) + O(\log(r)).
\]

Since \( G \) is of finite order, it follows that \( H \) is of finite order. To complete the proof we show that it is possible to choose \( H \) such that \( A \) is transcendental. Let \( g \) be any primitive of \( HG \). Suppose \( A \) defined by (6) is a polynomial. An application of the Clunie lemma to (6) shows that \( g \) is a polynomial \( p_1 \). This means \( H = p_1/G \).

We define \( \tilde{H} \) by (7) with \( n_k := 2[|z_k|^c] + 1 \). The same method as above shows that \( \tilde{H} \) is also of finite order. Now \( \tilde{H} \) cannot be of the form \( p_2/G \) with a polynomial \( p_2 \) since
\[
\tilde{H}(z) - H(z) = \sum_{k=1}^{\infty} \left( \frac{z}{z_k} - 1 \right) \left( \frac{z}{z_k} \right)^{n_k} \frac{c_k}{z - z_k}
\]
\[
= \sum_{k=1}^{\infty} c_k^{n_k+1} z^{n_k}.
\]

This is an entire function and thus either not of the form \((p_1 - p_2)/G\) or identically zero. In the latter case clearly \( c_k = 0 \) for all \( k \in \mathbb{N} \) and thus by the definition of \( c_k \) it follows \( \sigma_k = 0 \) for all \( k \in \mathbb{N} \). In this case simply set \( H \equiv 1 \), i.e. we choose \( g \) as a primitive of \( G \).

It seems to us that condition (3) is not necessary. Nonetheless we can show that a zero-sequence which does not fulfill (3) must have a rather special property.

**Theorem 2.** Let \( z_n \) be a zero-sequence of a solution of (1) such that the requirements of Theorem 1 are not fulfilled. Further let \( G \) be the canonical product formed by \( z_n \). Then there exists a subsequence \( z_{n_k} \to \infty \) such that for all \( j \in \mathbb{N}, b > 0 \) there exists \( k(j, b) \in \mathbb{N} \) with
\[
|G^{(j)}(z_{n_k})| \leq \exp \left( -|z_{n_k}|^b \right)
\]
for all \( k \geq k(j, b) \).
Proof. We assume without loss of generality \( b \in \mathbb{N} \). Since for all \( b > 0 \) and \( k_0 \in \mathbb{N} \) there exists \( k \geq k_0 \) such that \(|\mu_k| = e_p(1)/|z_k G'(z_k)| > \exp(|z_k|^b)\), we find for every \( b \in \mathbb{N} \) a subsequence \( z_{b,n} \) of \( z_n \) with \(|G'(z_{b,n})| \leq \exp(-|z_{b,n}|^b)\). We define \( z_{n_k} \) inductively by the usual diagonal method: Set \( z_{n_1} := z_{1,n} \) where \( l \) is chosen such that \(|z_{1,n}| \geq 1\). Now define \( z_{n_k} \) such that it is in the sequence \( z_{k,n} \) and \(|z_{n_k}| > |z_{n_{k-1}}|\). It follows \( |G'(z_{n_k})| \leq \exp(-|z_{n_k}|^k)\). By choosing \( k(1,b) := [b] \) the assertion follows for \( j = 1 \). An application of the Clunie lemma to (6) and standard order considerations show \( \rho(g) = \rho(A) \). Thus all coefficients in (4) are of finite order. Now from (4) we have
\[
|G''(z_{n_k})| = |2g'(z_{n_k})G'(z_{n_k})| \leq |2g'(z_{n_k})|\exp(-|z_{n_k}|^k)
\]
and thus for \( k \) large enough \(|G''(z_{n_k})| \leq \exp(|z_{n_k}|^{\gamma,b} + \varepsilon - |z_{n_k}|^k)\). By enlarging \( k \) if needed we get \(|G''(z_{n_k})| \leq \exp(-|z_{n_k}|^{k/2})\) and the assertion follows for \( j = 2 \). Differentiating (4) and an easy induction argument yield the statement.

Remarks. a) Let \( \gamma \) be the exponent of convergence of \( z_n \) and \( b \) as in (3). Then (8) shows that the constant \( c \) can be chosen as any number bigger than \( \max\{\gamma,b\} \). Thus it follows from (10) that \( H \) can be chosen such that \( \rho(H) \leq \max\{2\gamma,b+\gamma\} + \varepsilon \) for fixed \( \varepsilon > 0 \). Since \( \rho(G) = \gamma \), we get from (6) that \( \rho(A) \leq \max\{2\gamma,b+\gamma\} + \varepsilon \).

b) The mentioned sufficient condition \(|z_{n+1}| \geq K|z_n|\) in [B] is covered by (3) in the following sense: In this case the sequence \( \mu_k \) is bounded as can be seen from equation (42) in [B] and thus (3) holds with any \( b > 0 \). Since the convergence exponent of \( z_n \) is zero, we get from our order estimation in a) that \( A \) can be chosen to be of arbitrarily small order. In [B] it was shown that \( A \) can be constructed with \( \rho(A) = 0 \).

c) Of course (3) implies the necessary condition (2). We want to remark that this can be verified from \( \lambda_k = (z_k G''(z_k)\mu_k)/(2e_p(1)) + p/z_k \). This follows from \( \lambda_k = \sigma_k + p/z_k \) (see [B], equation (12)) and \( \mu_k = -e_p(1)/(z_k G'(z_k)) \).

References


Gerhard Mercator Universität, Fachbereich 11 Mathematik, Lothringerstrasse 65, D-47057 Duisburg, Federal Republic of Germany

E-mail address: sauer@math.uni-duisburg.de