PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 4, April 1997, Pages 1153–1159
S 0002-9939(97)03884-7

SEQUENTIAL TYPE KOROVKIN THEOREM ON L^∞ FOR QC-TEST FUNCTIONS

KEIJI IZUCHI

(Communicated by Theodore W. Gamelin)

Abstract. Let $\{T_n\}_n$ be a sequence of bounded linear operators on L^∞ such that $\|T_n\| \to 1$ and $\|T_n g - g\|_\infty \to 0$ for every $g \in QC$. It is proved that $\|T_n f - f\|_\infty \to 0$ for every $f \in L^\infty$.

1. Introduction

In 1953, Korovkin [8] (see also [9]) proved the following exciting approximation theorem: if $\{T_n\}_n$ is a sequence of positive linear operators on $C([0,1])$ such that $\|T_n t^j - t^j\|_\infty \to 0$ for $j = 0, 1, 2$ as $n \to \infty$, then $\|T_n f - f\|_\infty \to 0$ for every $f \in C([0,1])$. In 1968, Wulbert [14] proved that this theorem is also true if the condition "positivity" is replaced by the one "$\|T_n\| \to 1$". Recently, there has been much research on this subject, Korovkin type approximation theorems; see the monograph by Altomare and Campiti [1].

Let Ω be a compact Hausdorff space and let $C(\Omega)$ be the space of complex valued continuous functions on Ω. For a closed subset E of Ω and $f \in C(\Omega)$, let $\|f\|_E = \sup\{|f(x)|; x \in E\}$. When $E = \Omega$, we write $\|f\|_\infty = \|f\|_\Omega$. Let S be a closed subspace of $C(\Omega)$. We say that the sequential type Korovkin approximation theorem holds on $C(\Omega)$ for S if for every sequence of bounded linear operators $\{T_n\}_n$ on $C(\Omega)$ such that $\|T_n\| \to 1$ and $\|T_n g - g\|_\infty \to 0$ for $g \in S$, then it follows that $\|T_n f - f\|_\infty \to 0$ for every $f \in C(\Omega)$. S is called test functions. We can also consider the net type Korovkin approximation theorem by replacing the condition "a sequence $\{T_n\}_n$" by "a net $\{T_\alpha\}_\alpha$". The interesting problem is for which S the sequential type Korovkin theorem holds on $C(\Omega)$. Takahasi [13] (see [14]) proved that the net type Korovkin theorem holds on $C(\Omega)$ for S if and only if the Choquet boundary of S coincides with Ω. It is easy to see that if the net type Korovkin theorem holds for S then the sequential type Korovkin theorem holds for S. In [6], the author, Takagi and Watanabe show that if S is separable then the converse of the above fact is true. In [12], Scheffold gave the example of S such that the sequential type Korovkin theorem holds but the net type Korovkin theorem does not hold. The given S in [12] is the closed ideal of $C(\Omega)$ with some additional properties. As a completion of Scheffold's result, in [7] the author, Takagi and Watanabe prove that the sequential type Korovkin theorem holds on $C(\Omega)$ for a closed ideal S with $S = \{f \in C(\Omega); f = 0 \text{ on } \Gamma\}$, where Γ is a closed subset of Ω, if

Received by the editors February 23, 1996.
1991 *Mathematics Subject Classification.* Primary 41J35, 46J10.

and only if Γ does not contain any quasi G_δ-subsets of Ω, where a closed subset E is called quasi G_δ if there exists a sequence of open subsets $\{U_n\}_n$ of Ω such that $U_{n+1} \subset U_n$ and $E = \cap_{n=1}^{\infty} \overline{U_n}$, where $\overline{U_n}$ is the closure of U_n in Ω.

It seems very difficult to give a complete characterization of C^*-subalgebras S of $C(\Omega)$ for which the sequential type Korovkin approximation theorem holds. In this paper, we study the sequential type Korovkin theorem on the unit circle.

Let D be the open unit disk and ∂D the unit circle. Let H^∞ be the Banach algebra of boundary functions of bounded analytic functions on D. Then H^∞ is the essential supremum norm closed subalgebra of L^∞, the Banach algebra of bounded measurable functions on ∂D. We denote by $M(L^\infty)$ the maximal ideal space of L^∞. We consider that $L^\infty = C(M(L^\infty))$. It is well known that $H^\infty + C$ is the closed subalgebra of L^∞ [11], where C is the space of continuous functions on ∂D. Let

$$QC = (H^\infty + C) \cap \overline{(H^\infty + C)} \quad \text{and} \quad QA = H^\infty \cap QC.$$

Then $C \subset QC \subset L^\infty$ and the C^*- algebra QC is studied by Sarason [10] extensively. The purpose of this paper is to prove that the sequential type Korovkin theorem holds on L^∞ for test functions QC. Since QC does not separate the points in $M(L^\infty)$, by Takahasi's criterion the net type Korovkin theorem does not hold on L^∞ for QC. Also we note that the sequential type Korovkin theorem does not hold on L^∞ for C.

In the same way, we can prove that the sequential type Korovkin theorem holds on H^∞ for test functions QA.

2. Preliminaries

Let $M(H^\infty)$ be the maximal ideal space of H^∞. We identify a function in H^∞ with its Gelfand transform on $M(H^\infty)$. Then we may consider that $M(L^\infty) \subset M(H^\infty)$ and $M(L^\infty)$ is the Shilov boundary of H^∞. Also we can consider that $D \subset M(H^\infty)$, and by the corona theorem D is dense in $M(H^\infty)$. The references [2] and [3] are nice for the spaces H^∞ and L^∞.

For a subset E of $M(L^\infty)$, we denote by $\overline{E}$ the closure of E in $M(L^\infty)$. The outstanding topological property of $M(L^\infty)$ is that if U is an open subset of $M(L^\infty)$ then $\overline{U}$ is also open. For a point x in $M(H^\infty)$, there exists a unique probability measure μ_x on $M(L^\infty)$ such that

$$\int_{M(L^\infty)} f \, d\mu_x = f(x) \quad \text{for} \quad \text{every } f \in H^\infty.$$

We denote by $\mathrm{supp}\mu_x$ the closed support set of μ_x. The following is a characterization of functions in QC.

Lemma 1 ([10])**.** *Let $f \in L^\infty$. Then $f \in QC$ if and only if f is constant on* $\mathrm{supp}\,\mu_x$ *for every $x \in M(H^\infty) \setminus D$.*

For a point x in $M(L^\infty)$, let

$$Q_x = \{y \in M(L^\infty); f(y) = f(x) \ \text{for} \ \text{every } f \in QC\}.$$

The set Q_x is called the QC-level set associate with x. For a point ζ in $M(H^\infty) \setminus D$, there corresponds a QC-level set Q_ζ such that $\mathrm{supp}\mu_\zeta \subset Q_\zeta$. For a function f in L^∞, we denote by $N(f)$ the closure of

$$\cup\{\mathrm{supp}\,\mu_x; x \in M(H^\infty) \setminus D \ \text{and} \ f|\,\mathrm{supp}\,\mu_x \notin H^\infty|\mathrm{supp}\mu_x\}.$$

By the author [4, 5], the set $N(f)$ was investigated extensively. In this paper, the $N(f)$ plays the essential role.

Lemma 2 ([5, Corollary 2.1])**.** *Let $f \in L^\infty$. Then $N(f) = \cup\{Q_x; x \in N(f)\}$.*

Lemma 3 ([4, Corollary 7])**.** *Let $f_1, f_2 \in L^\infty$. Then $N(f_1) \cup N(f_2)$ does not contain any G_δ-subsets of $M(L^\infty)$.*

For $f \in L^\infty$, let

$$\tilde{N}(f) = N(f) \cup N(\bar{f}).$$

Then $\tilde{N}(f)$ coincides with the closure of $\cup\{\mathrm{supp}\mu_x; x \in M(H^\infty) \setminus D$ and $f|\mathrm{supp}\mu_x$ is not constant$\}$. If $f \in H^\infty$, then $\tilde{N}(f) = N(\bar{f})$. The following is a key to proving our theorem.

Lemma 4. *Let $f \in L^\infty$ and let $\{V_n\}_n$ be a sequence of open and closed subsets of $M(L^\infty)$ such that $V_n \cap \tilde{N}(f) = \emptyset$ for every n. Then there exist a subsequence $\{n_j\}_j$ of positive integers and $x_{n_j} \in V_{n_j}$ such that $\overline{\{x_{n_j}\}_j} \cap \tilde{N}(f) = \emptyset$.*

Proof. Let

$$W_n = \overline{\bigcup_{j=n}^{\infty} V_j}.$$

Then W_n is open and closed. By Lemma 3, the set $\cap_{n=1}^\infty W_n$ is not contained in $\tilde{N}(f)$. Take a point ζ_0 in $\cap_{n=1}^\infty W_n \setminus \tilde{N}(f)$, and take an open and closed subset U of $M(L^\infty)$ with $\zeta_0 \in U$ and $U \cap \tilde{N}(f) = \emptyset$. Then there is a subsequence $\{n_j\}_j$ such that

$$V_{n_j} \cap U \neq \emptyset \quad \text{for } j = 1, 2, \dots .$$

Take a point x_{n_j} in $V_{n_j} \cap U$. Then $\{x_{n_j}\}_j$ satisfies our assertion.

Here we show

Proposition 1. *Korovkin theorem does not hold on L^∞ for test functions C.*

Proof. Take points ζ_1 and ζ_2 in $M(L^\infty)$ such that

$$\zeta_1 \neq \zeta_2 \text{ and } g(\zeta_i) = g(1) \text{ for every } g \in C.$$

For each n, let $f_n(e^{i\theta}) = |e^{i\theta} + 1|^n / 2^n$ for $e^{i\theta} \in \partial D$ and let

$$T_n f = f(\zeta_1) f_n + (1 - f_n) f \quad \text{for } f \in L^\infty.$$

Then $\{T_n\}_n$ is a sequence of bounded linear operators on L^∞ with $\|T_n\| = 1$. Since $T_n f = f + f_n(f(\zeta_1) - f)$, it is not difficult to see that

$$\|T_n g - g\|_\infty \to 0 \quad \text{for } g \in C.$$

Take $f_0 \in L^\infty$ with $f_0(\zeta_1) = 0$ and $f_0(\zeta_2) = 1$. Then

$$\|T_n f_0 - f_0\|_\infty = \|f_n f_0\|_\infty \geq |f_n(\zeta_2) f_0(\zeta_2)| = 1 \text{ for every } n.$$

3. Theorems

The following is the main theorem in this paper. The proof is similar to the one of Theorem 2 in [7]. But our proof is deeply concerned with the property of $\tilde{N}(f)$ for $f \in L^\infty$.

Theorem 1. *Suppose $\{T_n\}_n$ is a sequence of bounded linear operators on $L^\infty = C(M(L^\infty))$ such that $\|T_n\| \to 1$ and $\|T_n g - g\|_\infty \to 0$ for every $g \in QC$ as $n \to \infty$. Then $\|T_n f - f\|_\infty \to 0$ for every $f \in L^\infty$ as $n \to \infty$.*

Proof. Let $\{T_n\}_n$ be a sequence of bounded linear operators such that $\|T_n\| \to 1$ and $\|T_n g - g\|_\infty \to 0$ for $g \in QC$. To prove that $\|T_n f - f\|_\infty \to 0$ for every $f \in C(M(L^\infty))$, suppose not. Then there exists a function f_0 in $C(M(L^\infty))$ with $\|f_0\|_\infty = 1$ and $\sigma > 0$ such that $\limsup_{n\to\infty} \|T_n f_0 - f_0\|_\infty > \sigma$. By considering a subsequence, we may assume that $\|T_n f_0 - f_0\|_\infty > \sigma$ for every n. So there exists a non-empty open and closed subset V_n of $M(L^\infty)$ such that

$$(1) \qquad |T_n f_0 - f_0| > \sigma \quad \text{on } V_n.$$

By Lemma 3, we may assume that $V_n \cap \tilde{N}(f_0) = \emptyset$. By Lemma 4, considering a subsequence we may assume the existence of x_n in V_n such that

$$(2) \qquad \overline{\{x_n\}_n} \cap \tilde{N}(f_0) = \emptyset.$$

By (1), we have

$$(3) \qquad |(T_n f_0)(x_n) - f_0(x_n)| > \sigma.$$

Also considering a subsequence, moreover we may assume that

$$(4) \qquad (T_n f_0)(x_n) \to a \ \text{ and } \ f_0(x_n) \to b \ \text{ as } n \to \infty.$$

Since $\|T_n\| \to 1$ and $\|f_0\|_\infty = 1$, $|a| \le 1$ and $|b| \le 1$. Also by (3), $|a - b| \ge \sigma$. Here we can find a complex number c such that

$$(5) \qquad |b - c| \le 1 \ \text{ and } \ |a - c| \ge 1 + \sigma.$$

Since $|a| \le 1$, we have $c \ne 0$.

Let w_0 be one of cluster points of $\{x_n\}_n$ in $M(L^\infty)$. By (2), $w_0 \notin \tilde{N}(f_0)$. Then by Lemma 2, there exists an open subset W of $M(L^\infty)$ such that

$$(6) \qquad w_0 \in W,\ W = \cup\{Q_y; y \in W\} \ \text{ and } \ \overline{W} \cap \tilde{N}(f_0) = \emptyset.$$

By (4), $f_0(w_0) = b$. By (6), there exists a function h in QC such that

$$(7) \qquad h(w_0) = 1 \ \text{ and } \ h = 0 \ \text{ on } \tilde{N}(f_0).$$

Then by Lemma 1 and the definition of $\tilde{N}(f_0)$, we have $hf_0 \in QC$. Hence by taking a smaller subset of W, we may assume that

$$(8) \qquad \|f_0 - b\|_W < \sigma/2.$$

By (6) and $c \ne 0$, there exists a function F in QC such that

$$(9) \qquad 0 \le F/c \le 1 \ \text{ on } M(L^\infty),\ F(w_0) = c \ \text{ and } \ F = 0 \ \text{ on } W^c.$$

Since $\|f_0\|_\infty = 1$, by (5) and (8) it is not difficult to see that

$$(10) \qquad \|F - f_0\|_\infty < \sigma/2 + 1.$$

Since $\|T_n\| \to 1$, we have

$$\limsup_{n\to\infty} \|T_n F - T_n f_0\|_\infty < \sigma/2 + 1.$$

Since $F \in QC$, by our assumption we have $\|T_n F - F\|_\infty \to 0$. Hence we obtain

$$\limsup_{n\to\infty} \|F - T_n f_0\|_\infty < \sigma/2 + 1. \tag{11}$$

Since w_0 is a cluster point of $\{x_n\}_n$, there exists a subsequence $\{x_{n_j}\}_j$ of $\{x_n\}_n$ such that

$$F(x_{n_j}) \to F(w_0) \quad \text{as} \quad j \to \infty. \tag{12}$$

Therefore we have

$$\begin{aligned} \limsup_{n\to\infty} \|F - T_n f_0\|_\infty &\geq \limsup_{j\to\infty} |F(x_{n_j}) - (T_{n_j} f_0)(x_{n_j})| \\ &= |c - a| \quad \text{by (4), (9) and (12)} \\ &\geq 1 + \sigma \quad \text{by (5).} \end{aligned}$$

This contradicts (11). Thus we get our assertion.

The key point of the proof of Theorem 1 is the following: for $f \in L^\infty$ the union set of all QC-level sets on which f have non-zero oscillations is a very small subset of $M(L^\infty)$. If its union set occupies a very big part of $M(L^\infty)$, we may not expect that the sequential type Korovkin theorem holds. We give an example such that the sequential type Korovkin theorem does not hold on some closed subsets of $M(L^\infty)$.

Example. Let $\{z_n\}_n$ be a sparse sequence in D, that is,

$$\lim_{k\to\infty} \prod_{n\neq k} \left| \frac{z_n - z_k}{1 - \bar{z}_n z} \right| = 1.$$

Let b be the associated sparse Blaschke product;

$$b(z) = \prod_{n=1}^{\infty} \frac{-\bar{z}_n}{|z_n|} \frac{z - z_n}{1 - \bar{z}_n z}, \qquad z \in D.$$

Let $Z(b) = \{x \in M(H^\infty) \setminus D; b(x) = 0\}$. Then by [4] we have that
(a) if $x, y \in Z(b)$ with $x \neq y$ then $Q_x \cap Q_y = \emptyset$,
(b) $N(\bar{b}) = \cup\{Q_x; x \in Z(b)\}$.
We shall prove that the sequential type Korovkin theorem does not hold on $C(N(\bar{b}))$ for test functions $QC|N(\bar{b})$. We note that b has non-zero oscillation on Q_x for each $x \in Z(b)$.

Let $\zeta \in N(\bar{b})$. By (a) and (b), there corresponds a point $\tau(\zeta)$ in $Z(b)$ such that $\zeta \in Q_{\tau(\zeta)}$. We note that $\tau : N(\bar{b}) \to Z(b)$ is a continuous map. For $f \in C(N(\bar{b}))$, let

$$(Tf)(\zeta) = \int_{N(\bar{b})} f d\mu_{\tau(\zeta)} \qquad \text{for } \zeta \in N(\bar{b}).$$

Then $Tf \in C(N(\bar{b}))$ and T is a bounded linear operator on $C(N(\bar{b}))$ with $\|T\| = 1$. By the definition of T, we know that T is the identity operator on $QC|N(\bar{b})$. Set $T_n = T$ for every n; then

$$\|T_n g - g\|_{N(\bar{b})} \to 0 \quad \text{for} \quad \text{every } g \in QC|N(\bar{b}).$$

Let $f_0 = b|N(\bar{b})$. Then $f_0 \in C(N(\bar{b}))$ and

$$(T_n f_0)(\zeta) = \int_{N(\bar{b})} b \, d\mu_{\tau(\zeta)} = b(\tau(\zeta)) = 0 \quad \text{for } \zeta \in N(\bar{b}).$$

Since $|b| = 1$ on $M(L^\infty)$, we have

$$\|T_n f_0 - f_0\|_{N(\bar{b})} = 1.$$

Hence $\|T_n f_0 - f_0\|_{N(\bar{b})}$ does not converge to 0 as $n \to \infty$.

In the same way as the proof of Theorem 1, we have the following theorems.

Theorem 2. *Let $\{T_n\}_n$ be a sequence of bounded linear operators on H^∞ such that $\|T_n\| \to 1$ and $\|T_n g - g\|_\infty \to 0$ for every $g \in QA$. Then $\|T_n f - f\|_\infty \to 0$ for every $f \in H^\infty$.*

Proof. We give a remark for the proof of this theorem.

(#) We cannot find F in QA which satisfies (9).

To overcome this difficulty, it is sufficient to show that for every $\epsilon > 0$ there exists F in QA such that $F(w_0) = c, |F| < \epsilon$ on W^c and $|F| + |c - F| < 1 + \epsilon$ on $M(L^\infty)$. The existence of such an F is proved in the proof of Theorem 2 in [6] for general function algebras.

Let I be the identity operator on L^∞. For a bounded linear operator T on L^∞, let $\|T\|_{QC} = \sup\{\|Tf\|_\infty; f \in QC, \|f\|_\infty \leq 1\}$.

Theorem 3. *Suppose $\{T_n\}_n$ is a sequence of bounded linear operators on $L^\infty = C(M(L^\infty))$ such that $\|T_n\| \to 1$ and $\|T_n - I\|_{QC} \to 0$ as $n \to \infty$. Then $\|T_n - I\| \to 0$ as $n \to \infty$.*

Proof. We give the outline. Suppose that there exists a sequence $\{f_n\}_n$ in $C(M(L^\infty))$ such that

$$\|f_n\|_\infty = 1 \quad \text{and} \quad \|T_n f_n - f_n\|_\infty > \sigma > 0.$$

By considering a subsequence, we may assume the existence of $\{x_n\}_n$ in $M(L^\infty)$ such that $x_n \notin \tilde{N}(f_n)$, $|(T_n f_n)(x_n) - f_n(x_n)| > \sigma$, $(T_n f_n)(x_n) \to a$, and $f_n(x_n) \to b$. Take c with $|b - c| \leq 1$ and $|a - c| \geq 1 + \sigma$. Find F_n in QC such that

$$\|F_n - f_n\|_\infty < \sigma/2 + 1 \quad \text{and} \quad F_n(x_n) = c.$$

By our assumption, we have

$$\limsup_{n\to\infty} \|F_n - T_n f_n\|_\infty < \sigma/2 + 1,$$

but

$$\|F_n - T_n f_n\|_\infty \geq |F_n(x_n) - (T_n f_n)(x_n)| \to |c - a| \geq 1 + \sigma.$$

This is a contradiction.

References

1. F. Altomare and M. Campiti, *Korovkin-type approximation theory and its applications*, W. de Gruyter, Berlin and New York, 1994. MR **95g:**41001
2. J. Garnett, *Bounded analytic functions*, Academic Press, New York and London, 1981. MR **83g:**30037
3. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N.J., 1962. MR **24:**A2844
4. K. Izuchi, *QC-level sets and quotients of Douglas algebras*, J. Funct. Anal. **65** (1986), 293-308. MR **87f:**46093
5. K. Izuchi, *Countably generated Douglas algebras*, Trans. Amer. Math. Soc. **299** (1987), 177-192. MR **88b:**46077
6. K. Izuchi, H. Takagi and S. Watanabe, *Sequential BKW-operators and function algebras*, J. Approx. Theory **85**(1996), 185–200. MR **97c:**46060

7. K. Izuchi, H. Takagi and S. Watanabe, *Sequential Korovkin type theorems and weighted composition operators*, Acta Sci. Math. (Szeged) **62** (1996), 161–174. CMP 97:02
8. P. P. Korovkin, *On convergence of linear positive operators in the space of continuous functions*, Dokl. Akad. Nauk SSSR (N.S) **90** (1953), 961-964. (in Russian) MR **15:**236a
9. P. P. Korovkin, *Linear operators and approximation theory*, Hindustan Publishing Corp., Delhi, 1960. MR **27:**561
10. D. Sarason, *Functions of vanishing mean oscillation*, Trans. Amer. Math. Soc. **207** (1975), 391-405. MR **51:**13690
11. D. Sarason, *Function theory on the unit circle*, Virginia Poly. and State Univ., Blacksburg, 1978. MR **80d:**30035
12. E. Scheffold, *Über die punktweise konvergenz von operatoren in $C(X)$*, Rev. Acad. Ci. Zaragoza **28**(1973), 5-12. MR **48:**948
13. S. -E. Takahasi, *(T, E)-Korovkin closures in normed spaces and BKW-operators*, J. Approx. Theory **82**(1995), 340-351. MR **96d:**41029
14. D. E. Wulbert, *Convergence of operators and Korovkin's theorem*, J. Approx. Theory **1** (1968), 381-390. MR **38:**3679

Department of Mathematics, Niigata University, Niigata 950-21, Japan
E-mail address: izuchi@scux.sc.niigata-u.ac.jp