ON POWER BOUNDED OPERATORS

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Abstract. In this paper we generalize the following consequence of a well-known result of Nagy: if $T$ and $T^{-1}$ are power bounded operators, then $T$ is a polynomially bounded operator.

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is called \textit{power bounded} (notation: $T \in \mathcal{PW}(\mathcal{H})$) if there exists a constant $M \geq 1$ such that
\[ \|T^n\| \leq M, \quad n \in \mathbb{N}, \] (1)
and $T$ is called \textit{polynomially bounded} (notation: $T \in \mathcal{PB}(\mathcal{H})$) if there exists a constant $M \geq 1$ such that
\[ \|p(T)\| \leq M\|p\|_\infty \] (2)
for every polynomial $p$, where $\|p\|_\infty = \sup\{|p(z)| : z \in \mathbb{C}, |z| \leq 1\}$. The smallest number $M$ satisfying (1) (resp., (2)) is called the \textit{power bound} (resp., the \textit{polynomial bound}) of $T$ and will be denoted by $M_w(T)$ (resp., $M_p(T)$), or simply $M_w$ (resp., $M_p$) when no confusion is possible. One knows (cf. [1, 2, 4]) that $\mathcal{PW}(\mathcal{H})$ strictly contains the class $\mathcal{PB}(\mathcal{H})$, but there is a theorem of Nagy [3] which says that every $T \in \mathcal{PW}(\mathcal{H})$ such that $T^{-1}$ exists and belongs to $\mathcal{PW}(\mathcal{H})$ is similar to a unitary operator, and therefore is polynomially bounded. The purpose of this note is to establish the following two stronger results than the above-mentioned consequence of Nagy’s theorem.

Theorem 1.1. Suppose $T \in \mathcal{PW}(\mathcal{H})$ (with $M_w(T) > 1$) and the following inequality holds for some positive number $\alpha$ and a strictly increasing sequence $\{n_k\} \subset \mathbb{N}$:
\[ 1/n_k \sum_{j=0}^{n_k} T^{*j}T^j \geq \alpha(I - P_{\ker(T)}), \] (3)
where $P_{\ker(T)}$ is the (orthogonal) projection on the kernel of $T$. Then $T \in \mathcal{PB}(\mathcal{H})$ and the polynomial bound $M_p$ of $T$ satisfies
\[ M_p \leq M_w(T)^3 \left( \frac{M_w(T)^2 - 1}{\alpha n M_w(T)} \right)^{1/2} + 1. \] (4)
Theorem 1.2. Suppose $T \in PW(H)$ and the following inequality holds for some positive number $\alpha$ and a strictly increasing sequence $\{t_k\}_{k \in \mathbb{N}}$ of real numbers converging to 1:

$$(1 - t_k) \sum_{j=0}^{\infty} t_k^j T^* T^j \geq \alpha (I - P_{\ker(T)}).$$

Then $T \in PB(H)$, and the polynomial bound $M_p$ of $T$ satisfies

$$M_p \leq \left( \frac{14}{\alpha} \right)^{1/2} M^3_w.$$  \hspace{1cm} (5)

As mentioned above, the following is an immediate consequence of either Theorem 1.1 or Theorem 1.2.

Corollary 1.3 (Nagy [4]). If $T \in PW(H)$ is invertible and $T^{-1} \in PW(H)$, then $T$ is polynomially bounded.

In order to prove Theorem 1.1, we use the following lemma, which is well known [5].

Lemma 1.4. Suppose $S \in \mathcal{L}(H)$ is such that $S^m$ is a contraction for some integer $m \geq 2$. Then $S$ is similar to a contraction, and, in particular,

$$A = (I + S^* S + ... + S^{(m-1)} S^{(m-1)})^{1/2}$$

is an invertible operator that satisfies

$$\|ASA^{-1}\| \leq 1.$$ \hspace{1cm} (7)

Proof. Clearly $A$ is an invertible selfadjoint operator. To establish (8) it is enough to check that

$$\|ASA^{-1} h\| \leq \|h\|, \quad h \in H.$$ \hspace{1cm} (9)

For a given $h$, define $g = A^{-1} h$, and hence (9) becomes equivalent to

$$\langle A^2 Sg, Sg \rangle \leq \langle A^2 g, g \rangle.$$ \hspace{1cm} (10)

Using (7), we see that (10) is equivalent to

$$\sum_{j=1}^{m} \|S^j g\|^2 \leq \sum_{j=0}^{m-1} \|S^j g\|^2,$$

which is true since $\|S^m g\| \leq \|g\|$.

Proof of Theorem 1.1. For brevity we write $M = M_w(T) > 1$. For each $n \in \mathbb{N}$, set $\alpha_n = M^{1/n}$, $\beta_n = \alpha_n^{-1}$, and note that $\beta_n < 1 < \alpha_n$. Since $\|(\beta_n T)^n\| \leq 1$ for each $n \in \mathbb{N}$, we may apply Lemma 1.4 to obtain for each $n \in \mathbb{N}$ a contraction $C_n$ such that

$$\beta_n T = A_n^{-1} C_n A_n,$$ \hspace{1cm} (11)

where $A_n = (\sum_{j=0}^{n-1} \beta_n^j T^{*j} T^j)^{1/2}$. Consider now an arbitrary polynomial $p(z) = a_0 + a_1 z + a_2 z^2 + ... + a_l z^l$. Then

$$p(T) = p(\alpha_n A_n^{-1} C_n A_n) = A_n^{-1} p(\alpha_n C_n) A_n.$$ \hspace{1cm} (12)
Applying the von Neumann inequality to $C_n$ and the polynomial $q_n(z) = p(\alpha_n z)$, we conclude from (12) that
\begin{equation}
\|p(T)\| \leq \|A_n^{-1}\| \|A_n\| \|q_n\|_{\infty}, \quad n \in \mathbb{N}.
\end{equation}
Let us observe now that for each $n \in \mathbb{N}$,
\begin{equation}
\|A_n\|^2 = \|A_n\| \leq \sum_{j=0}^{n-1} \beta_n^{2j} M^2 = M^2 (1 - \beta_n^{2n})/(1 - \beta_n^2) = M^2 (1 - M^{-2})/(1 - \beta_n^2),
\end{equation}
so
\begin{equation}
\|A_n\| \leq (M^2 - 1)^{1/2}/(1 - \beta_n^2)^{1/2}.
\end{equation}
Moreover, for each $n \in \mathbb{N}$, $\|A_n^{-1}\| = \gamma(A_n)^{-1}$, where $\gamma(A_n)$ is the greatest number $\gamma > 0$ with the property that $\|A_n h\| \geq \gamma \|h\|$ for all $h \in \mathcal{H}$. Equivalently,
\begin{equation}
\langle A_n^2 h, h \rangle \geq \gamma(A_n)^2 \langle h, h \rangle, \quad h \in \mathcal{H}, \quad n \in \mathbb{N}.
\end{equation}
Consider now the case that $\ker(T) = \{0\}$. Let $\{n_k\}$ be the sequence from (3). Then
\begin{equation}
A_{n_k}^2 = \sum_{j=0}^{n_k-1} \beta_{n_k}^{2j} T^* T^j \geq \beta_{n_k}^{2n_k} \sum_{j=0}^{n_k-1} T^* T^j \geq \beta_{n_k}^{2n_k} n_k \alpha I = n_k M^{-2} \alpha I, \quad k \in \mathbb{N}.
\end{equation}
Therefore, $\gamma(A_{n_k}) \geq n_k^{1/2} M^{-1} \alpha^{1/2}$ for each $k \in \mathbb{N}$, which implies that
\begin{equation}
\|A_{n_k}^{-1}\| = \gamma(A_{n_k})^{-1} \leq n_k^{-1/2} M \alpha^{-1/2}.
\end{equation}
Thus, from (14) and (15) we get
\begin{equation}
\|A_n\| \|A_n^{-1}\| \leq M(M^2 - 1)^{1/2} \alpha^{-1/2}/(n_k(1 - \beta_{n_k}^2))^{1/2}, \quad k \in \mathbb{N}.
\end{equation}
A simple continuity argument shows that for $p$ fixed we have
\begin{equation}
\lim_{n_k \to \infty} \|q_{n_k}\|_{\infty} = \|p\|_{\infty}.
\end{equation}
Going back to (13), and taking into account (16) and (17), we can let $k$ go to infinity and obtain the inequality
\begin{equation}
\|p(T)\| \leq M((M^2 - 1)/2 \ln M)^{1/2} \alpha^{-1/2} \|p\|_{\infty}
\end{equation}
by using the formula (from elementary calculus)
\begin{equation}
\lim_{n \to \infty} (M^{2/n} - 1)n = 2\ln M.
\end{equation}
Thus, in this case, $T \in \mathcal{PB}(\mathcal{H})$ and (4) is valid.

Let us consider now the general case. With respect to the decomposition $\mathcal{H} = (\ker T) \oplus (\ker T)^\perp$, $T$ has an operator matrix
\begin{equation}
T = \begin{bmatrix} 0 & S \\ 0 & Q \end{bmatrix}
\end{equation}
where $S : (\ker T)^\perp \to (\ker T)$ is a bounded linear operator, $Q \in \mathcal{PW}((\ker T)^\perp)$, and $M_w(Q) \leq M_w(T)$. For each polynomial $p$ one sees easily that
\begin{equation}
p(T) = \begin{bmatrix} p(0) I & S q(Q) \\ 0 & p(Q) \end{bmatrix}
\end{equation}
where \( q(z) = (p(z) - p(0))/z \). Therefore, since \( \|q\|_\infty \leq 2\|p\|_\infty \), it is sufficient to show that \( Q \) is polynomially bounded and has an appropriate polynomial bound. We want to use the first case, so let us observe that

\[
T^{*k}T^k \leq \begin{bmatrix}
0 & 0 \\
0 & (\|S\|^2 + \|Q\|^2)Q^{*k}Q^{k-1}
\end{bmatrix}, \quad k \in \mathbb{N}.
\]

But (3) and (20) together yield

\[
(\|S\|^2 + \|Q\|^2)/(n_k - 1) \sum_{j=0}^{n_k-1} Q^*Q^j \geq (\alpha - (\alpha + 1)/(n_k - 1))I_{(\ker T)^\perp}.
\]

In particular this says that if \( h \in \ker(Q) \cap (\ker T)^\perp \), then

\[
(\|S\|^2 + \|Q\|^2)/(n_k - 1)\langle h, h \rangle \geq (\alpha - (\alpha + 1)/(n_k - 1))\langle h, h \rangle,
\]

and letting \( k \) go to infinity we obtain that \( h = 0 \). Hence, \( Q \) satisfies the condition (3) in the case when \( \ker(Q) = \{0\} \) for \( \alpha' = (\alpha - \epsilon)/(\|S\|^2 + \|Q\|^2) > 0 \) and a subsequence \( \{n_k - 1\} \) for \( k \) large enough (depending upon \( \epsilon \)). Therefore, we obtain from the previous case,

\[
\|p(Q)\| \leq M((M^2 - 1)/2lnM)^{1/2}\alpha^{-1/2}(\|S\|^2 + \|Q\|^2)^{1/2}\|p\|_\infty,
\]

since \( \epsilon > 0 \) was arbitrary. Finally we get

\[
\|p(T)\| \leq (M((M^2 - 1)/2lnM)^{1/2}\alpha^{-1/2}(\|S\|^2 + \|Q\|^2)^{1/2}\|S\| + 1)\|p\|_\infty
\leq (M^3((M^2 - 1)/lnM)^{1/2}\alpha^{-1/2} + 1)\|p\|_\infty,
\]

which is what we wanted to show. \( \square \)

We want to consider now the continuous analog of Theorem 1.1.

**Proof of Theorem 1.2.** Let us define for \( T \in \mathcal{PW}(\mathcal{H}) \) and every \( t \in [0,1) \) the self-adjoint invertible operator

\[
A_t = (1-t)^{1/2}(\sum_{j=0}^{\infty} t^jT^*JT^j)^{1/2}.
\]

First, observe that this operator is well-defined for \( T \in \mathcal{PW}(\mathcal{H}) \), and moreover

\[
\|A_t\|^2 = \|A_t^2\| \leq (1-t)^{\sum_{j=0}^{\infty} t^j\|T^*JT^j\|} \leq M_w(T)^2.
\]

As before, let us consider the case when \( \ker(T) = \{0\} \). If (5) is satisfied, then \( \|A_t^{-1}\| \leq \alpha^{-1/2} \) at least for \( t = t_k \).

Now observe that for any \( h \in \mathcal{H} \) we have

\[
(1-t)\|A_t^{-1}h\|^2 + t\|A_tTA_t^{-1}h\|^2 = \|h\|^2,
\]

which, in particular, says that \( t^{1/2}A_tTA_t^{-1} \) is a contraction. Hence we can use the idea from the proof of Theorem 1.1 to get that

\[
\|p(T)\| \leq \|A_{t_k}\|\|A_{t_k}^{-1}\|\|q_k\|_\infty, \quad k \in \mathbb{N},
\]

where \( q_k(z) = p(t_k^{-1/2}z) \) for any given polynomial \( p \). Letting \( k \) go to infinity we get the inequality

\[
\|p(T)\| \leq M_w(T)\alpha^{-1/2}\|p\|_\infty,
\]
which is what we wanted to show in the case \( \ker(T) = 0 \). In the general case, if \( T \) has the decomposition (19), by using the inequality (20) and the hypothesis (5), we have that

\[
(\|S\|^2 + \|Q\|^2)(1 - t_k) \sum_{j=1}^{\infty} t_j^k Q^{j-1} Q^{j-1} \geq (\alpha - 1 + t_k) I_{(\ker T)^\perp},
\]

which says, first, that \( \ker(Q) = 0 \) and thus that \( Q \) is as in the first case. Therefore, we finally get

\[
\|p(T)\| \leq M_w(T) \left\{ (3 + 4\|S\|^2) \left( \frac{\|S\|^2 + \|Q\|^2}{\alpha} \right) \right\}^{1/2} \|p\|_\infty
\]

\[
\leq \left( \frac{14}{\alpha} \right)^{1/2} M_w(T) \|p\|_\infty,
\]

which was to be proved.

An easy corollary of Theorem 1.2 is the following generalization.

**Corollary 1.5.** Suppose \( T \in PW(\mathcal{H}) \) and the following inequality holds for some \( n \in \mathbb{N} \), some positive number \( \alpha \), and a strictly increasing sequence \( \{t_k\}_{k \in \mathbb{N}} \) of real numbers converging to \( 1 \):

\[
(1 - t_k) \sum_{j=0}^{\infty} t_j^k T^* j T^j \geq \alpha (I - P_{\ker(T^n)}).
\]

(23)

Then \( T \) is polynomially bounded.

**Proof.** With respect to the decomposition \( \mathcal{H} = \ker(T^n) \oplus (\ker(T^n))^\perp \), \( T \) has the operator matrix

\[
T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.
\]

Since \( \ker(T^n) \) is an invariant subspace for \( T \), the operator \( C \) must be zero. In addition, we have that

\[
T^n = \begin{bmatrix} 0 & E \\ 0 & F \end{bmatrix},
\]

where

(24) \( A^n = 0, \ F = D^n, \ E = \sum_{j=0}^{j=n} A^j BD^{n-j} \).

Now, for an arbitrary operator \( T, T \in PB(\mathcal{H}) \) if and only if \( T^n \in PB(\mathcal{H}) \) for some \( m \in \mathbb{N} \). This can be easily seen if we observe that for any polynomial \( p \), there exists a unique decomposition of the form

\[
p(z) = p_1(z) + z p_2(z) + z^2 p_3(z) + \ldots + z^{m-1} p_m(z).
\]
where \( p_1, p_2, p_3, \ldots, p_m \) are polynomials in \( z^m \) and \( \| p_j \|_\infty \leq \| p \|_\infty \) for \( j = 1, 2, \ldots, m \). Hence, it suffices to show that \( F \) and therefore \( D \) is polynomially bounded. But now, since for any integer \( k \geq 0 \)

\[
(I - P_{\ker(T^n)})T^s k T^k (I - P_{\ker(T^n)}) = \begin{bmatrix} 0 & 0 \\ 0 & D^s k D^k \end{bmatrix},
\]

which follows from (23) multiplying from the left and from the right by \( I - P_{\ker(T^n)} \), we obtain that \( D \) satisfies the hypothesis of Theorem 1.1. This means that \( D \) is polynomially bounded, and so \( F \) and \( T \) are also.

**Comments.** If we start with a contraction \( T \), let us show that the function \( A_t \) defined in (22) satisfies \( A_t \geq A_s \) for \( 0 \leq t < s \leq 1 \). Indeed, since \( A^2 \geq B^2 \) for positive simidefinite operators implies \( A \geq B \), it is enough to check that \( A^2_t \geq A^2_s \) \((t < s)\). This is equivalent to

\[
(1 - t) \sum_{j=0}^{\infty} v^j \| T^j h \|^2 \geq (1 - s) \sum_{j=0}^{\infty} s^j \| T^j h \|^2, \quad h \in \mathcal{H},
\]

and this can be written in the following equivalent form which is clearly true for \( T \) a contraction:

\[
(s - t)(\| h \|^2 - \| T h \|^2) + (s^3 - t^3)(\| T^2 h \|^2 - \| T^3 h \|^2) + (s^5 - t^5)(\| T^3 h \|^2 - \| T^4 h \|^2) + \ldots \geq 0.
\]

Therefore, it is interesting to ask: what is the class of operators for which the function \( t \to A_t \) is decreasing? In this connection one can easily prove, using ideas similar to those above, the following.

**Theorem 1.6.** Suppose \( T \in \mathcal{PW}(\mathcal{H}) \), the positive-operator-valued function

\[
t \to (1 - t)^{1/2} \left( \sum_{j=0}^{\infty} v^j T^j T^j \right)^{1/2}, \quad t \in [\varepsilon, 1), \ 0 < \varepsilon < 1,
\]

is decreasing, and the inequality (5) holds for some positive number \( \alpha \) and a strictly increasing sequence \( \{ t_k \} \) of real numbers converging to 1. Then \( T \) is similar to a contraction.

Another natural question is whether we can weaken the assumption (23) to

\[
(1 - t_k) \sum_{j=0}^{\infty} v_k^j T^j T^j \geq \alpha (I - P_{\cup_{n=0}^{\infty} \ker(T^n)}),
\]

and preserve the conclusion in Corollary 1.5. The same counterexample of Foguel in [2] shows that there exists an operator satisfying (25) which is not polynomially bounded.

**Remark.** This paper constitutes part of the author’s Ph.D. thesis written at Texas A&M University under the direction of Professor Carl Pearcy. We thank the referee of this paper who suggested the idea for improving Corollary 1.5.

**Added in proof.** Vern Paulsen has pointed out that the same techniques employed herein can be used to obtain the stronger result that under the hypotheses of Theorem 1.1 or 1.2, \( T \) is completely polynomially bounded, and thus is similar to a contraction.

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