THE EXPECTED VALUE OF THE NUMBER OF REAL ZEROS OF A RANDOM SUM OF LEGENDRE POLYNOMIALS

J. ERNEST WILKINS, JR.

(Communicated by Richard T. Durrett)

Abstract. It is known that the expected number of zeros in the interval \((-1, 1)\) of the sum \(a_0 \psi_0(t) + a_1 \psi_1(t) + \cdots + a_n \psi_n(t)\), in which \(\psi_k(t)\) is the normalized Legendre polynomial of degree \(k\) and the coefficients \(a_k\) are independent normally distributed random variables with mean 0 and variance 1, is asymptotic to \(3^{-1/2}n\) for large \(n\). We improve this result and show that this expected number is \(3^{-1/2}n + o(n^\delta)\) for any positive \(\delta\).

1. Introduction

Let \(k\) be a nonnegative integer, \(P_k(t)\) be the Legendre polynomial of degree \(k\), and \(\psi_k(t)\) be the normalised Legendre polynomial \((k+\frac{1}{2})^\frac{1}{2}P_k(t)\). Let \(n\) be a positive integer, and consider the random sum

\[ F(t) = \sum_{k=0}^{n} a_k \psi_k(t), \]

in which the coefficients \(a_k\) are independent normally distributed random variables with mean 0 and variance 1. If \(\nu_n\) is the expected value of the number of zeros of \(F(t)\) on the interval \((-1, 1)\), then Das [1] has shown that \(\nu_n \sim 3^{-1/2}n\) for large \(n\). (In fact, his analysis indicates that \(\nu_n = 3^{-1/2}n[1 + O((\log n)^{-3})]\). In this paper we will prove the somewhat better result that

\[ \nu_n = 3^{-1/2}n + o(n^\delta) \]

for any positive \(\delta\). Our analysis is similar to that of Das, but requires a more detailed treatment of the asymptotic expansion for \(P_n(t)\) when \(n\) is large.

2. Preliminary Analysis

Let \(\nu_n(a, b)\) be the expected number of zeros of \(F(t)\) on the subinterval \((a, b)\) of \((-1, 1)\). We know ([1] or [2, p. 111]) that

\[ \nu_n(a, b) = \pi^{-1} \int_{a}^{b} \left\{ A_n(t)C_n(t) - B_n^2(t) \right\}^{1/2} A_n^{-1}(t) \, dt, \]

\[ A_n(t) = P'_{n+1}(t)P_n(t) - P'_n(t)P_{n+1}(t), \]

Received by the editors November 1, 1995.

1991 Mathematics Subject Classification. Primary 60G99; Secondary 41A60.

Key words and phrases. Real zeros, random polynomials, Legendre polynomials.

©1997 American Mathematical Society
Stieltjes [3, 4] (or see [5, p. 195, Th. 8.21.5]) has shown that, if \(0 < \alpha < \pi\)

\[
2B_n(t) = P''_{n+1}(t)P_n(t) - P''_n(t)P_{n+1}(t),
\]

(5)

\[
6C_n(t) = 3\{P''_{n+1}(t)P'_n(t) - P''_n(t)P'_{n+1}(t)\} + \{P''_{n+1}(t)P_n(t) - P''_n(t)P_{n+1}(t)\}.
\]

Stieltjes [3, 4] (or see [5, p. 195, Th. 8.21.5]) has shown that, if \(0 < \alpha < \pi\) and \((n \sin \alpha)^{-1} = o(1)\),

(6)

then

\[
P_n(\cos \alpha) = \left\{ \frac{2}{(\pi \sin \alpha)} \right\}^{\frac{1}{2}} \left[ \sum_{s=0}^{m-1} \frac{\gamma_s n! \cos \beta_{ns}}{(2 \sin \alpha)^s \Gamma(n + s + 3/2)} + O((n \sin \alpha)^{-m}) \right],
\]

in which \(\gamma_s = \{\Gamma(s + \frac{1}{2})\}^2/(\pi s!)\), \(\beta_{ns} = \beta_{ns}(\alpha) = (n + s + \frac{1}{2})\alpha - \frac{1}{2}(s + \frac{1}{2})\pi\), and \(m\) is any positive integer. Moreover, the upper bound implicit in the \(O\) symbol depends only on \(m\), not only in (7), but in the later identities (9), (10), (11), (14), (16), (18), (21), (22), (23) and (24). If we define \(G_{kn}(\alpha)\) so that

(8)

\[
G_{kn}(\alpha) = \sum_{s=0}^{k} \gamma_s D_{k-s}(1, s + 3/2)(2 \sin \alpha)^{k-s} \cos \beta_{ns},
\]

in which the coefficients \(D_h(x, y)\) are those that appear in the asymptotic expansion [6, p. 119, Eq. 5.02]

\[
n^{x-y} \Gamma(n + x)/\Gamma(n + y) = \sum_{h=0}^{n-1} D_h(x, y)n^{-h} + O(n^{-p}),
\]

then \(G_{kn} = G_{kn}(\alpha)\) is uniformly bounded in \(n\) and \(\alpha\), and we deduce from (7) that

(9)

\[
P_n(\cos \alpha) = \left\{ \frac{2}{(n \pi \sin \alpha)} \right\}^{\frac{1}{2}} \left[ \sum_{k=0}^{m-1} (2n \sin \alpha)^{-k} G_{kn} + O(n^{-m} \sin^{-m} \alpha) \right].
\]

Because \(D_0(x, y) = 1\), we see that \(G_{n0}(\alpha) = \cos \beta_{n0}\). Moreover,

(10)

\[
P_{n+1}(\cos \alpha) = \left\{ \frac{2}{(n \pi \sin \alpha)} \right\}^{\frac{1}{2}} \left[ \sum_{k=0}^{m-1} (2n \sin \alpha)^{-k} H_{kn} + O(n^{-m} \sin^{-m} \alpha) \right],
\]

(11)

\[
P_{n-1}(\cos \alpha) = \left\{ \frac{2}{(n \pi \sin \alpha)} \right\}^{\frac{1}{2}} \left[ \sum_{k=0}^{m-1} (2n \sin \alpha)^{-k} J_{kn} + O(n^{-m} \sin^{-m} \alpha) \right],
\]

(12)

\[
H_{kn} = \sum_{s=0}^{k} -s-\frac{1}{2} C_{k-s}(2 \sin \alpha)^{k-s} G_{s,n+1},
\]

(13)

\[
J_{kn} = \sum_{s=0}^{k} -s-\frac{1}{2} C_{k-s}(-2 \sin \alpha)^{k-s} G_{s,n-1},
\]

in which \(qC_j = \Gamma(j + 1)/\{\Gamma(q + 1)\Gamma(j - q + 1)\}\). The functions \(H_{kn}\) and \(J_{kn}\) are uniformly bounded in \(n\) and \(\alpha\), and \(H_{on} = \cos \beta_{n+1,0}, J_{on} = \cos \beta_{n-1,0}\).

We use the identity [7, p. 309, eq. V]

\[
(1 - t^2)P'_n(t) = n\{P_{n-1}(t) - tP_n(t)\}
\]
in conjunction with (9) and (11) to see that

\begin{equation}
P'_{n}(\cos \alpha) = \left\{ \frac{2n}{(\pi \sin^{3} \alpha)} \right\} \frac{1}{2} \sum_{k=0}^{m-1} (2n \sin \alpha)^{-k} K_{kn} + O(n^{-m} \sin^{-m-1} \alpha),
\end{equation}

\begin{equation}
L_{kn} = \sum_{s=0}^{k} \frac{1}{2-s} C_{k-s} (2 \sin \alpha)^{k-s} K_{s,n+1}.
\end{equation}

Therefore, \( L_{kn} \) is uniformly bounded in \( n \) and \( \alpha \), and \( L_{on} = \sin \beta_{n+1,o} \).

It now follows from (3), (9), (10), (14) and (16) that, if \( t = \cos \alpha \),

\begin{equation}
A_{n}(t) = \left\{ \frac{2n}{(\pi \sin^{3} \alpha)} \right\} \frac{1}{2} \sum_{k=0}^{m-1} (2n \sin \alpha)^{-k} M_{kn} + O(n^{-m} \sin^{-m-2} \alpha),
\end{equation}

\begin{equation}
M_{kn} = \sum_{s=0}^{k} (L_{sn} G_{k-s,n} - K_{sn} H_{k-s,n})/\sin \alpha.
\end{equation}

We deduce from (17), (8) and (12) that

\[ L_{sn} G_{k-s,n} - K_{sn} H_{k-s,n} \equiv \gamma_{k-s}(K_{s,n+1} \cos \beta_{n,k-s} - K_{sn} \cos \beta_{n+1,k-s}) \]

\[ \equiv \gamma_{k-s}(K_{s,n+1} - K_{sn} \cos \alpha) \cos \beta_{n,k-s} \].

We infer from (15), (13) and (8) that \( K_{sn} = U'_{sn} + U''_{sn} \), in which

\[ U'_{sn} = (J_{sn} - G_{s,n-1})/\sin \alpha = \sum_{k=0}^{s-1} (-2 \sin \alpha)^{s-1-k} G_{k,n-1} \]

\[ = -(s - \frac{1}{2}) G_{s-1,n-1} - (s - \frac{1}{2}) \cos \beta_{n-1,s-1}, \]

\[ U''_{sn} = (G_{s,n-1} - G_{sn} \cos \alpha)/\sin \alpha \]

\[ = \sum_{k=0}^{s} \gamma_{r} D_{s-k}(1, k + 3/2)(2 \sin \alpha)^{s-k} \sin \beta_{nk} \equiv \gamma_{s} \sin \beta_{ns}. \]

We next observe that

\[ U'_{s,n+1} - U'_{sn} \cos \alpha = -(s - \frac{1}{2})(\cos \beta_{n,s-1} - \cos \beta_{n-1,s-1} \cos \alpha) \]

\[ = (s - \frac{1}{2}) \sin \beta_{n-1,s-1} \sin \alpha \equiv 0, \]

\[ U''_{s,n+1} - U''_{sn} \cos \alpha \equiv \gamma_{s}(\sin \beta_{n+1,s} - \sin \beta_{ns} \cos \alpha) \]

\[ = \gamma_{s} \cos \beta_{ns} \sin \alpha \equiv 0. \]
It follows from (19) that $M_{kn}$ is uniformly bounded in $n$ and $\alpha$, and $M_{on} = 1$. It is a consequence of the differential equation [7, p. 304]

$$(1 - t^2)P_n''(t) - 2tP_n'(t) + n(n + 1)P_n(t) = 0,$$

satisfied by $P_n(t)$, and the definitions (3) and (4) that

$$(1 - t^2)B_n(t) = tA_n(t) - (n + 1)P_n(t)P_{n+1}(t).$$

It then follows from (18), (9) and (10) that

$$B_n(t) = \sum_{k=0}^{m-1} \frac{2}{(\pi \sin^2 \alpha)} (2n \sin \alpha)^{-k} N_{kn} + O(n^{-m} \sin^{-m-2} \alpha),$$

in which $N_{kn}$ is a function, uniformly bounded in $n$ and $\alpha$, whose explicit expression is not needed.

An additional consequence of (20) is that (it is convenient to suppress the dependence on $t$ of the Legendre polynomials and their derivatives)

$$(1 - t^2)(P_{n+1}'' - P_n'' P_{n+1}) = (n + 1)(nA_n - 2P_{n+1}P_n').$$

If we differentiate (20) and use the definitions (3) and (4), we find that

$$(1 - t^2)(P_{n+1}'' - P_n'' P_{n+1}) = n^2A_n + 8tB_n + (n - 2)P_{n+1}'P_n - 3nP_{n+1}'P_n.$$ 

It then follows from (5) and the last two equations that

$$6(1 - t^2)C_n = (2n^2 + 3n)A_n + 8tB_n - (5n + 8)P_{n+1}'P_n - 3nP_{n+1}'P_n.$$ 

An appeal to (18), (21), (10), (14), (16) and (7) shows that

$$C_n(t) = \sum_{k=0}^{m-1} \frac{2n^2}{(3\pi \sin^3 \alpha)} (2n \sin \alpha)^{-k} Q_{kn} + O(n^{-m} \sin^{-m-2} \alpha),$$

in which $Q_{kn}$ is a function, uniformly bounded in $n$ and $\alpha$, whose explicit expression is not needed, except for the case $Q_{on} = 1$.

With the help of (18), (21) and (22), we now find that

$$\{A_n(t)C_n(t) - B_n^2(t)\}^{1/2} A_n^{-1}(t)$$

$$= \left( \frac{3^{1/2}}{n \sin \alpha} \right) \sum_{k=0}^{m-1} (2n \sin \alpha)^{-k} R_{kn} + O(n^{-m} \sin^{-m-2} \alpha),$$

in which $R_{kn}$ is a function, uniformly bounded in $n$ and $\alpha$, whose explicit expression is not needed, except for the case $R_{on} = 1$.

3. Proof of (1)

Suppose that $\varepsilon = n^{-2m/(m+4)}$, and that $|t| \leq 1 - \varepsilon$. Then $n \sin \alpha = n(1 - t^2)^{1/2} > n\varepsilon^{1/2} = n^{4/(m+4)}$. Hence (6) is true. We conclude from (2) and (23) that

$$\nu_n(-1 + \varepsilon, 1 - \varepsilon) = 2\nu_n(0, 1 - \varepsilon)$$

$$= \left\{ \frac{2n}{(3\pi \sin \alpha)} \right\} \int_0^{1 - \varepsilon} [(1 - t^2)^{-1/2} + \sum_{k=1}^{m-1} O(n^{-k}(1 - t^2)^{-(k+1)/2})$$

$$+ O(n^{-m}(1 - t^2)^{-(m+5)/2})] dt.$$
We observe that
\[
\int_0^{1-\varepsilon} (1-t^2)^{-\frac{1}{2}} dt = \frac{\pi}{2} - \cos^{-1}(1-\varepsilon) = (\pi/2) + O(\varepsilon^{\frac{3}{2}}),
\]
\[
\int_0^{1-\varepsilon} (1-t^2)^{-1} dt = \frac{1}{2} \log \{ (2-\varepsilon)/\varepsilon \} = O(\log \varepsilon^{-1}),
\]
\[
\int_0^{1-\varepsilon} (1-t^2)^{-3/2} dt = (1-\varepsilon) (2\varepsilon - \varepsilon^2)^{-\frac{1}{2}} = O(\varepsilon^{-\frac{5}{2}}).
\]
Because $1-t^2 > \varepsilon$ when $0 \leq t \leq 1-\varepsilon$, we see that
\[
\int_0^{1-\varepsilon} (1-t^2)^{(-h-\frac{1}{2})} dt < \varepsilon^{(1-h)} \int_0^{1-\varepsilon} (1-t^2)^{-3/2} dt = O(\varepsilon^{-h+\frac{3}{2}})
\]
when $h \geq 1$. Therefore,
\[
\nu_n(-1+\varepsilon, 1-\varepsilon) = 3^{-\frac{1}{4}} n [1 + O(\varepsilon^{\frac{1}{2}}) + O(n^{-1} \log \varepsilon^{-1})]
\]
\[
+ \varepsilon\frac{1}{2} \sum_{k=2}^{m-1} O\{ (n\varepsilon^{\frac{1}{4}})^{-k} \} + O(n^{-m}(m+3)/2)],
\]
(24)
\[
\nu_n(-1+\varepsilon, 1-\varepsilon) = 3^{-\frac{1}{4}} n [1 + O\{ n^{-m/m+4}] \}].
\]
Let $\mu(\varepsilon)$ be the number of complex zeros of $F(t)$ in the circle $|t-1| < \varepsilon$. Then $\nu_n(1-\varepsilon, 1)$ does not exceed the expected value of $\mu(\varepsilon)$. It follows from Jensen’s Theorem [8, p. 187, Eq. 25]
\[
\mu(\varepsilon) \leq (2\pi)^{-1} \int_0^{2\pi} \log_2 |F(1+2\varepsilon e^{i\theta})/F(1)| d\theta.
\]
(25)
We next use the identity [7, p. 312],
\[
P_k(z) = \pi^{-1} \int_0^{\pi} \{z + (z^2-1)\frac{1}{2} \cos \varphi\}^k d\varphi,
\]
to see that
\[
|P_k(1+2\varepsilon e^{i\theta})| \leq \{ 1 + 2\varepsilon + 2(\varepsilon + \varepsilon^2)^{\frac{1}{2}} \}^k < (1 + A\varepsilon^{\frac{1}{2}})^k,
\]
in which $A = 2(1 + 2^{\frac{1}{2}}) < 5$. Therefore, when $0 \leq k \leq n$,
(26)
\[
|P_k(1+2\varepsilon e^{i\theta})| < \exp \{ n \log (1 + 5\varepsilon^{\frac{1}{2}}) \} < \exp (5n\varepsilon^{\frac{1}{2}}).
\]
The Chebyshev inequality [9, p. 219, Eq. 61] shows that $Prob(|a_k| \leq n) > 1 - n^{-2}$ for each $k$, such that
(27)
\[
Prob(|a_k| \leq n \text{ when } 0 \leq k \leq n) > 1 - (n+1)n^{-2} \geq 1 - 2n^{-1}.
\]
Because the Schwarz inequality implies that
\[
\sum_{k=0}^{n} \left( k + \frac{1}{2} \right)^{\frac{1}{2}} \leq \left \{ \left( n+1 \right)^{\frac{1}{2}} \sum_{k=0}^{n} \left( k + \frac{1}{2} \right) \right \}^{\frac{1}{2}} = \left \{ (n+1)^{3/2} \right \}^{\frac{1}{2}} = \sqrt{n^{3/2}}
\]
when $n \geq 4$, it follows from (26) and (27) that
(28)
\[
Prob\{ |F(1+2\varepsilon e^{i\theta})| < n^{5/2} \exp (5n\varepsilon^{\frac{1}{2}}) \} > 1 - 2n^{-1}.
\]
Moreover, $F(1)$ is a normally distributed random variable with mean 0 and variance $\zeta^2 = (n+1)^2/2$. Therefore,

\begin{equation}
\text{Prob}\{|F(1)| < 1\} = (2\pi\zeta^2)^{-\frac{1}{2}} \int_{-1}^{1} \exp\left(-u^2/2\zeta^2\right) du < 2\pi^{-\frac{1}{2}}/(n+1) < 2n^{-1}.
\end{equation}

We infer from (25), (28) and (29) that $\mu(\varepsilon) < \log_2\{n^{5/2}\exp(5\varepsilon^2)\}$ with probability greater than $1 - 4n^{-1}$. Because $\mu(\varepsilon) \leq n$ for all $F(t)$, we see that the expected value of $\mu(\varepsilon)$ is, and so also $\nu_n(1-\varepsilon, 1)$ and $\nu_n(-1, -1+\varepsilon)$ are, $O(\log n) + O(n^{\varepsilon^2}) + O(1) = O(n^{4/(m+4)})$. When this result is combined with (24), we find that $\nu_n = \nu_n(-1, 1) = 3^{-\frac{1}{2}}n + O\{n^{4/(m+4)}\}$. If $\delta$ is any positive number and we choose the integer $m$ so large that $m+4 > 4/\delta$, we can finally conclude that (1) is true, in the sense that for any positive $\eta$ and $\delta$ there exists an integer $n(\eta, \delta)$ such that $|\nu_n - 3^{-\frac{1}{2}}n|n^{-\delta} < \eta$ when $n > n(\eta, \delta)$.

**References**