EXTREME POINTS OF UNIT BALLS IN LIPSCHITZ FUNCTION SPACES

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Abstract. We give a new characterization of the set ext \((B_X^\#)\) of all extreme points of the unit ball \(B_X^\#\) in the Banach space \(X^\#\) of all Lipschitz functions on a metric space \(X\). This result is applied to get a total variation characterization of ext \((B_X^\#)\) in the particular case when \(X\) is a convex subset of a Banach space.

Let \(0 \in X\) be an arbitrarily chosen point of a metric space \(X = (X, d)\) which consists of at least two distinct points. Following Lindenstrauss [3] denote by \(X^\#\) the Banach space of all functions \(f : X \to \mathbb{R}\) such that \(f(0) = 0\) and

\[
\|f\| = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X, x \neq y \right\} < \infty.
\]

In other words, the Banach space \(X^\#\) consists of all real-valued Lipschitz functions defined on \(X\), which are equal zero at the distinguished point 0. In the following, we always assume that the distinguished point 0 is equal to the origin of the Banach space \(E\), whenever \(X\) is a subset of \(E\) containing the origin of \(E\).

In the study of geometric Banach space theory and its various applications it is important to have a good characterization of the extreme points of unit balls. The investigation of the set of all extreme points \(\text{ext}(B_X^\#)\) of the unit ball \(B_X^\#\) of \(X^\#\) has been originated by Rolewicz [4] who has proved the following theorem.

**Theorem A.** Let \(f\) be a function in \([0, 1]^\#\) with \(\|f\| = 1\). Then \(f \in \text{ext}(B_{[0, 1]^\#})\) if and only if \(|f'(x)| = 1\) a.e. on \([0, 1]\).

Moreover, he has shown in [5] that a similar result cannot hold for the space \(X = [0, 1] \times [0, 1]\) with Euclidean metric. Next, Cobzas [1] has characterized the extreme points in \(X^\#\) for a rather restricted class of metric spaces \(X\). Recently, Farmer [2] has presented a new characterization of the set \(\text{ext}(B_X^\#)\) without any additional restrictions on \(X\). More precisely, he proved the following theorem.

**Theorem B.** Let \(X\) be a metric space, and let \(f\) be a function in \(X^\#\) with the norm \(\|f\| = 1\). Then \(f \in \text{ext}(B_X^\#)\) if and only if (i) \(\epsilon_x^f = 0\) for all \(x, y \in X\),
where
\[
\epsilon^f_{x,y} = \inf \left\{ \epsilon > 0 : d(x_{i-1}, x_i) - \epsilon_i \leq |f(x_{i-1}) - f(x_i)| \ (i = 1, \ldots, n) \right\}
\]

with the infimum taken over all finite sequences \(\epsilon_1, \ldots, \epsilon_n > 0\) and \(x_1, \ldots, x_{n-1} \in X\) satisfying the above inequalities.

Moreover, he noted that condition (i) is equivalent to the condition
(ii) \(\epsilon^f_{x,0} = 0\) for every \(x \in X\),
which is an immediate consequence of the triangle inequality
\[
(1)
\]
In this paper, we first apply Theorem B to derive a new characterization of \(\text{ext}(B_X^\#)\). Next, we use this result to obtain the following

**Theorem 1.** Let \(X\) be a convex subset of a normed linear space \(E = (E, \|\cdot\|)\), and let \(f\) be a function in \(X^\#\) such that \(\|f\| = 1\). Then \(f \in \text{ext}(B_X^\#)\) if and only if

\[
(i) \quad \inf \left\{ \sum_{i=1}^{n} \left( \|x_i - x_{i-1}\| - \int_{0}^{1} |f'_{x_i,x_{i-1}}(t)| dt \right) : x_0 = x, x_n = y \right\} = 0
\]

for all \(x, y \in X\), where the infimum is taken over all finite sequences \(x_1, \ldots, x_{n-1} \in X\), and
\[
f_{x_i,x_{i-1}}(t) = f((1 - t)x_{i-1} + tx_i), \ 0 \leq t \leq 1.
\]

For this purpose, let
\[
\langle x, y \rangle = \{ z \in X : d(x, y) = d(x, z) + d(z, y) \}
\]
be the metric interval with endpoints \(x, y \in X\). Additionally, let \((x_i)_{0}^{n}\) be a metric subdivision of \(\langle x, y \rangle\) with \(x \neq y\), i.e., let \(x_0 = x, x_n = y, x_i \in \langle x, y \rangle, x_i \neq x_j\) for \(i \neq j\), and
\[
(2) \quad d(x, y) = \sum_{i=1}^{n} d(x_{i-1}, x_i).
\]

Then we define
\[
(3) \quad \rho_f(x, y) = \inf \left\{ \|f\| \ d(x, y) - \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \right\},
\]

where the infimum is taken over all finite metric subdivisions \((x_i)_{0}^{n}\) of the interval \(\langle x, y \rangle\). Additionally, we put \(\rho_f(x, x) = 0\). Since points \(x_0 = x\) and \(x_1 = y\) form a subdivision of \(\langle x, y \rangle\), it follows from (3) that
\[
(4) \quad \rho_f(x, y) \leq \|f\| \ d(x, y) - |f(x) - f(y)|
\]
for all \(x, y \in X\). Further, we have
\[
\rho_f(x, y) \leq \|f\| \ d(x, y) - \sum_{i=1}^{n+m} |f(x_i) - f(x_{i-1})|.
\]
for all metric subdivisions \((x_i)_{i=0}^n\) of \((x, z)\) and \((x_i)_{i=n+1}^{n+m}\) of \((z, y)\), where \(z \in (x, y)\).
Hence one can take the first infimum over \(x_1, \ldots, x_{n-1}\) and the second over \(x_{n+2}, \ldots, x_{n+m-1}\) to get
\[
\rho_f(x, y) \leq \rho_f(x, z) + \rho_f(z, y), \quad z \in (x, y).
\]
In general, \(\rho_f\) does not satisfy the triangle inequality. For example, let \(\|x\|_p\) (\(1 < p < \infty\)) denote \(l^p\)-norm of \(x = (x_1, x_2) \in X = \mathbb{R}^2\). Then we have
\[
\rho_f(x, y) = \|x - y\|_p - |x_1 - y_1|
\]
for the function \(f(x) = x_1\). Hence we get
\[
1 = \rho_f(x, y) > \rho_f(x, z) + \rho_f(z, y) = 2^{\frac{1}{p}} - 1,
\]
whenever \(x = (0, 0), y = (0, 1)\) and \(z = (1, 0)\).
In view of this example, we define
\[
\sigma_f(x, y) = \inf \{\rho_f(x, z_1) + \rho_f(z_1, z_2) + \cdots + \rho_f(z_n, y) : z_1, \ldots, z_n \in X, n \in \mathbb{N}\}
\]
for all \(x, y \in X\) and \(f \in X^\#\). Clearly, \(\sigma_f\) is a symmetric function such that \(\sigma_f(x, x) = 0\) and
\[
0 \leq \sigma_f \leq \rho_f.
\]
In particular, this together with (4) gives
\[
|f(x) - f(y)| \leq \|f\|d(x, y) - \sigma_f(x, y); \quad x, y \in X.
\]
Further, taking the infimum over \((z_i)_{i=1}^n\) and \((y_i)_{i=1}^m\) of the right-hand side of the inequality
\[
\sigma_f(x, y) \leq [\rho_f(x, z_1) + \rho_f(z_1, z_2) + \cdots + \rho_f(z_n, z)]
+ [\rho_f(z, y_1) + \rho_f(y_1, y_2) + \cdots + \rho_f(y_m, y)],
\]
we derive
\[
\sigma_f(x, y) \leq \sigma_f(x, z) + \sigma_f(z, y),
\]
and therefore
\[
|\sigma_f(x, y) - \sigma_f(x, z)| \leq \sigma_f(y, z)
\]
for all \(x, y, z \in X\). Note also that
\[
\sigma_f \leq \mu \leq \rho_f \implies \sigma_f = \mu,
\]
whenever the function \(\mu : X \times X \to \mathbb{R}\) satisfies the triangle inequality on \(X\). Indeed, note that
\[
\rho_f(x, z_1) + \rho_f(z_1, z_2) + \cdots + \rho_f(z_n, y) \geq \mu(x, z_1) + \mu(z_1, z_2)
+ \cdots + \mu(z_n, y) \geq \sigma_f(x, y),
\]
and take the infimum over \((z_i)_{i=0}^n\) to get \(\sigma_f = \mu\).

**Theorem 2.** Let \(X\) be a metric space, and let \(f\) be a function in \(X^\#\) with the norm \(\|f\| = 1\). Then \(f \in \text{ext}(B_{X^\#})\) if and only if
\[
(i) \quad \sigma_f(x, y) = 0 \quad \text{for all} \quad x, y \in X.
\]
Proof. Suppose first that $\sigma_f (x, y) = 0$ for all $x, y \in X$. Moreover, take an arbitrary $\epsilon > \rho_f (x, y)$. Then it follows from (2) – (3) that there exists a metric subdivision $(x_i)_{i=0}^n$ of $(x, y)$ for which

$$d (x, y) - \epsilon = \left( \sum_{i=1}^n d_i \right) - \epsilon < \sum_{i=1}^n c_i,$$

where

$$d_i = d (x_{i-1}, x_i) > 0 \quad \text{and} \quad c_i = |f (x_i) - f (x_{i-1})|.$$

Since $\|f\| = 1$, we have $c_i \leq d_i$. Moreover, by (11) one can find $n$ numbers $e_i$ ($i = 1, ..., n$) such that $0 \leq e_i \leq e_i$ (if $c_i > 0$), $e_i = 0$ (if $c_i = 0$), and

$$\left( \sum_{i=1}^n d_i \right) - \epsilon = \sum_{i=1}^n c_i.$$

Now denote $e_i = d_i - c_i$. Then we have $e_i > 0$, $\sum_{i=1}^n e_i = \epsilon$, and $c_i \geq e_i = d_i - e_i$, i.e.,

$$d (x_{i-1}, x_i) - \epsilon \leq |f (x_i) - f (x_{i-1})| \quad (i = 1, ..., n).$$

Hence it follows from the definition of $\epsilon^f_{x,y}$ that $\epsilon^f_{x,y} \leq \epsilon$. Since $\epsilon > \rho_f (x, y)$ was arbitrary, we conclude that

$$0 = \sigma_f (x, y) \leq \epsilon^f_{x,y} \leq \rho_f (x, y)$$

for all $x, y \in X$. This in conjunction with (1) enables to apply (10) in order to get $\epsilon^f_{x,y} = \sigma_f (x, y) = 0$. Thus Theorem B yields $f \in \text{ext} (B_X)$, which completes the proof of necessity.

For the proof of sufficiency, suppose that there exist $f \in X^\#$ and $z \in X$ for which $\|f\| = 1$ and $Y = \{ y : \sigma_f (z, y) > 0 \} \neq \emptyset$. Then the triangle inequality and symmetry of $\sigma_f$ yield

$$(12) \quad \sigma_f (x, y) = \sigma_f (z, y)$$

for all $x \in X \setminus Y$ and $y \in Y$. This together with (8) and (9) enables to repeat mutatis mutandis Farmer’s proof [2] of sufficiency of Theorem B, with $\epsilon^f_{x,y}$ replaced by $\sigma_f (x, y)$, in order to show that $f \notin \text{ext} (B_X^\#)$.

From now on, we will assume that $X$ is a convex subset of a normed linear space $(E, \|\|)$. In this case, we define

$$\hat{\rho}_f (x, y) = \inf \left\{ \|f\| \|x - y\| - \sum_{i=1}^n |f (x_i) - f (x_{i-1})| \right\},$$

where the infimum is taken only over all finite subdivisions $(x_i)_{i=0}^n$ of the form

$$x_i = (1 - t_i) x + t_i y \quad (0 = t_0 < t_1 < ... < t_n = 1).$$

It is clear that (2) holds for these algebraic subdivisions of the algebraic interval $[x, y] = \{(1 - t) x + t y : 0 \leq t \leq 1\}$, and that $[x, y] = (x, y)$ and $\hat{\rho}_f (x, y) = \rho_f (x, y)$ for all $x, y \in X$, whenever $E$ is a strictly convex space. In general, we have only $\rho_f \leq \hat{\rho}_f$.

If $\hat{\sigma}_f (x, y) = \inf_{\ell \in \mathcal{L}} \hat{\rho}_f (x, y)$ is defined by formula (6) with $\rho_f$ replaced by $\hat{\rho}_f$, then $\sigma_f \leq \hat{\sigma}_f$. By the same arguments as above, one can also prove that $\hat{\rho}_f$ and $\hat{\sigma}_f$ satisfy inequality

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(4) and the triangle inequality, respectively. In particular, by using (2) and (4) we obtain

\[ \hat{\sigma}_f (x, y) \leq \hat{\rho}_f (x_0, x_1) + \ldots + \hat{\rho}_f (x_{n-1}, x_n) \leq \| f \| \| x - y \| - \sum_{i=1}^{n} | f(x_i) - f(x_{i-1}) | \]

for all metric subdivisions \((x_i)_{i=0}^n\) of \((x, y)\). Hence we derive \( \hat{\sigma}_f \leq \rho_f \). Therefore, one can apply (10) with \( \mu = \hat{\sigma}_f \) in order to get \( \hat{\sigma}_f = \sigma_f \).

**Lemma 1.** Let \( X \) be a convex subset of a normed linear space \( E = (E, \| \cdot \|) \), and let \( f \in X^\# \). Then we have

\[ \hat{\rho}_f (x, y) = \| f \| \| x - y \| - V^1_0 (f_{x,y}) = \| f \| \| x - y \| - \int_0^1 | f'_{x,y}(t) | \, dt, \]

where \( V^1_0 (f_{x,y}) \) denotes the total variation of the function \( f_{x,y} \) defined by

\[ f_{x,y}(t) = f ((1-t)x + ty) \quad (0 \leq t \leq 1). \]

**Proof.** By (13) we obtain

\[ \hat{\rho}_f (x, y) = \| f \| \| x - y \| - \sup \left\{ \sum_{i=1}^{n} | f(x_i) - f(x_{i-1}) | \right\} \]

\[ = \| f \| \| x - y \| - V^1_0 (f_{x,y}), \]

where the supremum is taken over all finite algebraic subdivisions \((x_i)_{i=0}^n\) of \([x, y]\). Since \( f \in X^\# \), we have

\[ | f_{x,y}(t) - f_{x,y}(s) | \leq \| f \| \| x - y \| | t - s | \quad (0 \leq t, s \leq 1). \]

Hence the derivative \( f'_{x,y}(t) \) exists almost everywhere on \([0, 1]\), and the function \( t \rightarrow f'_{x,y}(t) \) is integrable. Moreover, we have

\[ V^1_0 (f_{x,y}) = \int_0^1 | f'_{x,y}(t) | \, dt. \]

This in conjunction with (14) completes the proof.

In view of the fact that \( \hat{\sigma}_f = \sigma_f \), Theorem 1 is an immediate consequence of Lemma 1 and Theorem 2. Moreover, it follows from the triangle inequality for \( \sigma_f \) that Theorems 1 and 2 remain true, whenever we put either \( x = 0 \) or \( y = 0 \) into them. In particular, if the interval \( X = [0, 1] \) is equipped with the metric \( d(x, y) = |x - y| \), then Lemma 1 yields

\[ \rho_f (x, y) = \hat{\rho}_f (x, y) = |x - y| \left( 1 - \int_0^1 | f'(s) | \, ds \right) \]

for all \( x, y \in [0, 1] \). On the other hand, by (5) and (7) one can apply (10) with \( \mu = \rho_f \) to get \( \sigma_f = \rho_f \). Hence Theorem A follows directly from Theorem 2.

Finally, we present another application of Theorem 2 which shows that the set \( \text{ext} (B_{X^\#}) \) of all extreme points of the unit ball \( B_{X^\#} \) of \( X^\# \) is quite rich, whenever
X is a normed linear space. For this purpose, denote by $X^*$ the dual space of $X$, and note that

$$\sigma_f (x, x + \alpha z) = \sigma_f (0, \alpha z) = |\alpha| \cdot \sigma_f (0, z) \quad (\alpha \in \mathbb{R}; \ x, z \in X)$$

(15) for every functional $f \in X^*$. To prove these identities, we need only to change variables $z_k \rightarrow z_k + x$ ($z_k \rightarrow \alpha z_k$) in the definition of $\sigma_f = \sigma_f$ applied to $y = x + \alpha z$ ($y = \alpha z$, respectively), and use the identity

$$\hat{\rho}_f (x, y) = \|f\| \|x - y\| - |f(x - y)| \quad (x, y \in X),$$

which is a direct consequence of Lemma 1 and linearity of $f$. Since $\sigma_f$ satisfies the triangle inequality, it follows from (15) that

$$\sigma_f (0, z_1 + z_2) \leq \sigma_f (0, z_1) + \sigma_f (z_1, z_1 + z_2) = \sigma_f (0, z_1) + \sigma_f (0, z_2).$$

This in conjunction with (15) means that the function $z \rightarrow \sigma_f (0, z)$ ($z \in X$) is a seminorm on $X$.

**Theorem 3.** Let $X$ be a normed linear space. Then we have

$$\text{ext } (B_{X^*}) \cap X^* = \text{ext } (B_{X^*}).$$

**Proof.** In view of definition of extreme points, we directly have

$$\text{ext } (B_{X^*}) \cap X^* \subseteq \text{ext } (B_{X^*}).$$

Conversely, let a functional $f \in X^*$ be such that $\|f\| = 1$ and $f \notin \text{ext } (B_{X^*})$. We need only to prove that $f \notin \text{ext } (B_{X^*})$. By Theorem 2 the set

$$Y = \{y: \sigma_f (0, y) > 0\}$$

is nonempty. Moreover, it follows from (15) that the set $X \setminus Y$ is a linear subspace of $X$ which, in view of (12), has the property

$$\sigma_f (x, y) = \sigma_f (0, y) \quad (x \in X \setminus Y, \ y \in Y).$$

(16) Now take a point $y_0 \in Y$, and denote by $X_0$ the linear subspace spanned by $y_0$ and $X \setminus Y$. Next, define the linear functional $g$ on $X_0$ by the formula

$$g(x + \alpha y_0) = \alpha \sigma_f (0, y_0) \quad (x \in X \setminus Y, \ \alpha \in \mathbb{R}).$$

Then it follows from (15) and (16) that

$$|g(x + \alpha y_0)| = \sigma_f (x, x + \alpha y_0) = \sigma_f (0, x + \alpha y_0) \quad (x \in X \setminus Y, \ \alpha \in \mathbb{R}),$$

whenever $x + \alpha y_0 \in Y$. Otherwise, if $x + \alpha y_0 \notin Y$ then $\alpha = 0$ and (17) is obvious. Since the function $z \rightarrow \sigma_f (0, z)$ is a seminorm on $X$ and $g$ satisfies condition (17) on $X_0$, it follows from the Hahn-Banach theorem that the functional $g : X_0 \rightarrow \mathbb{R}$ has an extension to the whole space $X$, which satisfies the inequality

$$|g(z)| \leq \sigma_f (0, z), \ z \in X.$$

Consequently, one can apply (4) and (7) to get

$$|g(z)| \leq \|z\| - |f(z)|, \ z \in X.$$

Thus $g \in X^*$ and $f_k \in X^*$ ($k = 1, 2$), where functionals $f_k$ are defined by

$$f_k (z) = f(z) + (-1)^k g(z).$$

Therefore, we obtain

$$|f_k(z)| \leq |f(z)| + |g(z)| \leq \|z\|.$$
for every $z \in X$. Hence we have $\|f_k\| \leq 1$ ($k = 1, 2$) and $f_k(y_0) \neq f(y_0)$, which in conjunction with the identity $f = (f_1 + f_2)/2$ shows that $f \notin \text{ext}(B_{X^*})$. Thus the proof is completed.

\begin{thebibliography}{99}


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