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# NONSTANDARD MODELS AND ANALYTIC EQUIVALENCE RELATIONS

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Abstract. We improve a result of Hjorth concerning the nature of thin analytic equivalence relations. The key lemma uses a weakly compact cardinal to construct certain nonstandard models, which Hjorth obtained using #'s.

The purpose of this note is to improve the following result of Hjorth [2].

**Theorem** (Hjorth)**.** *Suppose that for every real $x, x^{\#}$ exists. Let $E$ be an analytic equivalence relation, $\Sigma^1_1$ in parameter $x_0$. Then either there exists a perfect set of pairwise $E$-inequivalent reals or every $E$-equivalence class has a representative in a set-generic extension of $L[x_0]$.*

Hjorth's proof makes use of his analysis of nonstandard Ehrenfeucht-Mostowski models built from #'s. Instead, we construct the necessary nonstandard models using infinitary model theory, assuming only the existence of weak compacts.

**Theorem 1.** *Suppose that for every real $x$ there is a weakly compact cardinal in $L[x]$. Then the conclusion of Hjorth's Theorem still holds.*

The main lemma is the following.

**Lemma 2.** *Suppose that there is a weakly compact cardinal in $L[x]$, $x$ a real. Then there is a countable nonstandard $\omega$-model $M_x$ of $ZF$ such that $x \in M_x$ and $L(M_x) =$ ($L$ in the sense of $M_x$) has an isomorphic copy in a set-generic extension of $L[x_0]$, for any real $x_0$.*

It is not known if the conclusion of Lemma 2 holds in $ZFC$ alone, for arbitrary $x$ (with $ZF$ replaced by an arbitrary finite subtheory).

*Proof of Theorem 1 from Lemma 2* (*as in Hjorth* [2]). Suppose that $E$ is an analytic equivalence relation, $\Sigma^1_1$ in the parameter $x_0$, and choose an $x_0$-recursive tree $T$ on $\omega \times \omega \times \omega^\omega$ such that $xEy \longleftrightarrow T(x, y)$ has a branch. For each countable ordinal $\alpha$ we define $xE_\alpha y \longleftrightarrow \mathrm{rank}(T(x, y))$ is at least $\alpha$; then $E_\alpha$ is Borel in $(x_0, c)$ where $c$ is any real coding $\alpha$ and $E$ is the intersection of the $E_\alpha$'s. We may assume that each $E_\alpha$ is an equivalence relation (see Theorem 1.4 of Hjorth [2]). A theorem of Harrington and Silver says that a $\Pi^1_1$-equivalence relation has a perfect set of pairwise inequivalent reals or each equivalence class has a representative constructible from the parameter defining the relation. As $E_\alpha$ is Borel in $(x_0, c)$ where $c$ is a

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real coding $\alpha$, and as we may assume that $E$ and hence each $E_\alpha$ has no perfect set of pairwise inequivalent reals, we know that each $E_\alpha$-equivalence class has a representative in $L[x_0, c]$ where $c$ is any real coding $\alpha$.

Now let $x$ be arbitrary and by Lemma 2 choose a countable nonstandard $\omega$-model $M_x$ of $ZF$ containing $(x_0, x)$ such that $L(M_x)$ has an isomorphic copy in a set-generic extension $N$ of $L[x_0]$. Let $a \in \mathrm{ORD}(M_x)$ be nonstandard and let $c$ be a code for $a$, generic over $M_x$; then by applying Harrington-Silver in $M_x[c]$ we conclude that there is $y$ in $L(M_x)[x_0, c]$ such that $yE_a x$. By choosing $c$ to be generic over $N$ as well we get that $y$ belongs to a set-generic extension of $L[x_0]$. Finally, $yEx$ since, if not, $yE_\alpha x$ would fail for some $\alpha$ admissible in $(y, x)$ and hence for some (standard) $\alpha < a$. $\dashv$

To prove Lemma 2 we discuss infinitary logic. Fix a real $x$ and assume $V = L[x]$. Let $\kappa$ be weakly compact and introduce the language $\mathcal{L}$ consisting of the formulas in the language of set theory with constants $\underline{a}$ for $a \in L_\kappa[x]$, closed under conjunctions and disjunctions of size less than $\kappa$ (however we allow a formula to have only finitely many free variables). Let $T$ be the theory of $\langle L_\kappa[x], a\rangle$, $a \in L_\kappa[x]$, in this language. An *n-type* is a set of formulas $\Gamma$ with free variables $v_1 \dots v_n$, and such a $\Gamma$ is *consistent with* $T$ if there is a model $M$ of $T$ and $m_1 \dots m_n$ in $M$ such that $M \models \varphi(m_1 \dots m_n)$ for each $\phi \in \Gamma$, where $M$ exists in a set-generic extension of $V = L[x]$. $\Gamma$ is *complete* if for every $\varphi(v_1 \dots v_n)$ either $\varphi$ or $\sim \varphi$ belongs to $\Phi$.

Now work in the Lévy collapse $L[x, c]$, where $c$ is a real coding $\kappa^+$ of $L[x]$. Let $d_1, d_2, \dots$ be $\omega$-many new constant symbols and for $D \subseteq \{d_1, d_2, \dots\}$ let the language $\mathcal{L}_D$ be defined like $\mathcal{L}$ but with the new constant symbols from $D$. Define $T_0 = T \subseteq T_1 \subseteq \dots$ and $D_0 = \phi \subseteq D_1 \subseteq D_2 \subseteq \dots$ inductively as follows: if $T_n, D_n$ have been defined select a complete $k$-type $\Gamma_n(\vec{v})$ in $L[x]$ consistent with $T_n$, choose $D_n \subseteq D_{n+1}$ so that $\mathrm{card}(D_{n+1} - D_n) = k$, and let $T_{n+1} = T_n \cup \Gamma_n(\vec{d})$, where $\vec{d}$ enumerates $D_{n+1} - D_n$. This can be done in such a way that $\bigcup_n T_n = T^*$ is *L[x]-saturated*: if $\Gamma(\vec{v}, \vec{w})$ is an $L[x]$-type, $\vec{d}$ a finite sequence from $D$, and $\Gamma(\vec{d}, \vec{w})$ consistent with $T^*$, then $T^*$ includes $\Gamma(\vec{d}, \vec{e})$ for some $\vec{e}$. And note that each $T_n$ belongs to $L[x]$ (though of course $T^*$ itself makes use of the Lévy collapse $c$).

Let $M_x$ be the model determined by $T^*$, whose universe consists of (equivalence classes of) the constants $d_n, n \in \omega$. Note that a set in $L[x]$ of sentences in some $\mathcal{L}_D$ is consistent iff each subset of $L[x]$-cardinality $< \kappa$ is, by $\Pi^1_1$-reflection and the equivalence of consistency with existence of a model after performing a Lévy collapse of $\kappa$. An easy consequence is that $M_x$ is nonstandard with standard ordinal $\kappa$.

Now consider $L(M_x)$: every $n$-type in the language $\mathcal{L}_0 =$ (same as $\mathcal{L}$ but restricted to $L_\kappa$) that is realized in $L(M_x)$ belongs to $L$, as each of its initial segments (obtained by restricting to some $L_\alpha, \alpha < \kappa$) belongs to $L$ and $\kappa$ is weakly compact. Also, just as $M_x$ is saturated for types in $L[x]$, $L(M_x)$ is saturated for types in $L$, since again by weak compactness any $\mathcal{L}_0$-type in $L$ consistent with $T$ can be extended to a complete $\mathcal{L}$-type consistent with $T$ in $L[x]$.

Now it is clear that $L(M_x)$ has an isomorphic copy in $L[c]$ : using $c$ we can do the same construction as we did above in $L[x, c]$, obtaining $M_0$, a model that is saturated for $\mathcal{L}_0$-types in $L$ and realizing only types in $L$. Now construct an isomorphism via a back and forth argument in $\omega$ steps between $M_0$ and $L(M_x)$.

Finally, note that by absoluteness the desired model $M_x$ exists not only in $L[x, c]$ but in $L[x]$. $\dashv$

*Remark.* Lemma 2 can also be used to establish the following improvement of the Glimm-Effros style dichotomy theorem of Hjorth and Kechris [3]: Let $E$ be a $\Sigma^1_1$ equivalence relation. Assume that for every real $x$ there is a weakly compact cardinal in $L[x]$. Then either $E_0$ is continuously reducible to $E$, or $E$ is reducible to $2^{<\omega_1}$ by a function $\Delta^1_2$ in the codes.

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