AN ERGODIC THEOREM FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN THE INTERMEDIATE SENSE

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Abstract. This paper is concerned with an ergodic theorem for asymptotically nonexpansive mappings in the intermediate sense in Banach spaces.

1. Introduction and the main result

Throughout this paper X denotes a uniformly convex real Banach space, C a nonempty closed convex subset of X, and T a mapping from C into itself.

Recently the asymptotic behavior of asymptotically nonexpansive mappings has been studied by many authors (see [4], [14], [15] and [16]). There appear in the literature two definitions of an asymptotically nonexpansive mapping. One is due to Kirk [6]:

\[ \limsup_{n \to \infty} \sup_{y \in K} (\|T^n x - T^n y\| - \|x - y\|) \leq 0 \]  

(1.1)

for each \( x \in C \) and each bounded set \( K \subset C \), and \( T^N \) is continuous for some \( N \geq 1 \). The other is due to Goebel and Kirk [5]: There exists a sequence \( \{k_n\} \) with \( \lim_{n \to \infty} k_n = 1 \) such that

\[ \|T^n x - T^n y\| \leq k_n \|x - y\| \text{ for } x, y \in C \text{ and } n = 0, 1, 2, \ldots. \]

Bruck, Kuczumow and Reich [4] have introduced a definition between these two: \( T \) is called \textit{asymptotically nonexpansive in the intermediate sense} if \( T \) is continuous and

\[ \limsup_{n \to \infty} \sup_{x, y \in K} (\|T^n x - T^n y\| - \|x - y\|) \leq 0 \]  

(1.2)

for any bounded subset \( K \subset C \). The purpose of the present paper is to prove the ergodic theorem for such a mapping in the class of Banach spaces in which the nonlinear mean ergodic theorem is usually set.

Here we summarize the notations used in the sequel. Denote by \( F(T) \) the set of fixed points of a mapping \( T \). The convex hull of a subset \( M \) of \( X \) is denoted by
coM, and the closed convex hull by clcoM. We put
\[ \Delta^{n-1} = \{ \lambda = (\lambda_1, \cdots, \lambda_n) : \lambda_i \geq 0 \ (i = 1, 2, \cdots, n) \text{ and } \sum_{i=1}^{n} \lambda_i = 1 \} \]
and
\[ B_r = \{ x \in X : \|x\| \leq r \} \text{ for } r > 0. \]
Also, \( \omega_w(\{z_n\}) \) denotes the set of weak subsequential limits of a sequence \( \{z_n\}_{n \geq 0} \) in \( X \).

**Definition 1.1.** A sequence \( \{x_n\}_{n \geq 0} \) in \( C \) is called an almost-orbit of \( T \) if
\[ \lim_{n \to \infty} \sup_{m \geq 0} \|x_{n+m} - T^m x_n\| = 0. \]  

**Definition 1.2.** A sequence \( \{z_n\}_{n \geq 0} \) in \( X \) is said to be weakly almost convergent to \( z \in X \) if \( \frac{1}{n} \sum_{i=0}^{n-1} z_i + k \) converges weakly as \( n \to \infty \) to \( z \) uniformly in \( k \geq 0 \).

The main result in this paper is stated as follows:

**Theorem 1.1.** Suppose that \( T : C \to C \) is asymptotically nonexpansive in the intermediate sense with \( F(T) \neq \emptyset \), and that \( \{x_n\} \) is an almost-orbit of \( T \). If the norm of \( X \) is Fréchet differentiable, then \( \{x_n\} \) is weakly almost convergent to the unique point of \( F(T) \cap \text{clco} \omega_w(\{x_n\}) \).

2. **Proof of theorem**

In what follows, a mapping \( T : C \to C \) is assumed to be asymptotically nonexpansive in the intermediate sense with \( F(T) \neq \emptyset \). Take \( f \in F(T) \) and let \( K \) be a bounded closed convex subset of \( C \) including the set \( \{f\} \). Put \( D_K = \text{diameter } K \).

The key point in proving mean ergodic theorems is to estimate the difference between \( T^k(\sum^n_{i=1} \lambda_i z_i) \) and \( \sum^n_{i=1} \lambda_i T^k z_i \) for \( \lambda \in \Delta^{n-1} \), \( z_1, \cdots, z_n \in K \) and \( k \geq 1 \), as done previously in [15, Proposition 3.1] and [8, Lemma 3]. However, we cannot make use of Bruck’s inequality [3, Theorem 2.1] as used in [15] and [8], because our operator \( T \) is not Lipschitz continuous. Therefore our argument is different from theirs.

**Lemma 2.1.** For \( \varepsilon > 0 \) there exist an integer \( N_\varepsilon \geq 1 \) and \( \delta_{2,\varepsilon} > 0 \) such that if \( k \geq N_\varepsilon \), \( z_1, z_2 \in K \) and if \( \|z_1 - z_2\| - \|T^k z_1 - T^k z_2\| \leq \delta_{2,\varepsilon} \), then
\[ \|T^k(\lambda_1 z_1 + \lambda_2 z_2) - \lambda_1 T^k z_1 - \lambda_2 T^k z_2\| < \varepsilon \]
for all \( \lambda = (\lambda_1, \lambda_2) \in \Delta^1 \).

**Proof.** Let \( \delta \) be the modulus of uniform convexity of \( X \) and define a function \( d : \mathbb{R}^+ \to \mathbb{R}^+ \) by
\[ d(t) = \begin{cases} 
\frac{1}{2} \int_0^t \delta(s) ds & \text{if } 0 \leq t \leq 2, \\
\frac{1}{2} \delta(2)(t-2) & \text{if } t > 2.
\end{cases} \]
It is then well known that \( d \) is a strictly increasing, continuous convex function, and that it satisfies
\[ 2\lambda_1 \lambda_2 d(\|u - v\|) \leq 1 - \|\lambda_1 u + \lambda_2 v\| \]
for \( \lambda = (\lambda_1, \lambda_2) \in \Delta^1 \), \( \|u\| \leq 1 \) and \( \|v\| \leq 1 \).
For $\varepsilon > 0$ choose $\eta_{\varepsilon} > 0$ such that $\frac{D_K}{2} d^{-1} \left( \frac{2\eta_{\varepsilon}}{D_K} \right) < \varepsilon$ and put $\delta_{2,\varepsilon} = \min\{\eta_{\varepsilon}, \frac{D_K}{2}\}$. By (1.2) there exists an integer $N_{\varepsilon} \geq 1$ (depending on the set $K$) such that if $k \geq N_{\varepsilon}$,

$$\|T^k x - T^k y\| - \|x - y\| < \delta_{2,\varepsilon} \text{ for all } x, y \in K.$$ 

Let $k \geq N_{\varepsilon}$ and let $z_1, z_2 \in K$ with $\|z_1 - z_2\| - \|T^k z_1 - T^k z_2\| \leq \delta_{2,\varepsilon}$. It suffices to show Lemma 2.1 in the case of $0 < \lambda_i < 1$ ($i = 1, 2$).

Put

$$u = \frac{T^k z_2 - T^k (\lambda_1 z_1 + \lambda_2 z_2)}{\lambda_1 (\|z_1 - z_2\| + \delta_{2,\varepsilon})} \quad \text{and} \quad v = \frac{T^k (\lambda_1 z_1 + \lambda_2 z_2) - T^k z_1}{\lambda_2 (\|z_1 - z_2\| + \delta_{2,\varepsilon})}.$$ 

Then we have $\|u\| \leq 1, \|v\| \leq 1$ and

$$\lambda_1 u + \lambda_2 v = \frac{T^k z_2 - T^k z_1}{\|z_1 - z_2\| + \delta_{2,\varepsilon}}.$$ 

Since

$$u - v = \frac{\lambda_1 T^k z_1 + \lambda_2 T^k z_2 - T^k (\lambda_1 z_1 + \lambda_2 z_2)}{\lambda_1 \lambda_2 (\|z_1 - z_2\| + \delta_{2,\varepsilon})}$$

and $\frac{2}{D_K} \lambda_1 \lambda_2 (\|z_1 - z_2\| + \delta_{2,\varepsilon}) \leq \frac{2}{D_K} \frac{1}{\lambda_2} \left( D_K + \frac{D_K}{4} \right) < 1$, we have by (2.1) and (2.2)

$$d \left( \frac{2}{D_K} \|\lambda_1 T^k z_1 + \lambda_2 T^k z_2 - T^k (\lambda_1 z_1 + \lambda_2 z_2)\| \right)$$

$$\leq \frac{2}{D_K} \lambda_2 (\|z_1 - z_2\| + \delta_{2,\varepsilon}) d(\|u - v\|)$$

$$\leq \frac{2}{D_K} \lambda_1 \lambda_2 (\|z_1 - z_2\| + \delta_{2,\varepsilon}) \frac{1}{2 \lambda_1 \lambda_2} \left( 1 - \frac{\|T^k z_1 - T^k z_2\|}{\|z_1 - z_2\| + \delta_{2,\varepsilon}} \right)$$

$$= \frac{1}{D_K} (\|z_1 - z_2\| - \|T^k z_1 - T^k z_2\| + \delta_{2,\varepsilon}) \leq \frac{2\delta_{2,\varepsilon}}{D_K} \leq \frac{2\eta_{\varepsilon}}{D_K}.$$ 

Here we have used the convexity of a function $d$ and the fact that $d(0) = 0$. Consequently, we obtain from the choice of $\eta_{\varepsilon}$

$$\|T^k (\lambda_1 z_1 + \lambda_2 z_2) - \lambda_1 T^k z_1 - \lambda_2 T^k z_2\| \leq \frac{D_K}{2} d^{-1} \left( \frac{2\eta_{\varepsilon}}{D_K} \right) < \varepsilon. \quad \square$$

**Lemma 2.2.** For each $\varepsilon > 0$ and each integer $n \geq 2$ there exist an integer $N_{\varepsilon} \geq 1$ and $\delta_{n,\varepsilon} > 0$, where $N_{\varepsilon}$ is independent of $n$, such that if $k \geq N_{\varepsilon}, z_1, \cdots, z_n \in K$ and if $\|z_i - z_j\| - \|T^k z_i - T^k z_j\| \leq \delta_{n,\varepsilon}$ for $1 \leq i, j \leq n$, then

$$\left\| T^k \left( \sum_{i=1}^n \lambda_i z_i \right) - \sum_{i=1}^n \lambda_i T^k z_i \right\| < \varepsilon$$

for all $\lambda = (\lambda_1, \cdots, \lambda_n) \in \Delta^{n-1}$.

**Proof.** Let $\varepsilon > 0$ and let $n \geq 2$ be an arbitrary integer. Choose an integer $N_{\varepsilon} \geq 1$ in Lemma 2.1. We shall construct $\delta_{n,\varepsilon}(n = 2, 3, \cdots)$ inductively. Let $\delta_{2,\varepsilon}$ be as in Lemma 2.1. Suppose that all $\delta_{q,\varepsilon}$ are constructed for $q = 2, \cdots, p$. Let $\varepsilon' = \min \left( \frac{1}{3} \delta_{p,\varepsilon}, \frac{\varepsilon}{2^p} \right)$ and put $\delta_{p+1,\varepsilon} = \min(\delta_{2,\varepsilon}', \varepsilon')$. 


Let $\lambda \in \Delta^p, z_1, \ldots, z_{p+1} \in K, k \geq N_\varepsilon$ and $\|z_i - z_j\| - \|T^k z_i - T^k z_j\| \leq \delta_{p+1, \varepsilon}$ for $1 \leq i, j \leq p+1$. The case $\lambda_{p+1} = 1$ is trivial and so we assume $\lambda_{p+1} \neq 1$. Putting

$$u_j = (1 - \lambda_{p+1})z_j + \lambda_{p+1}z_{p+1}, \quad \mu_j = \frac{\lambda_j}{1 - \lambda_{p+1}}$$

and

$$u_j' = (1 - \lambda_{p+1})T^k z_j + \lambda_{p+1}T^k z_{p+1}$$

for $j = 1, 2, \ldots, p$, we have

$$\sum_{i=1}^{p+1} \lambda_i z_i = \sum_{j=1}^{p} \frac{\lambda_j}{1 - \lambda_{p+1}} \{(1 - \lambda_{p+1})z_j + \lambda_{p+1}z_{p+1}\} = \sum_{j=1}^{p} \mu_j u_j$$

and hence

$$\sum_{i=1}^{p+1} \lambda_i T^k z_i = \sum_{j=1}^{p} \mu_j u_j'$$

and

$$\|T^k \left( \sum_{i=1}^{p+1} \lambda_i z_i \right) - \sum_{i=1}^{p+1} \lambda_i T^k z_i \| = \|T^k \left( \sum_{j=1}^{p} \mu_j u_j \right) - \sum_{j=1}^{p} \mu_j u_j' \|$$

$$\leq \left\| T^k \left( \sum_{j=1}^{p} \mu_j u_j \right) - \sum_{j=1}^{p} \mu_j T^k u_j \right\| + \sum_{j=1}^{p} \mu_j \|T^k u_j - u_j'\|.$$

Since $\|z_j - z_{p+1}\| - \|T^k z_j - T^k z_{p+1}\| \leq \delta_{p+1, \varepsilon} \leq \delta_{2, \varepsilon}$, we have by Lemma 2.1

$$\|u_j' - T^k u_j\| = \|\{(1 - \lambda_{p+1})T^k z_j + \lambda_{p+1}T^k z_{p+1}\} - T^k \{(1 - \lambda_{p+1})z_j + \lambda_{p+1}z_{p+1}\}\| \leq \varepsilon'$$

for $1 \leq j \leq p$ and

$$\|u_j - u_l\| - \|u_j' - u_l'\| = (1 - \lambda_{p+1})\|z_j - z_l\| - \|T^k z_j - T^k z_l\|$$

$$\leq (1 - \lambda_{p+1})\delta_{p+1, \varepsilon} \leq \varepsilon' = \min\left(\frac{1}{3} \delta_{p, \varepsilon} + \varepsilon', \frac{\varepsilon}{2}\right)$$

for $1 \leq j, l \leq p$. Therefore we obtain

$$\|u_j - u_l\| - \|T u_j - T u_l\| \leq \|u_j - u_l\| - \|u_j' - u_l'\| + \|u_j' - T u_l\| + \|u_j' - T u_j\|$$

$$\leq \frac{1}{3} \delta_{p, \varepsilon} + \varepsilon' + \varepsilon' \leq \delta_{p, \varepsilon}$$

for $1 \leq j, l \leq p$, and thus by the inductive assumption and (2.3) the desired conclusion holds.

Since $X$ is uniformly convex, it has the convex approximation property (C.A.P.), i.e. for each $\varepsilon > 0$ there exists an integer $p(= p(\varepsilon)) \geq 1$ such that for all subsets $M$ in $X$ whose diameters are uniformly bounded,

$$coM \subset co_p M + B_\varepsilon,$$

where $co_p M := \{\lambda_1 z_1 + \cdots + \lambda_p z_p : \lambda \in \Delta^{p-1}, \ z_1, \cdots, z_p \in M\}$ (see [3, Theorem 1.1]).

The following lemma shows that the positive number $\delta_{n, \varepsilon}$ in Lemma 2.2 can be chosen independently of $n$, thanks to this property of the space $X$.\qed
Lemma 2.3. For every $\varepsilon > 0$ and every integer $n \geq 2$ there exist an integer $N_{\varepsilon} \geq 1$ and $\delta_{\varepsilon} > 0$, where both $N_{\varepsilon}$ and $\delta_{\varepsilon}$ are independent of $n$, such that if $k \geq N_{\varepsilon}$, $z_1, \ldots, z_n \in K$ and if $\|z_i - z_j\| - \|T^k z_i - T^k z_j\| \leq \delta_{\varepsilon}$ for $1 \leq i, j \leq n$, then

$$\left\| T^k \left( \sum_{i=1}^{n} \lambda_i z_i \right) - \sum_{i=1}^{n} \lambda_i T^k z_i \right\| < \varepsilon$$

for all $\lambda \in \Delta^{n-1}$.

Proof. Fix $\varepsilon > 0$ and an integer $n \geq 2$ arbitrarily. Denote by $N_{1,\varepsilon}$ the integer $N_{\varepsilon}$ in Lemma 2.2. By (1.2) there is an integer $N_{2,\varepsilon} \geq 1$ such that if $k \geq N_{2,\varepsilon}$, then we have

$$\|T^k x - T^k y\| - \|x - y\| < \varepsilon/4 \text{ for all } x, y \in K.$$

Put $N_{\varepsilon} = \max(N_{1,\varepsilon}, N_{2,\varepsilon})$. Let $\delta_{n,\varepsilon}(n = 2, 3, \ldots)$ be positive numbers determined in Lemma 2.2. As pointed out in the proof of [3, Theorem 2.1], $X \times X$ has the C.A.P. and hence we can choose an integer $p(= p(\varepsilon)) \geq 1$ such that

$$coM \subset co_p M + B_{\varepsilon/4} \times B_{\varepsilon/4}$$

for all subsets $M$ in $X \times X$ whose diameters are uniformly bounded. Note that this integer $p$ is independent of $n$. Put $\delta_{\varepsilon} = \delta_{p,\varepsilon}$.

Let $k \geq N_{\varepsilon}, z_1, \ldots, z_n \in K$, and $\|z_i - z_j\| - \|T^k z_i - T^k z_j\| \leq \delta_{\varepsilon}$ (1 $\leq i, j \leq n$). Consider $M = \{[z_i, T^k z_i] \in X \times X : i = 1, 2, \ldots, n\}$. Note that there exists $r > 0$, independent of $k$ and $n$, such that $\sup_{(x,y) \in M} \|(x,y)\|_{X \times X} \leq r$ because of (2.5) and $f \in F(T) \cap K$. Then for each $\lambda \in \Delta^{n-1}$ there exist $\mu \in \Delta^{p-1}$ and $i_1, \ldots, i_p \in \{1, 2, \ldots, n\}$ such that

$$\left\| \sum_{i=1}^{n} \lambda_i z_i - \sum_{j=1}^{p} \mu_j z_{i_j} \right\| < \varepsilon/4$$

and

$$\left\| \sum_{i=1}^{n} \lambda_i T^k z_i - \sum_{j=1}^{p} \mu_j T^k z_{i_j} \right\| < \varepsilon/4.$$

Therefore we have by (2.5) and the choice of $\delta_{\varepsilon}$

$$\left\| T^k \left( \sum_{i=1}^{n} \lambda_i z_i \right) - \sum_{i=1}^{n} \lambda_i T^k z_i \right\| \leq \left\| T^k \left( \sum_{i=1}^{n} \lambda_i z_i \right) - T^k \left( \sum_{j=1}^{p} \mu_j z_{i_j} \right) \right\| + \left\| T^k \left( \sum_{j=1}^{p} \mu_j z_{i_j} \right) - \sum_{j=1}^{p} \mu_j T^k z_{i_j} \right\| + \left\| \sum_{j=1}^{p} \mu_j T^k z_{i_j} - \sum_{i=1}^{n} \lambda_i T^k z_i \right\| < \varepsilon. \quad \square$$

For each $\varepsilon > 0$ and each integer $k \geq 1$ set

$$F_{\varepsilon}(T^k) = \{ x \in C : \|T^k x - x\| \leq \varepsilon \}.$$
Lemma 2.4. For each \( \varepsilon > 0 \) there exist an integer \( N(\varepsilon) \geq 1 \) and \( \delta(= \delta(\varepsilon)) > 0 \) such that

\[
(2.6) \quad \text{clco}(F_k(T^k) \cap K) \subset F_k(T^k) \cap K
\]

for all \( k \geq N(\varepsilon) \).

Proof. For each \( \varepsilon > 0 \) choose an integer \( N(\varepsilon) := N_\varepsilon \geq 1 \) and \( \delta(\varepsilon) > 0 \) in Lemma 2.3.

Let \( k \geq N(\varepsilon) \) and put \( \delta := (\delta(\varepsilon)) = \min\left(\frac{1}{2} \delta(\varepsilon), \frac{1}{2}\right) \). Let \( y \in \text{co}(F_k(T^k) \cap K) \), say

\[
y = \sum_{i=1}^{n} \beta_i y_i \quad (y_i \in F_k(T^k) \cap K, \ i = 1, 2, \ldots, \ n; \ \beta_i \in \Delta^n).
\]

By Lemma 2.3,

\[
\|T^k \left( \sum_{i=1}^{n} \beta_i y_i \right) - \sum_{i=1}^{n} \beta_i y_i \| \leq \|T^k y_i - y_i\| \leq \varepsilon,
\]

because \( \|y_i - y_j\| - \|T^k y_i - T^k y_j\| \leq \|y_i - T^k y_i\| + \|y_j - T^k y_j\| \leq 2\delta \leq \delta(\varepsilon) \) for \( 1 \leq i, j \leq n \). Therefore \( \text{co}(F_k(T^k) \cap K) \subset F_k(T^k) \), which implies (2.6). \( \square \)

Lemma 2.5. Let \( \{z_n\} \) be a sequence in \( K \) such that \( w\lim_{n \to \infty} z_n = z \). Suppose that for each \( \varepsilon > 0 \) there exists an integer \( N(\varepsilon) \geq 1 \) such that for \( k \geq N(\varepsilon) \) there is an integer \( N_k, \varepsilon \geq 1 \) satisfying \( \|T^k z_n - z_n\| < \varepsilon \) for all \( n \geq N_k, \varepsilon \). Then we have \( z \in F(T) \).

Proof. We shall show that \( \lim_{k \to \infty} \|T^k z - z\| = 0 \). For \( \varepsilon > 0 \) choose an integer \( N(\varepsilon) \geq 1 \) and \( \delta(\varepsilon) > 0 \) in Lemma 2.3. By (1.2) there exists an integer \( N_1(\varepsilon) \geq 1 \) such that if \( k \geq N_1(\varepsilon) \),

\[
(2.7) \quad \|T^k u - T^k v\| - \|u - v\| < \frac{\varepsilon}{5}
\]

for all \( u, v \in K \). Put \( \varepsilon' = \min\left(\frac{1}{2} \delta(\varepsilon), \frac{1}{2}\right) \). By assumption we can take an integer \( N(\varepsilon') \geq 1 \). Let \( N_2(\varepsilon) := \max(N_\varepsilon, N(\varepsilon)) \) and let \( k \geq N_2(\varepsilon) \).

Since \( z \in \text{clco}\{z_n : n \geq N_k, \varepsilon\} \), where \( N_k, \varepsilon \) is the integer determined by assumption, there exists a sequence \( \{\sum_{i=1}^{l_n} \lambda^{(i)}_n z_{\phi_n(i)}\} \subset \text{co}\{z_n : n \geq N_k, \varepsilon\} \) such that \( \lim_{n \to \infty} \sum_{i=1}^{l_n} \lambda^{(i)}_n z_{\phi_n(i)} = z \).

Since

\[
\|z_{\phi_n(i)} - z_{\phi_n(j)}\| - \|T^k z_{\phi_n(i)} - T^k z_{\phi_n(j)}\|
\leq \|z_{\phi_n(i)} - T^k z_{\phi_n(i)}\| + \|z_{\phi_n(j)} - T^k z_{\phi_n(j)}\| \leq \delta(\varepsilon) \quad (1 \leq i, j \leq l_n)
\]

by assumption, Lemma 2.3 implies

\[
(2.8) \quad \left\|T^k \left( \sum_{i=1}^{l_n} \lambda^{(i)}_n z_{\phi_n(i)} \right) - \sum_{i=1}^{l_n} \lambda^{(i)}_n T^k z_{\phi_n(i)} \right\| < \frac{\varepsilon}{5}.
\]

There is also \( N_3(k, \varepsilon) \geq 1 \) such that

\[
(2.9) \quad \left\|\sum_{i=1}^{l_n} \lambda^{(i)}_n z_{\phi_n(i)} - z \right\| < \frac{\varepsilon}{5}.
\]
for all $n \geq N_3(k, \varepsilon)$. Since $z \in K$, the combination of the above inequalities with (2.7) gives

$$\|T^k z - z\| \leq \left\| T^k z - T^k \left( \sum_{i=1}^{l_1} \lambda_n^{(i)} z_{\phi_n(i)} \right) \right\| + \left\| T^k \left( \sum_{i=1}^{l_1} \lambda_n^{(i)} z_{\phi_n(i)} \right) - \left( \sum_{i=1}^{l_1} \lambda_n^{(i)} T^k z_{\phi_n(i)} \right) \right\| + \left\| \sum_{i=1}^{l_1} \lambda_n^{(i)} (T^k z_{\phi_n(i)} - z_{\phi_n(i)}) \right\| + \left\| \sum_{i=1}^{l_1} \lambda_n^{(i)} z_{\phi_n(i)} - z \right\| < \varepsilon,$$

whenever $n \geq N_3(k, \varepsilon)$. This shows that $\|T^k z - z\| < \varepsilon$ for $k \geq N_3(\varepsilon)$. 

\[ \Box \]

**Lemma 2.6.** Suppose that $\{x_n\}_{n \geq 0}$ and $\{y_n\}_{n \geq 0}$ are almost-orbits of $T$. Then $\{\|x_n - y_n\|\}$ converges as $n \to \infty$.

**Proof.** Put $a_n = \sup_{m \geq 0} \|x_{n+m} - T^m x_n\|$ and $b_n = \sup_{m \geq 0} \|y_{n+m} - T^m y_n\|$ for $n \geq 0$. Then $a_n \to 0$ and $b_n \to 0$ as $n \to \infty$. By (1.2) for each $\varepsilon > 0$ and each integer $n \geq 1$ there exists an integer $N(\varepsilon, n) \geq 1$ such that if $m \geq N(\varepsilon, n)$, then $\|T^m x_n - T^m y_n\| - \|x_n - y_n\| < \varepsilon$. Since

$$\|x_{n+m} - y_{n+m}\| \leq \|x_{n+m} - T^m x_n\| + \|T^m x_n - T^m y_n\| + \|T^m y_n - y_{n+m}\| \leq a_n + b_n + \|x_n - y_n\| + \varepsilon$$

for $m \geq N(\varepsilon, n)$, letting $m \to \infty$, and then $n \to \infty$ and $\varepsilon \downarrow 0$, we have

$$\limsup_{m \to \infty} \|x_m - y_m\| \leq \liminf_{n \to \infty} \|x_n - y_n\|$$

and so the conclusion holds.

\[ \Box \]

**Lemma 2.7.** Suppose that $\{x_j^{(p)}\}_{j \geq 0}$ ($p = 1, 2, \ldots$) are almost-orbits of $T$ satisfying $\sup\{\|x_j^{(p)}\| : j \geq 0, p \geq 1\} < \infty$. Then for each $\varepsilon > 0$ and each integer $n \geq 2$ there exist positive integers $N_\varepsilon$ and $i_n(\varepsilon)$, where $N_\varepsilon$ is independent of $n$, such that

$$\left\| T^k \left( \sum_{p=1}^{n} \lambda_p x_i^{(p)} \right) - \sum_{p=1}^{n} \lambda_p T^k x_i^{(p)} \right\| < \varepsilon$$

for all $k \geq N_\varepsilon$, $i \geq i_n(\varepsilon)$ and $\lambda \in \Delta^{n-1}$.

**Proof.** Take $f \in F(T)$ and set $K = \text{clco}\{x_j^{(p)} : j \geq 0, p \geq 1\} \cup \{f\}$. For $\varepsilon > 0$ take an integer $N_\varepsilon \geq 1$ and $\delta_\varepsilon > 0$ in Lemma 2.3.

Since $\{\|x_j^{(p)} - x_j^{(q)}\|\}_{j \geq 0}$ converges as $j \to \infty$ by Lemma 2.6, for each $p, q$ there exists an integer $i_0(\varepsilon, p, q) \geq 1$ such that $\|x_i^{(p)} - x_i^{(q)}\| - \|x_{i+k}^{(p)} - x_{i+k}^{(q)}\| < \delta_\varepsilon/3$ if $i \geq i_0(\varepsilon, p, q)$ and $k \geq 0$. Moreover there is an integer $i_1(\varepsilon, p, q) \geq 1$ such that $a_i^{(p)} < \delta_\varepsilon/3$ for all $i \geq i_1(\varepsilon, p, q)$, where $a_i^{(p)} = \max_{j \geq 0} \|x_{i+j}^{(p)} - T^j x_i^{(p)}\|$ for $i \geq 0$. Put $i_n(\varepsilon) = \max\{i_0(\varepsilon, p, q), i_1(\varepsilon, p, q) : 1 \leq p, q \leq n\}$. If $i \geq i_n(\varepsilon)$ and $k \geq N_\varepsilon$, then

$$\|x_i^{(p)} - x_i^{(q)}\| - \|T^k x_i^{(p)} - T^k x_i^{(q)}\|$$

$$\leq \|x_i^{(p)} - x_i^{(q)}\| - \|x_{i+k}^{(p)} - x_{i+k}^{(q)}\| + a_i^{(p)} + a_i^{(q)} < \delta_\varepsilon$$

for $1 \leq p, q \leq n$ and so Lemma 2.3 gives the desired conclusion.  

\[ \Box \]
By Lemma 2.6 an almost-orbit \( \{x_n\} \) of \( T \) is bounded, because of \( F(T) \neq \emptyset \).

In what follows take \( f \in F(T) \) and set \( K = \text{clco}(\{x_i : i \geq 0\} \cup \{f\}) \) for an almost-orbit \( \{x_n\} \) of \( T \).

**Lemma 2.8.** Suppose that \( \{x_i\}_{i \geq 0} \) is an almost-orbit of \( T \). Then for each \( \varepsilon > 0 \) there exists an integer \( N_{\varepsilon} \geq 1 \) such that for each \( k \geq N_{\varepsilon} \), there is an integer \( N_{k,\varepsilon} \geq 1 \) satisfying

\[
\frac{1}{n} \sum_{i=0}^{n-1} x_{i+l} \in F_{x}^{k} \quad \text{for all } n \geq N_{k,\varepsilon} \text{ and } l \geq 0.
\]

**Proof.** Let \( \varepsilon > 0 \). By Lemma 2.4 there exist an integer \( N_1(\varepsilon) \geq 1 \) and \( \delta_1(=\delta_1(\varepsilon)) > 0 \) such that

\[
\text{clco}(F_{\delta_1}(T^k) \cap K) \subset F_{\varepsilon}(T^k) \cap K
\]

for \( k \geq N_1(\varepsilon) \). Let \( \delta(=\delta(\varepsilon)) = \min \left( \frac{\delta_1}{2D_{K}}, \delta_1 \right) \).

Also, from (1.2) we can choose an integer \( N_2(\varepsilon) \geq 1 \) such that if \( k \geq N_2(\varepsilon) \),

\[
\|T^k u - T^k v\| - \|u - v\| < \frac{\varepsilon}{3}
\]

for all \( u, v \in K \).

Moreover by Lemma 2.7 for any integer \( p \geq 1 \) there are positive integers \( N_3(\varepsilon) \) and \( i_p(\varepsilon) \), where \( N_3(\varepsilon) \) is independent of \( p \), such that

\[
\text{clco}(T^k K) \cap F_{\varepsilon}(T^k) \cap K
\]

for \( k \geq N_3(\varepsilon) \), \( i \geq i_p(\varepsilon) \) and \( l \geq 0 \).

Choose \( p(=p(k,\varepsilon)) \geq 1 \) so that \( \frac{D_{K}}{p} \leq \frac{\delta_2}{2} \). Since \( \{x_i\}_{i \geq 0} \) is an almost-orbit of \( T \), there exists an integer \( N_4(\varepsilon) \geq 1 \) such that \( \sup_{q \geq 0}\|T^q x_m - T^q x_0\| < \frac{\delta_2}{8} \) for \( m \geq N_4(\varepsilon) \). Set \( w_i = \frac{1}{p} \sum_{j=0}^{p-1} x_{i+j} \) for \( i \geq 0 \).

If \( i \geq i_p(\varepsilon) + N_4(\varepsilon) \), by (2.13) we have

\[
\|w_{i+k+l} - T^k w_{i+l}\|
\]

\[
\leq \left| \frac{1}{p} \sum_{j=0}^{p-1} (x_{i+j+k+l} - T^k x_{i+j+l}) \right| + \left| \frac{1}{p} \sum_{j=0}^{p-1} T^k x_{i+j+l} - T^k \left( \frac{1}{p} \sum_{j=0}^{p-1} x_{i+j+l} \right) \right| < \frac{\delta_2}{4}
\]

for all \( l \geq 0 \). We also have from (2.12)

\[
\|w_{i+k} - T^k w_i\| \leq \|w_{i+k} - f\| + \|T^k f - T^k w_i\|
\]

\[
\leq \|w_{i+k} - f\| + \|f - w_i\| + \frac{\varepsilon}{3} \leq 2D_{K} + \frac{\varepsilon}{3} =: D_3(\varepsilon)
\]

for \( i \geq 0 \).
Choose \( N_5(k, \varepsilon) \geq i_p(\varepsilon) + N_4(\varepsilon) + 1 \) such that \( \frac{D_k(\varepsilon)(i_p(\varepsilon) + N_4(\varepsilon))}{n} \leq \frac{\varepsilon^2}{4} \) for all \( n \geq N_5(k, \varepsilon) \). If \( n \geq N_5(k, \varepsilon) \), it then follows from (2.14) and (2.15) that
\[
\frac{1}{n} \sum_{i=0}^{n-1} \|w_{i+l} - T^k w_{i+l}\|
\leq \frac{1}{n} \sum_{i=0}^{n-1} \|w_{i+l} - w_{i+k+l}\| + \frac{1}{n} \left( \sum_{i=0}^{i_p + N-1} + \sum_{i=i_p + N}^{n-1} \right) \|w_{i+k+l} - T^k w_{i+l}\|
\leq \frac{D_k}{p} + \left( \frac{i_p + N}{n} \right) D_k(\varepsilon) + \frac{\varepsilon^2}{4} \leq \delta^2
\]
for all \( l \geq 0 \), where \( N = N_4(\varepsilon) \) and \( i_p = i_p(\varepsilon) \). Finally, choose \( N_6(k, \varepsilon) \geq 1 \) so that \( \frac{(p-1)D_k}{2n} < \frac{\varepsilon}{12} \) for all \( n \geq N_6(k, \varepsilon) \). Put \( N_{k, \varepsilon} = \max(N_5(k, \varepsilon), N_6(k, \varepsilon)) \). Let \( n \geq N_{k, \varepsilon} \) and \( l \geq 0 \).

Set \( A(k, n, l) = \{ i \in Z : 0 \leq i \leq n-1 \} \) and \( B(k, n, l) = \{ 0, 1, \ldots, n-1 \} \setminus A(k, n, l) \).

By (2.16), \( \sharp A(k, n, l) \leq n \delta \), where \( \sharp \) denotes cardinality. Since \( \delta D_k < \frac{\varepsilon}{12} \) and \( \frac{(p-1)D_k}{2n} < \frac{\varepsilon}{12} \), by (2.11) we have
\[
\frac{1}{n} \sum_{i=0}^{n-1} x_{i+l} = \frac{1}{n} \sum_{i=0}^{n-1} w_{i+l} + \frac{1}{np} \sum_{i=1}^{p-1} (p - i)(x_{i+l-1} - x_{i+l+n-1})
\]
\[
= \left[ \frac{1}{n} (\sharp A(k, n, l)) \cdot f + \frac{1}{n} \sum_{i \in B(k, n, l)} w_{i+l} \right] + \left[ \frac{1}{n} \sum_{i \in A(k, n, l)} (w_{i+l} - f) \right]
+ \frac{1}{np} \sum_{i=1}^{p-1} (p - i)(x_{i+l-1} - x_{i+l+n-1})
\in \text{clco}(F_\varepsilon(T^k) \cap K) + B_{\Delta \varepsilon} + B_{\Delta \varepsilon} \subset (F_\varepsilon(T^k) \cap K) + B_{\Delta \varepsilon}
\]
for all \( l \geq 0 \). Combining this with (2.12) we find the desired claim.

\[ \square \]

Lemma 2.9. Suppose that the norm of \( X \) is Fréchet differentiable and that \( \{x_n\} \) is an almost-orbit of \( T \). Then the following hold:

(i) \( \{ x_n, J(f - g) \} \) converges as \( n \to \infty \) for every \( f, g \in F(T) \), where \( J \) is the normalized duality map of \( X \).

(ii) \( F(T) \cap \text{clco} \omega_x(\{x_n\}) \) is at most a singleton.

Proof. Let \( \lambda \in (0, 1) \) and \( f, g \in F(T) \). By Lemma 2.7 and (1.2) for \( \varepsilon > 0 \) there exist an integer \( N_\varepsilon \geq 1 \) and \( i_2(\varepsilon) \geq 1 \) such that if \( k \geq N_\varepsilon \) and \( n \geq i_2(\varepsilon) \),
\[
\|T^k (\lambda x_n + (1 - \lambda) f) - \lambda T^k x_n - (1 - \lambda) f\| < \varepsilon,
\|T^k u - T^k v\| - \|u - v\| < \varepsilon
\]
for all \( u, v \in K \). Then we have
\[
\|\lambda x_{n+m} + (1 - \lambda) f - g\| \leq \lambda \|x_{n+m} - T^m x_n\|
+ \|T^m x_n + (1 - \lambda) f - T^m (\lambda x_n + (1 - \lambda) f)\| + \|\lambda x_n + (1 - \lambda) f - g\| + \varepsilon
\]
\[
\leq \sup_{l \geq 0} \|x_{n+l} - T^l x_n\| + \|\lambda x_n + (1 - \lambda) f - g\| + 2 \varepsilon
\]
for $m \geq N_\varepsilon$ and $n \geq i_2(\varepsilon)$. Letting $m \to \infty$, and then $n \to \infty$ and $\varepsilon \downarrow 0$, we get
\[
\limsup_{m \to \infty} \|\lambda x_m + (1 - \lambda)f - g\| \leq \liminf_{n \to \infty} \|\lambda x_n + (1 - \lambda)f - g\|
\]
and so $\|\lambda x_n + (1 - \lambda)f - g\|$ converges as $n \to \infty$. The claim (ii) is shown by the same way as in [7, Lemma 3.6].

Using Lemmas 2.5, 2.8 and 2.9, by the same argument used in the proof of [8, Theorem] we can easily prove Theorem 1.1.

We also have the following result on the weak convergence of an almost-orbit $\{x_n\}$ of $T$.

**Theorem 2.10.** Let $\{x_n\}_{n \geq 0}$ be an almost-orbit of $T$. Suppose that
\[
\lim_{n \to \infty} w-\lim (x_n - x_{n+1}) = 0.
\]
Then we have $\omega_w(\{x_n\}) \subset F(T)$. Further if the norm of $X$ is Fréchet differentiable, $\{x_n\}$ converges weakly to an element in $F(T)$.

**Proof.** Let $u \in \omega_w(\{x_n\})$. Then by definition there is a subsequence $\{p_m\}$ of $\{p\}$ such that $w-\lim_{m \to \infty} x_{p_m} = u$. We have from assumption $w-\lim_{m \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} x_{p_{i+1}} = w-\lim_{m \to \infty} x_{p_m} = u$ for any fixed $n \geq 1$. Thus Lemmas 2.5 and 2.8 show $u \in F(T)$. The last assertion follows from Lemma 2.9 (ii).

**Remark 2.1.** (i) By using the same argument as in [9] with the aid of Lemma 2.5, 2.7 and 2.8 we see that in Theorem 1.1 the assumption that the norm of $X$ is Fréchet differentiable may be replaced by the assumption that $X$ satisfies Opial’s property.

(ii) We can also prove more general ergodic theorems corresponding to [10, Theorems 1 and 2] for semigroups of asymptotically nonexpansive mappings in the intermediate sense. As a consequence we see that [14, Corollary 4] holds for asymptotically nonexpansive mappings in the intermediate sense.

(iii) We do not know whether Theorem 2.10 is valid for asymptotically nonexpansive mappings satisfying (1.1).

**References**


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