AMENABLE REPRESENTATIONS AND FINITE INJECTIVE VON NEUMANN ALGEBRAS

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Abstract. Let $U(M)$ be the unitary group of a finite, injective von Neumann algebra $M$. We observe that any subrepresentation of a group representation into $U(M)$ is amenable in the sense of Bekka; this yields short proofs of two known results—one by Robertson, one by Haagerup—concerning group representations into $U(M)$.

A unitary representation $\pi$ of a group $\Gamma$ on a Hilbert space $H$ is amenable if there exists on $B(H)$ an $\text{Ad}\pi$-invariant state, i.e. a state $\phi$ on $B(H)$ such that, for any $T \in B(H)$, $g \in \Gamma$:

$$\phi(\pi(g)T\pi(g^{-1})) = \phi(T).$$

This notion was introduced and studied by Bekka in [Be].

In the present paper, $M$ will always denote a finite, injective von Neumann algebra. We start with the observation that, if $\pi(\Gamma)$ is contained in the unitary group $U(M)$, then any subrepresentation of $\pi$ is amenable. We use this to give short, hopefully new proofs of two known results. The first, due to Robertson ([Ro], Theorem C and remark (4) on p. 554), states that for any representation $\pi$ of a group $\Gamma$ with Kazhdan’s property (T) into $U(M)$, the closure of $\pi(\Gamma)$ in the $L^2$-norm topology on $U(M)$ is compact. The second, due to Haagerup ([Ha], Lemma 2.2), says that for any $n \in \mathbb{N}$, $U_1, U_2, \ldots, U_n \in U(M)$ and $P$ a non-zero projector in the commutant $M'$ of $M$:

$$\left\| \sum_{i=1}^n PU_i \otimes \bar{PU}_i \right\| = n$$

(where the bar denotes the same operator, but acting on the conjugate Hilbert space).

**Proposition 1.** Let $\pi$ be a representation of a group $\Gamma$ into $U(M)$. Then any subrepresentation of $\pi$ is amenable.

Proof. We may assume that $M = \pi(\Gamma)'$. Let $\rho$ be a subrepresentation of $\pi$ on a closed subspace $\mathcal{H}_\rho$ which is the range of a projector $p \in M'$. To construct an $\text{Ad}\rho$-invariant state on $B(\mathcal{H}_\rho) = pB(\mathcal{H})p$, choose a conditional expectation

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$E: B(\mathcal{H}_\rho) \to pM$, a trace $\tau$ on $pM$, and set $\phi = \tau \circ E$. Then, for any $T \in B(\mathcal{H}_\rho)$, $g \in \Gamma$:

$$\phi(\rho(g)T\rho(g^{-1})) = \tau(\rho(g)E(T)\rho(g^{-1})) = \phi(T),$$

which concludes the proof.

For a representation $\pi$ of $\Gamma$ into $U(M)$, the closure of $\pi(\Gamma)$ in $U(M)$ is compact in the $L^2$-norm if and only if $\pi$ decomposes as a direct sum of finite-dimensional representations. Indeed, if $\pi(\Gamma)$ is relatively compact, the identity representation of the compact group $\overline{\pi(\Gamma)}$ decomposes into finite-dimensional representations; conversely, using the fact that the $L^2$ and strong topologies coincide on $U(M)$, it is easy to see that any unitary representation that decomposes into finite-dimensional ones, has relatively compact range. We then state Robertson’s result mentioned above in the following equivalent form.

**Proposition 2.** Let $\Gamma$ be a group with Kazhdan’s property (T). Any representation $\pi$ of $\Gamma$ into $U(M)$ decomposes as a direct sum of finite-dimensional representations.

**Proof.** By Zorn’s lemma, find a subrepresentation $\rho$ of $\pi$ that decomposes as a direct sum of finite-dimensional representations, and maximal with respect to that property. We have to show that $\rho = \pi$. If this is not the case, consider the subrepresentation $\rho^\perp$ on the orthogonal complement of $H_\rho$. By Proposition 1, $\rho^\perp$ is an amenable representation of $\Gamma$. Because $\Gamma$ has property (T), it follows from Corollary 5.9 of [Be] that any amenable representation of $\Gamma$ has a (non-zero) finite-dimensional subrepresentation. This, applied to $\rho^\perp$, contradicts maximality of $\rho$. \qed

**Remarks.** (1) Proposition 2 makes precise an earlier result of Kirchberg ([Ki], Corollary 1.2): any Kazhdan group that admits a faithful representation into $U(M)$, must be residually finite. Kirchberg also mentions in the same paper [Ki] that his proof of residual finiteness for certain subgroups of $U(M)$ works for a bigger class of subgroups than just subgroups with property (T) (e.g. it works for non-abelian free groups). However, the proof of Proposition 2 does not extend, in view of Theorem 1 of [BV]: Kazhdan groups are characterized by the fact that any amenable representation has a finite-dimensional subrepresentation.

(2) Robertson proved ([Ro], Lemma 4.2) a result more general than our Proposition 2: any representation of a Kazhdan group into the unitary group of a finite von Neumann algebra with Haagerup’s approximation property, decomposes as a direct sum of finite-dimensional representations.

We now turn to Haagerup’s result mentioned in the beginning.

**Proposition 3.** For any $n \in \mathbb{N}$, $U_1, U_2, \ldots, U_n \in U(M)$ and $P$ a non-zero projector in the commutant $M'$ of $M$:

$$\left\| \sum_{i=1}^n PU_i \otimes \overline{P} U_i \right\| = n.$$

**Proof.** Let $\Gamma$ be a finitely generated group, $S$ a finite generating subset, and $\pi$ a unitary representation of $\Gamma$. It follows from Theorem 5.1 of [Be] and remark (b) on p. 74 of [HRV], that $\pi$ is amenable if and only if 1 belongs to the spectrum of the operator $\frac{1}{|S|} \sum_{s \in S} (\pi \otimes \overline{\pi})(s)$, where $\overline{\pi}$ denotes the contragredient representation of $\pi$. If this is the case, then $\| \sum_{s \in S} (\pi \otimes \overline{\pi})(s) \| = |S|$. The result then follows by
considering the $U_i$'s as generators of a representation of the free group $\mathbb{F}_n$ on $n$ generators, and appealing to Proposition 1.

**Remark.** Haagerup has proved that Proposition 3 actually yields a characterization of finite, injective von Neumann algebras. Indeed, it is enough to assume $\| \sum_{i=1}^{n} PU_i \otimes \bar{P} U_i \| = n$ for any $n$-tuple of unitaries and any non-zero central projection, to make sure that $M$ is finite and injective (see [Ha], Lemma 2.2). For a factor acting on a separable Hilbert space, this characterization of the hyperfinite $\text{II}_1$-factor is due to Connes ([Co], Remark 5.29).

**References**


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