WEIGHTED HARDY-LITTLEWOOD INEQUALITY
FOR $A$-HARMONIC TENSORS

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Abstract. In this paper we prove a local weighted integral inequality for conjugate $A$-harmonic tensors similar to the Hardy and Littlewood integral inequality for conjugate harmonic functions. Then by using the local weighted integral inequality, we prove a global weighted integral inequality for conjugate $A$-harmonic tensors in John domains.

1. Introduction and notation

Conjugate harmonic functions have wide applications in many fields, such as potential theory, harmonic analysis and the theory of $H^p$-spaces. Conjugate $A$-harmonic tensors are interesting and important generalizations of conjugate harmonic functions and $p$-harmonic functions, $p > 1$. Many interesting results of conjugate $A$-harmonic tensors and their applications in fields such as quasiregular mappings and the theory of elasticity have been found recently, see [N3], [I], [IM], [S], [B] and [BM]. In this paper, we prove local weighted inequalities and global weighted integral inequalities for conjugate $A$-harmonic tensors in John domains.

Let $e_1, e_2, \cdots, e_n$ denote the standard unit basis of $\mathbb{R}^n$. For $l = 0, 1, \cdots, n$, the linear space of $l$-vectors, spanned by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l}$, corresponding to all ordered $l$-tuples $I = (i_1, i_2, \cdots, i_l)$, $1 \leq i_1 < i_2 < \cdots < i_l \leq n$, is denoted by $\wedge^l = \wedge^l(\mathbb{R}^n)$. The Grassmann algebra $\wedge = \bigoplus \wedge^l$ is a graded algebra with respect to the exterior products. For $\alpha = \sum \alpha_I e_I \in \wedge$ and $\beta = \sum \beta_I e_I \in \wedge$, the inner product in $\wedge$ is given by $\langle \alpha, \beta \rangle = \sum \alpha_I \beta_I$ with summation over all $l$-tuples $I = (i_1, i_2, \cdots, i_l)$ and all integers $l = 0, 1, \cdots, n$.

We define the Hodge star operator $\star$: $\wedge \to \wedge$ by the rule

$$\star 1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n$$

and $\alpha \wedge \star \beta = \beta \wedge \star \alpha = \langle \alpha, \beta \rangle (\star 1)$

for all $\alpha, \beta \in \wedge$.

Hence the norm of $\alpha \in \wedge$ is given by the formula $|\alpha|^2 = \langle \alpha, \alpha \rangle = (\alpha \wedge \star \alpha) \in \wedge^0 = \mathbb{R}$. The Hodge star is an isometric isomorphism on $\wedge$ with $\star : \wedge^l \to \wedge^{n-l}$ and $\star \star (-1)^{l(n-l)} : \wedge^l \to \wedge^l$.

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Throughout this paper, we always assume $\Omega$ is a connected open subset of $\mathbb{R}^n$. We write $\mathbb{R} = \mathbb{R}^1$. Cubes or balls are denoted by $Q$ and $\sigma Q$ is the cube or ball with the same center as $Q$ and with $\text{diam}(\sigma Q) = \sigma \text{diam}(Q)$. The $n$-dimensional Lebesgue measure of a set $E \subseteq \mathbb{R}^n$ is denoted by $|E|$. Suppose that $w \in L^1_{\text{loc}}(\mathbb{R}^n)$, $w > 0$ a.e., and $0 < p < \infty$. We denote the weighted $L^p$-norm of a measurable function $f$ over $E$ by

$$
||f||_{p,E,w} = \left(\int_E |f(x)|^p w(x)dx\right)^{1/p}.
$$

A differential $l$-form $\omega$ on $\Omega$ is a Schwartz distribution on $\Omega$ with values in $\wedge^l(\mathbb{R}^n)$. We denote the space of differential $l$-forms by $D'(\Omega, \wedge^l)$. We write $L^p(\Omega, \wedge^l)$ for the $l$-forms $\omega(x) = \sum_I \omega_I(x)dx_I = \sum \omega_{i_1,i_2,\ldots,i_l}(x)dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}$ with $\omega_I \in L^p(\Omega, \mathbb{R})$ for all ordered $l$-tuples $I$. Thus $L^p(\Omega, \wedge^l)$ is a Banach space with norm

$$
||\omega||_{p,\Omega} = \left(\int_\Omega |\omega(x)|^pdx\right)^{1/p} = \left(\int_\Omega \left(\sum_I |\omega_I(x)|^2\right)^{p/2}dx\right)^{1/p}.
$$

Similarly, $W^l_p(\Omega, \wedge^l)$ are those differential $l$-forms on $\Omega$ whose coefficients are in $W^1_p(\Omega, \mathbb{R})$. The notations $W^l_{p,\text{loc}}(\Omega, \mathbb{R})$ and $W^1_{p,\text{loc}}(\Omega, \wedge^l)$ are self-explanatory. We denote the exterior derivative by $d : D'(\Omega, \wedge^l) \to D'(\Omega, \wedge^{l+1})$ for $l = 0, 1, \ldots, n$. Its formal adjoint operator $d^* : D'(\Omega, \wedge^{l+1}) \to D'(\Omega, \wedge^l)$ is given by $d^* = (-1)^{n-l+1} \ast d \ast$ on $D'(\Omega, \wedge^{l+1})$, $l = 0, 1, \ldots, n$.

**Definition 1.1.** We call $u$ a $p$-harmonic function if $u$ satisfies the $p$-harmonic equation

$$
\text{div}(\nabla u|\nabla u|^{p-2}) = 0
$$

with $p > 1$. Its conjugate in the plane is a $q$-harmonic function $v$, $p^{-1} + q^{-1} = 1$, which satisfies

$$
\nabla u|\nabla u|^{p-2} = \left(\frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x}\right).
$$

Note that if $p = q = 2$, we get the usual conjugate harmonic functions. If $\omega : \Omega \to \wedge^l$, then the value of $\omega(x)$ at the vectors $\xi_1, \ldots, \xi_l \in \mathbb{R}^n$ will be denoted by $\omega(x; \xi_1, \ldots, \xi_l)$. The following lemma appears in [IL].

**Lemma 1.3.** Let $Q \subset \mathbb{R}^n$ be a cube. To each $y \in Q$ there corresponds a linear operator $K_y : C^\infty(Q, \wedge^l) \to C^\infty(Q, \wedge^{l-1})$ defined by

$$
(K_y\omega)(x; \xi_1, \ldots, \xi_l) = \int_0^1 t^{l-1} \omega(tx + y - ty; x - y, \xi_1, \ldots, \xi_{l-1})dt
$$

and the decomposition

$$
\omega = d(K_y\omega) + K_y(d\omega).
$$

We define another linear operator $T_Q : C^\infty(Q, \wedge^l) \to C^\infty(Q, \wedge^{l-1})$ by averaging $K_y$ over points $y$ in $Q$

$$
T_Q\omega = \int_Q \varphi(y)K_y\omega dy,
$$

where
where \( \varphi \in C_0^\infty(Q) \) is normalized by \( \int_Q \varphi(y)dy = 1 \). We define the \( l \)-form \( \omega_Q \in D'(Q, \wedge^l) \) by
\[
(1.5) \quad \omega_Q = |Q|^{-1} \int_Q \omega(y)dy, \quad l = 0, \quad \text{and} \quad \omega_Q = d(T_Q\omega), \quad l = 1, 2, \cdots, n,
\]
for all \( \omega \in L^p(Q, \wedge^l), \; 1 \leq p < \infty \).

In recent years there has been new interest developed in the study of the \( A \)-harmonic equation for differential forms, largely pertaining to applications in quasiconformal analysis and nonlinear elasticity, that is:
\[
(1.6) \quad d^* A(x, d\omega) = 0,
\]
where \( A : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n) \) satisfies the following conditions:
\[
(1.7) \quad |A(x, \xi)| \leq a|\xi|^{p-1} \quad \text{and} \quad \langle A(x, \xi), \xi \rangle \geq |\xi|^p
\]
for almost every \( x \in \Omega \) and all \( \xi \in \wedge^l(\mathbb{R}^n) \). Here \( a > 0 \) is a constant and \( 1 < p < \infty \) is a fixed exponent associated with (1.6). A solution to (1.6) is an element of the Sobolev space \( W^{1,p,\text{loc}}(\Omega, \wedge^{l-1}) \) such that
\[
\int_{\Omega} \langle A(x, d\omega), d\varphi \rangle = 0
\]
for all \( \varphi \in W^1_{\text{loc}}(\Omega, \wedge^{l-1}) \) with compact support.

In order to formulate the Hardy-Littlewood type estimate it is required first of all that the equation is written in the form of a first order differential system:
\[
(1.8) \quad A(x, du) = d^* v .
\]

In this way we obtain a pair \((u, v)\) of \((l-1)\)-form \( u \) and \((l+1)\)-form \( v \), called the conjugate \( A \)-harmonic fields. Example: \( du = d^* v \) is an analogue of a Cauchy-Riemann system in \( \mathbb{R}^n \). Clearly, the \( A \)-harmonic equation is not affected by adding a closed form to \( u \) and coclosed form to \( v \). Therefore, any type of estimates between \( u \) and \( v \) must be modulo such forms. Suppose that \( u \) is a solution to (1.6) in \( \Omega \). Then by Poincaré’s lemma, at least locally in a ball \( B \), there exists a form \( v \in W^1_{q,\text{loc}}(B, \wedge^{l+1}) \), \( \frac{1}{p} + \frac{1}{q} = 1 \), such that (1.8) holds.

**Definition 1.9.** When \( u \) and \( v \) satisfy (1.8) in \( \Omega \), and \( A^{-1} \) exists in \( \Omega \), we call \( u \) and \( v \) conjugate \( A \)-harmonic tensors in \( \Omega \).

Hardy and Littlewood in [HL] proved the following result.

**Theorem A.** For each \( p > 0 \), there is a constant \( C \) such that
\[
\int_D |u - u(0)|^p dx dy \leq C \int_D |v - v(0)|^p dx dy
\]
for all analytic functions \( f = u + iv \) in the unit disk \( D \).

Craig A. Nolder proves the similar results about \( K \)-quasiregular mappings in [N1] and [N2]. Recently, Craig A. Nolder generalized the above result and proved the following important Theorem B and Theorem C about conjugate \( A \)-harmonic tensors [N3].
**Theorem B.** Let \( u \) and \( v \) be conjugate \( A \)-harmonic tensors in \( \Omega \subset \mathbb{R}^n, \sigma > 1, \) and \( 0 < s, t < \infty \). Then there exists a constant \( C \), independent of \( u \) and \( v \), such that
\[
\| u - u_Q \|_{s, Q} \leq C|Q|^\beta \| v - c_1 \|_{t, \sigma Q}^{q/p}
\]
and
\[
\| v - v_Q \|_{t, Q} \leq C|Q|^{-\beta p/q} \| u - c_2 \|_{s, \sigma Q}^{p/q}
\]
for all cubes \( Q \) with \( \sigma Q \subset \Omega \). Here \( c_1 \) is any form in \( W^1_{p, loc}(\Omega, \Lambda) \) with \( d^* c_1 = 0 \), \( c_2 \) is any form in \( W^1_{q, loc}(\Omega, \Lambda) \) with \( d c_2 = 0 \) and \( \beta = 1/s + 1/n - (1/t + 1/n)q/p \).

**Theorem C.** Let \( u \in D^\prime(\Omega, \Lambda^0) \) and \( v \in D^\prime(\Omega, \Lambda^2) \) be conjugate \( A \)-harmonic tensors and \( 0 < s, t < \infty \). If \( \Omega \) is a \( \delta \)-John domain, \( q \leq p \), \( v - c \in L^t(\Omega, \Lambda^2) \) and
\[
s = \Phi(t) = \frac{npt}{nq + t(q - p)}, \quad 0 < t < \infty,
\]
then \( u \in L^s(\Omega, \Lambda^0) \) and moreover, there exists a constant \( C \), independent of \( u \) and \( v \), such that
\[
\| u - u_{Q_0} \|_{s, \Omega} \leq C \| v - c \|_{t, \Omega}^{q/p}.
\]
Here \( c \) is any form in \( W^1_{q, loc}(\Omega, \Lambda) \) with \( d^* c = 0 \).

Our main results Theorem 2.4 and Theorem 3.4 generalize (1.10) and (1.12), respectively.

2. **The local weighted integral inequality**

We will use the following generalized Hölder’s inequality.

**Lemma 2.1.** Let \( 0 < \alpha < \infty, 0 < \beta < \infty \) and \( s^{-1} = \alpha^{-1} + \beta^{-1} \). If \( f \) and \( g \) are measurable functions on \( \mathbb{R}^n \), then
\[
\| fg \|_{s, \Omega} \leq \| f \|_{\alpha, \Omega} \cdot \| g \|_{\beta, \Omega}
\]
for any \( \Omega \subset \mathbb{R}^n \).

The following definition appears in [G].

**Definition 2.2.** We say the weight \( w(x) > 0 \) satisfies the \( A_r \) condition, \( r > 1 \), if
\[
\sup_Q \left( \frac{1}{|Q|} \int_Q w(x) dx \right) \left( \frac{1}{|Q|} \int_Q \left( \frac{1}{w} \right)^{1/(r - 1)} dx \right)^{r^{-1}} < \infty
\]
for any cube \( Q \subset \mathbb{R}^n \).

We also need the following lemma [G].

**Lemma 2.3.** If \( w \in A_r, r > 1 \), then there exist constants \( \beta > 1 \) and \( C \), independent of \( w \), such that
\[
\| w \|_{\beta, Q} \leq C|Q|^{(1 - \beta)/\beta} \| w \|_{1, Q}
\]
for all cubes \( Q \subset \mathbb{R}^n \).

Now, we can prove the following local weighted result.
**Theorem 2.4.** Let $u$ and $v$ be conjugate $A$-harmonic tensors in a domain $\Omega \subset \mathbb{R}^n$ and $w \in A_r$. Let $s = \Phi(t)$ be defined by (1.11). Then there exists a constant $C$, independent of $u$ and $v$, such that

\[
(2.5) \quad \left( \int_Q |u - u_Q|^s w dx \right)^{1/s} \leq C \left( \int_{\sigma Q} |v - c|^t w^{pt/qs} dx \right)^{q/p},
\]

for all cubes $Q$ with $\sigma Q \subset \Omega \subset \mathbb{R}^n$ and $\sigma > 1$. Here $c$ is any form in $W_{q,loc}^1(\Omega, \Lambda)$ with $d^*c = 0$.

**Proof.** By Lemma 2.3, there exist constants $\alpha > 1$ and $C_1$, independent of $w$, such that

\[
(2.6) \quad \| w \|_{\alpha,\sigma Q} \leq C_1 |Q|^{(1-\alpha)/\alpha} \| w \|_{1,\sigma Q}.
\]

Since $1/\alpha s + (\alpha - 1)/\alpha s = 1/s$, then by Lemma 2.1, we have

\[
(2.7) \quad \| u - u_Q \|_{s,\sigma Q, w} \leq \| w \|_{\alpha, Q}^{1/s} \| u - u_Q \|_{\alpha s/(\alpha - 1), Q}.
\]

By Theorem B, there is a constant $C_2$, independent of $u$ and $v$, such that for any $t' > 0$, we have

\[
(2.8) \quad \| u - u_Q \|_{\alpha s/(\alpha - 1), Q} \leq C_2 |Q|^\beta' \| v - c \|_{\alpha, \sigma Q}^{q/p},
\]

where $\beta' = (\alpha - 1)/\alpha s + 1/n - (1/t' + 1/n)q/p$. Combining (2.7) and (2.8), we obtain

\[
(2.9) \quad \| u - u_Q \|_{s,\sigma Q, w} \leq C_2 |Q|^\beta' \| w \|_{\alpha, Q}^{1/s} \| v - c \|_{\alpha, \sigma Q}^{q/p}.
\]

Now, choose $t' = t/k$, where $k$ is to be determined later, and note $|v - c| = w^{-n/q} |v - c| w^{pt/qs}$; by Lemma 2.1, we get

\[
(2.10) \quad \| v - c \|_{t', \sigma Q} \leq \| (1/w)^{pt/qs} \|_{1/(k-1), \sigma Q}^{1/t} \left( \int_{\sigma Q} |v - c|^t w^{pt/qs} dx \right)^{1/k}.
\]

From (2.6), (2.9) and (2.10) we have

\[
(2.11) \quad \| u - u_Q \|_{s,\sigma Q, w} \leq C_3 \bigg| Q \bigg|^{\beta' + (1-\alpha)/\alpha s} \| w \|_{1,\sigma Q}^{1/s} \cdot \| (1/w)^{pt/qs} \|_{1/(k-1), \sigma Q}^{q/p} \left( \int_{\sigma Q} |v - c|^t w^{pt/qs} dx \right)^{q/p}.
\]

We choose $k = 1 + pt(r - 1)/qs$. Then $(k - 1)qs/pt = r - 1$, and by $w \in A_r$, we know

\[
(2.12) \quad \| w \|_{1,\sigma Q}^{1/s} \| (1/w)^{pt/qs} \|_{1/(k-1), \sigma Q}^{q/p} = \bigg| Q \bigg|^{1/s + (k-1)q/pt} \left( \frac{1}{\sigma Q} \right)^{1/(k-1)} \left( \frac{1}{\sigma Q} \right)^{1/(r-1)} \left( \frac{1}{w} \right)^{1/(r-1)} dx \right)^{1/s} \leq C_4 |Q|^{1/s + (k-1)q/pt}.
\]

By (2.11) and (2.12), we have

\[
(2.13) \quad \| u - u_Q \|_{s,\sigma Q, w} \leq C_5 |Q|^r \left( \int_{\sigma Q} |v - c|^t w^{pt/qs} dx \right)^{q/p}.
\]
where $\gamma = \beta^* + (1 - \alpha)/\alpha s + 1/s + q(k - 1)/pt = -(nq + t(q - p))/npt + 1/s = 0$ by (1.11). So (2.13) becomes
\[
\| u - u_Q \|_{s,Q,w} \leq C \left( \int_{\sigma Q} |v - c|^q w^{pt/qs} \, dx \right)^{q/pt},
\]
that is,
\[
\left( \int_Q |u - u_Q|^s w \, dx \right)^{1/s} \leq C \left( \int_{\sigma Q} |v - c|^q w^{pt/qs} \, dx \right)^{q/pt}.
\]
We have completed the proof of Theorem 2.4.

3. THE GLOBAL WEIGHTED INTEGRAL INEQUALITY

**Definition 3.1.** We call $\Omega$, a proper subdomain of $\mathbb{R}^n$, a $\delta$-John domain, $\delta > 0$, if there exists a point $x_0 \in \Omega$ which can be joined with any other point $x \in \Omega$ by a continuous curve $\gamma \subset \Omega$ so that
\[
d(\xi, \partial \Omega) \geq \delta|x - \xi|
\]
for each $\xi \in \gamma$. Here $d(\xi, \partial \Omega)$ is the Euclidean distance between $\xi$ and $\partial \Omega$.

We know that a $\delta$-John domain has the following properties [N3].

**Lemma 3.2.** Let $\Omega \subset \mathbb{R}^n$ be a $\delta$-John domain. Then there exists a covering $\mathcal{V}$ of $\Omega$ consisting of open cubes such that:

i) $\sum_{Q \in \mathcal{V}} \chi_Q(x) \leq N \chi_{\Omega}(x)$, $x \in \mathbb{R}^n$.

ii) There is a distinguished cube $Q_0 \in \mathcal{V}$ (called the central cube) which can be connected with every cube $Q \in \mathcal{V}$ by a chain of cubes $Q_0, Q_1, \cdots, Q_k = Q$ from $\mathcal{V}$ such that for each $i = 0, 1, \cdots, k - 1$,
\[
Q \subset NQ_i.
\]
There is a cube $R_i \subset \mathbb{R}^n$ (this cube does not need to be a member of $\mathcal{V}$) such that
\[
R_i \subset Q_i \cap Q_{i+1}, \quad \text{and} \quad Q_i \cup Q_{i+1} \subset NR_i.
\]

The following lemma appears in [IN].

**Lemma 3.3.** If $\mathcal{V}$ is a collection of cubes in $\mathbb{R}^n$ and $C_Q$ are non-negative numbers associated with the cubes $Q \in \mathcal{V}$ and $w \in A_r$, $d\mu(x) = w(x) \, dx$, then for $1 \leq p < \infty$ and $N \geq 1$ we have
\[
\left( \int_{\mathbb{R}^n} \left( \sum_{Q \in \mathcal{V}} C_Q \chi_{NQ} \right)^p \, d\mu(x) \right)^{1/p} \leq B_p \left( \int_{\mathbb{R}^n} \left( \sum_{Q \in \mathcal{V}} C_Q \chi_Q \right)^p \, d\mu(x) \right)^{1/p},
\]
where $B_p$ is independent of the collection $\mathcal{V}$ and the numbers $C_Q$.

**Theorem 3.4.** Let $u \in D'(\Omega, \Lambda^0)$ and $v \in D'(\Omega, \Lambda^2)$ be conjugate $A$-harmonic tensors. Let $q \leq p$, $v - c \in L'(\Omega, \Lambda^2)$, $w \in A_r$ and $s = \Phi(t)$ is defined in (1.11). Then there exists a constant $C$, independent of $u$ and $v$, such that
\[
\left( \int_\Omega |u - u_{Q_0}|^s w \, dx \right)^{1/s} \leq C \left( \int_\Omega |v - c|^q w^{pt/qs} \, dx \right)^{q/pt}
\]
for any $\delta$-John domain $\Omega \subset \mathbb{R}^n$. Here $c$ is any form in $W^1_{q, \text{loc}}(\Omega, \Lambda)$ with $d^* c = 0$ and $Q_0 \subset \Omega$ is the cube appearing in Lemma 3.2.
Proof. Since $w \in A_r$, we can write $d\mu(x) = w(x)dx$; then (2.5) can be written as

$$\int_{Q} |u - u_Q|^s d\mu(x) \leq C \left( \int_{\sigma Q} |v - c|^t w^{pt/q^*} dx \right)^{qs/pt}. \tag{3.5}$$

We use the notations and the covering $\mathcal{V}$ described in the above Lemma 3.2 and the properties of the measure $d\mu(x) = w(x)dx$: if $w \in A_r$, then

$$\mu(NQ) \leq MN^{nr} \mu(Q) \tag{3.6}$$

for each cube $Q$ with $NQ \subset \mathbb{R}^n$ (see [G]) and

$$\max(\mu(Q_i), \mu(Q_{i+1})) \leq MN^{nr} \mu(Q_i \cap Q_{i+1}) \tag{3.7}$$

for the sequence of cubes $Q_i, Q_{i+1}, i = 0, 1, \ldots, k - 1$ described in ii). We will use the elementary inequality $|a + b|^s \leq 2^s(|a|^s + |b|^s)$ for all $s > 0$. In particular we have

$$\int_{\Omega} |u - u_Q|^s wdx = \int_{\Omega} |u - u_Q|^s d\mu(x) \leq 2^s \sum_{Q \in \mathcal{V}} \int_{Q} |u - u_Q|^s d\mu(x) + 2^s \sum_{Q \in \mathcal{V}} \int_{Q} |u_Q - u_Q|^s d\mu(x). \tag{3.8}$$

The first sum can be estimated by (3.5) and the condition i):

$$\sum_{Q \in \mathcal{V}} \int_{Q} |u - u_Q|^s d\mu(x) \leq C_1 \sum_{Q \in \mathcal{V}} \left( \int_{\sigma Q} |v - c|^t w^{pt/q^*} dx \right)^{qs/pt} \tag{3.9}$$

$$\leq C_1 N \left( \int_{\Omega} |v - c|^t w^{pt/q^*} dx \right)^{qs/pt}.$$

Now we estimate the second sum in (3.8). Fix a cube $Q \in \mathcal{V}$ and let $Q_0, Q_1, \ldots, Q_k = Q$ be the chain from ii). We have

$$|u_{Q_0} - u_Q| \leq \sum_{i=0}^{k-1} |u_{Q_i} - u_{Q_{i+1}}|. \tag{3.10}$$

From (3.5) and (3.7) we have

$$|u_{Q_i} - u_{Q_{i+1}}|^s = \frac{1}{\mu(Q_i \cap Q_{i+1})} \int_{Q_i \cap Q_{i+1}} |u_{Q_i} - u_{Q_{i+1}}|^s d\mu(x) \leq \frac{1}{\max(\mu(Q_i), \mu(Q_{i+1}))} \int_{Q_i \cap Q_{i+1}} |u_{Q_i} - u_{Q_{i+1}}|^s d\mu(x) \tag{3.11}$$

$$\leq C_2 \sum_{j=i}^{i+1} \frac{1}{\mu(Q_j)} \int_{Q_j} |u - u_Q|^s d\mu(x) \leq C_3 \sum_{j=i}^{i+1} \frac{1}{\mu(Q_j)} \left( \int_{\sigma Q_j} |v - c|^t w^{pt/q^*} dx \right)^{qs/pt} \tag{3.12}$$

Since $Q \subset NQ_j$ for $j = i, i + 1, 0 \leq i \leq k - 1$ (see ii)), we have

$$|u_{Q_i} - u_{Q_{i+1}}|^s \chi_Q(x) \leq C_3 \sum_{j=i}^{i+1} \frac{X_{NQ_j}(x)}{\mu(Q_j)} \left( \int_{\sigma Q_j} |v - c|^t w^{pt/q^*} dx \right)^{qs/pt} \tag{3.13}.$$
By (3.10) we have (note $|a + b|^{1/s} \leq 2^{1/s}(|a|^{1/s} + |b|^{1/s})$)

$$|u_{Q_0} - u_Q| \chi_Q(x) \leq C_4 \sum_{R \in V} \left( \frac{1}{\mu(R)} \left( \int_{\sigma R} |v - c|^t w^{pt/q^s} \, dx \right)^{qs/pt} \right)^{1/s} \chi_{NR}(x)$$

for every $x \in \mathbb{R}^n$. Hence

(3.11)

$$\sum_{Q \in V} \int_Q |u_{Q_0} - u_Q|^s d\mu(x)$$

$$\leq C_5 \int_{\mathbb{R}^n} \left| \sum_{R \in V} \left( \frac{1}{\mu(R)} \left( \int_{\sigma R} |v - c|^t w^{pt/q^s} \, dx \right)^{qs/pt} \right)^{1/s} \chi_{NR}(x) \right|^s \, d\mu(x).$$

If $0 \leq s \leq 1$, we use the inequality $|\sum t_\alpha|^s \leq \sum |t_\alpha|^s$, (3.6) and the condition i) to get

$$\sum_{Q \in V} \int_Q |u_{Q_0} - u_Q|^s d\mu(x) \leq C_6 \sum_{R \in V} \frac{\mu(N R)}{\mu(R)} \left( \int_{\sigma R} |v - c|^t w^{pt/q^s} \, dx \right)^{qs/pt}$$

$$\leq C_7 \sum_{R \in V} \left( \int_{\sigma R} |v - c|^t w^{pt/q^s} \, dx \right)^{qs/pt}.$$

Note $qs/pt \geq 1$ and $\sum t_\alpha^p \leq (\sum t_\alpha)^p$ for $p \geq 1$ and $t_\alpha > 0$; then

$$\sum_{Q \in V} \int_Q |u_{Q_0} - u_Q|^s d\mu(x) \leq C_7 \sum_{R \in V} \left( \int_{\Omega} |v - c|^t w^{pt/q^s} \chi_{\sigma R}(x) \, dx \right)^{qs/pt}$$

$$\leq C_7 \left( \int_{\Omega} |v - c|^t w^{pt/q^s} \chi_{\sigma R}(x) \, dx \right)^{qs/pt} \leq C_7 \left( \int_{\Omega} |v - c|^t w^{pt/q^s} \chi_{\sigma R}(x) \, dx \right)^{qs/pt} \leq C_7 \left( \int_{\Omega} |v - c|^t w^{pt/q^s} \chi_{\sigma R}(x) \, dx \right)^{qs/pt}.$$
with the elementary inequality $|\sum_{i=1}^{N} t_i|^s \leq N^{s-1} \sum_{i=1}^{N} |t_i|^s$; we obtain

$$\sum_{Q \in V} \int_Q |u_{Q,0} - u_Q|^s d\mu(x) \leq C_{10} \int_{R^n} \left( \sum_{R \in V} \frac{1}{\mu(R)} \left( \int_{\sigma R} |v - c|^t u^{pt/qs} dx \right)^{qs/pt} \chi_R(x) \right) d\mu(x)$$

(3.13)

$$= C_{10} \sum_{R \in V} \left( \int_{\sigma R} |v - c|^t u^{pt/qs} dx \right)^{qs/pt} \leq C_{11} \left( \int_{\Omega} |v - c|^t u^{pt/qs} dx \right)^{qs/pt}$$

by the condition i). Combining (3.8), (3.9) and (3.13), we have proved the theorem for the case $1 \leq s < \infty$; thus, we have completed the proof of Theorem 3.4.

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**REFERENCES**


