ISOTOPY AND IDENTITIES IN ALTERNATIVE ALGEBRAS

M. BABIKOV

(Communicated by Lance W. Small)

Abstract. In this paper we show how to construct an isomorphism between an alternative algebra $A$ over a field of characteristic $\neq 3$ and its isotope $A^{(1+c)}$, where $c$ is an element of Zhevlakov’s radical of $A$. This leads to the equivalence of any polynomial identity $f = 0$ in alternative algebras and the isotope identity $f^{(s)} = 0$.

Given an invertible element $s$ of an alternative algebra $A$, we can form a new algebra by taking the same linear structure but a new multiplication

$$x *_s y = (xs)y.$$

The resulting algebra, denoted $A^{(s)}$, is also alternative (see [1]) and is called an $s$-isotope. Associative and Cayley isotopes are always isomorphic [2], and an isotope of a finite-dimensional alternative algebra over an algebraically closed field of characteristic $\neq 3$ is isomorphic to the original algebra as well [1]. We consider the case of an arbitrary alternative algebra over a field of characteristic $\neq 3$ with nonzero Zhevlakov radical, and a particular choice of $s$, and construct an explicit isomorphism between $A$ and $A^{(s)}$. As a consequence, we derive the equivalence of any polynomial identity and its isotope in an arbitrary alternative algebra over a field of characteristic $\neq 3$. The paper has benefited from many discussions with J. Ferrar, and I would like to thank him for his valuable help and encouragement.

Theorem 1. There is a set of coefficients $t^i_j, i, j \geq 0$, such that $t^0_0 = 1$ and for any integer $m$, any alternative algebra $A$ over a field of characteristic $\neq 3$ and any elements $a, b, c$ from $A$ such that $\text{Id}_A(c)^m = 0$, the polynomial $t(x) = \sum_{i,j} t^i_j c^i x^j$ satisfies

$$t(a) *_{1+c} t(b) = t(ab).$$

Proof. We will show the way to calculate the coefficients $t^i_j$ such that (1) holds for any alternative algebra $A$. Clearly,

$$t(a) *_{1+c} t(b) = t(a)t(b) + (t(a)c)t(b) = \sum t^i_j t^k_i (c^i ac^j)(c^k bc^l) + \sum t^k_i (c^i ac^{j+1})(c^k bc^l) = \sum t^k_i (t^i_j + t^i_{j-1})(c^i ac^j)(c^k bc^l),$$

Received by the editors March 28, 1995.
1991 Mathematics Subject Classification. Primary 17D05.

©1997 American Mathematical Society

1571
where $t_{i-1} = 0$ and $0 \leq i, j, k, l \leq m - 1$. For brevity we shall denote $t^k_i (t^j_j + t^i_{i-1})$

simply by $t^k_{ij}$; then

$$t(a) *_{1+e} t(b) - t(ab) = \sum_{i,j} t^k_{ij} ((c^i a c^j + k)(bc^l) + (c^i a c^j, c^k, bc^l))$$

$$- \sum_{i,j} t^k_{ij} ((c^i a)(bc^l) - (c^i, a, b)c^l + (c^i a, b, c^l))$$

$$= \sum_{i,l} \sum_{j,k=0}^{m-1} t^k_{ij} (c^i a c^j)(bc^l) + \sum_{j,k=0}^{m-1} t^k_{ij} c^i + \sum_{j,k=0}^{m-1} t^k_{ij} (c^i a, b, c^l)$$

(2)

We need (1) to be true for any alternative algebra, in particular for an associative one, for which all the associators in (2) are zeroes and (1) is equivalent to

$$\sum_{i,l} \sum_{j,k=0}^{m-1} t^k_{ij} c^i + \sum_{j,k=0}^{m-1} t^k_{ij} (c^i a, b, c^l)$$

(3)

For the cases $n = 0$ and $n \geq 1$ we get that

$$t^0_{il} = t^0_{i},$$

(4)

$$\sum_{j+k=n} t^k_{ij} = 0,$$

(5)

for any $i$ and $l$. It is easy to see that (4) yields

$$t^i_i = t^0_{i} = t^0_{i}.$$  

(6)

But then

$$\sum_{j+k=n} t^k_{ij} = t^1_i \sum_{j+k=n} (t^k_{ij} + t^k_{i-1}),$$

and therefore (5) yields

$$\sum_{j+k=n} t^k_{ij} = - \sum_{j+k=n-1} t^k_{ij}.$$  

Consequently, since $\sum_{j+k=1} t^k_{ij} = t^0_{i} = 1$ and by induction on $n$,

$$\sum_{j+k=n} t^k_{ij} = (-1)^n.$$  

(7)

Now we assume (6) and (7), which implies (3). Going back to the alternative case we note that (4) and (5) also imply

$$\sum_{i,l} \sum_{j,k=0}^{m-1} t^k_{ij} (c^i a c^j)(bc^l) + \sum_{j,k=0}^{m-1} t^k_{ij} (c^i a, b, c^l) = 0.$$  

So in the expression (2) only sums with the associators are left. It is easy to see that in any alternative algebra the Moufang identities imply

$$(a, b, c^{k+l}) = -(a, c^l, c^{k+b}) + b(a, c^l, c^k) + c^l(a, b, c^k)$$

$$= (a, b, c^l)c^k + c^l(a, b, c^k).$$
Using this identity we get
\[ t(a) * t(b) - t(ab) = \sum t^k_{ij}(-a(b, c^{j+k})c^{k+i} + (a, b, c^{j+k})c^j) \]
\[ - \sum t^i_j(a, b, c^j)c^i + \sum t^i_j(a, b, c^i)c^j \]
\[ = \sum T_{pq}(a, b, c^p)c^q, \]

where
\[ T_{pq} = t^p_q - t^q_p - \sum_{l+j=p, k+i=q} t^k_{ij} + \sum_{l+j+k=p, i=q} t^k_{ij}, \]

Let us simplify this expression for \( T_{pq} \). Clearly by (7)
\[ \sum_{l+j=k=p} t^k_{ij} = \sum_{k+l=p-j} t^k_j \sum_{j} i_{0q} = \sum_{j} (-1)^{p-j} i_{0j} \]
\[ = \sum_{j} (-1)^{p-j}(t^q_j + t^q_{j-1}) = t^q_p, \]

and so
\[ T_{pq} = t^p_q - \sum_{l+j=p, k+i=q} t^k_{ij}. \]

Now we use induction on \( d = p + q \) to prove the existence of \( t^d_0 \) and \( t^d_0 \) such that \( T_{pq} = 0 \) for any \( p + q \leq d \). Note that although we do it for any \( p \) and \( q \), we need \( T_{pq} = 0 \) only for \( p \neq 0 \), because otherwise in (8) we have \( (a, b, c^p)c^q = 0 \). First, we consider the case \( q = 0 \) and find \( t^d_0 \):
\[ t^d_0 + t^0_0 = \sum_{i+j=d} t^i_j - \sum_{i+j=d, i \neq d, j \neq d} t^j_i = (-1)^d - \sum_{i+j=d, i \neq d, j \neq d} t^0_{ij} \cdot t^0_{ij}. \]

So, since \( T_{pq} = 0 \),
\[ t^d_0 = \sum_{l+j=d} t^0_i(t^j_0 + t^0_{j-1}) = \sum_{l+j=d, l \neq d, j \neq d} t^0_i(t^0_j + t^0_{j-1}) + t^0_{d-1} + 2t^0_d \]
\[ = \sum_{l+j=d, l \neq d, j \neq d} t^0_{ij} + t^0_{d-1}2 \left( (-1)^d - \sum_{i+j=d, i \neq d, j \neq d} t^0_{ij} \right), \]

and we get
\[ t^d_0 = \frac{1}{3} \left( \sum_{l+j=d, l \neq d, j \neq d} t^0_{ij} + t^0_{d-1} + 2 \left( (-1)^d - \sum_{i+j=d, i \neq d, j \neq d} t^0_{ij} \right) \right). \]

To complete the proof we must show that \( T_{qp} = 0 \) for any \( p, q \) such that \( p + q = d \) based on the assumption that this is true for \( p + q \leq d - 1 \). If \( p \neq 0 \) and \( q \neq 0 \), then
\[ T_{pq} = t^p_q - \sum_{k+i=q} t^k_{ij} \sum_{l+j=p} t^j_0 = t^p_q - t^p_0 t^0_q = 0. \]
This completes the proof.

Theorem 2. Any alternative algebra \( A \) over a field of characteristic \( p \neq 3 \) has a polynomial identity. Let \( \phi \) be a homomorphism of \( A \) onto \( A^{(1+c)} \).

Proof. Theorem 1 states that \( t \) is a homomorphism. To prove that \( t \) is surjective, we have to find a polynomial \( T = t^{-1} \) such that \( t(T(a)) = a \). Let us use induction on \( m = \deg(c) \). For \( m = 1 \) the statement is obvious: \( T = 1 \). Assume that \( T_1 \) is the required polynomial for the case of \( \deg(c) = m - 1 \); then for \( \deg(c) = m \) we have \( t(T_1(a)) = a + R(a) \), where \( R \) is a homogeneous polynomial linear in \( a \) and of degree \( m - 1 \) in \( c \). We set

\[
T(a) = T_1(a) - T_1(R(a)).
\]

This proves the theorem.

Corollary 1. Let \( A, c \) and \( t(x) \) be the same as in the Theorem 1; then the mapping \( t : x \to t(x) \) is an isomorphism of \( A \) onto \( A^{(1+c)} \).

Proof. Theorem 1 states that \( t \) is a homomorphism. To prove that \( t \) is surjective, we have to find a polynomial \( T = t^{-1} \) such that \( t(T(a)) = a \). Let us use induction on \( m = \deg(c) \). For \( m = 1 \) the statement is obvious: \( T = 1 \). Assume that \( T_1 \) is the required polynomial for the case of \( \deg(c) = m - 1 \); then for \( \deg(c) = m \) we have \( t(T_1(a)) = a + R(a) \), where \( R \) is a homogeneous polynomial linear in \( a \) and of degree \( m - 1 \) in \( c \). We set

\[
T(a) = T_1(a) - T_1(R(a)).
\]

This proves the theorem.

Let \( x_1, \ldots, x_k \) be some generators of a free alternative algebra \( A_0 \), and \( f \) a polynomial in \( x_1, \ldots, x_k \). Since \( A_0 \) is free, for any \( s \in A_0 \) there is a homomorphism \( \phi_s : A_0 \to A_0^{(s)} \) such that \( x_i \mapsto x_i \). We denote the image of \( f \) by \( f^{(s)} \).

Theorem 2. Any alternative algebra \( A \) over a field of characteristics \( p \neq 3 \) with a polynomial identity \( f = 0 \) satisfies also \( f^{(s)} = 0 \) for any \( s \in A \).

Proof. The variety of alternative algebras is homogeneous ([3], page 8); therefore we need to consider only the case of homogeneous polynomial \( f \). Let \( A_0 \) be a free alternative algebra on \( k + 1 \) generators \( x_1, \ldots, x_k, c \). By Theorem 1 and the corollary,

\[
f^{(1+c)}(x_1, \ldots, x_k) = t(f(t^{-1}x_1, \ldots, t^{-1}x_k)) + c_m,
\]

where \( c_m \in \text{Id}(c)^m \) and \( m = \deg(f) \). Consider the homogeneous component of degree \( m - 1 \) in \( c \):

\[
f^{(c)}(x_1, \ldots, x_k) = \Delta^m_c t(f(t^{-1}x_1, \ldots, t^{-1}x_k)) \in \text{Id}(f).
\]

Here \( \Delta^m_c \) is the linearization operator (see [3]), and \( \text{Id}(f) \) denotes the ideal generated by all values of \( f \). Since \( A_0 \) is free and \( x_1, \ldots, x_k, c \) are its generators, (10) holds for any algebra \( A \) and for any elements \( x_1, \ldots, x_k, c \) in \( A \). This proves the theorem.
For algebras over a field of characteristics 3 Theorem 2 is false. Consider a commutative but not associative alternative algebra; it satisfies $[x, y] = 0$. On the other hand,

$$[x, y]^{(c)} = (xc)y - (yc)x = (x, c, y).$$

So, generally speaking, $[x, y]^{(c)} \neq 0$.

**References**


Department of Mathematics, Ohio State University, Columbus, Ohio 43202

E-mail address: brkvch@math.ohio-state.edu