ISOTOPY AND IDENTITIES IN ALTERNATIVE ALGEBRAS

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Abstract. In this paper we show how to construct an isomorphism between an alternative algebra $A$ over a field of characteristic $\neq 3$ and its isotope $A^{(1+c)}$, where $c$ is an element of Zhevlakov’s radical of $A$. This leads to the equivalence of any polynomial identity $f = 0$ in alternative algebras and the isotope identity $f^{(s)} = 0$.

Given an invertible element $s$ of an alternative algebra $A$, we can form a new algebra by taking the same linear structure but a new multiplication

$$x \ast_s y = (xs)y.$$ 

The resulting algebra, denoted $A^{(s)}$, is also alternative (see [1]) and is called an $s$-isotope. Associative and Cayley isotopes are always isomorphic [2], and an isotope of a finite-dimensional alternative algebra over an algebraically closed field of characteristic $\neq 3$ is isomorphic to the original algebra as well [1]. We consider the case of an arbitrary alternative algebra over a field of characteristic $\neq 3$ with nonzero Zhevlakov radical, and a particular choice of $s$, and construct an explicit isomorphism between $A$ and $A^{(s)}$. As a consequence, we derive the equivalence of any polynomial identity and its isotope in an arbitrary alternative algebra over a field of characteristic $\neq 3$. The paper has benefited from many discussions with J. Ferrar, and I would like to thank him for his valuable help and encouragement.

Theorem 1. There is a set of coefficients $t^i_j$, $i, j \geq 0$, such that $t^{0}_0 = 1$ and for any integer $m$, any alternative algebra $A$ over a field of characteristic $\neq 3$ and any elements $a, b, c$ from $A$ such that $\text{Id}_A(c)^m = 0$, the polynomial $t(x) = \sum_{i,j} t^i_j c^ix^j$ satisfies

$$(1) \quad t(a) \ast_{1+c} t(b) = t(ab).$$

Proof. We will show the way to calculate the coefficients $t^i_j$ such that (1) holds for any alternative algebra $A$. Clearly,

$$t(a) \ast_{1+c} t(b) = t(a)t(b) + (t(a)c)t(b) = \sum t^i_j t^k_l (c^iac^j)(c^kb^l) + \sum t^i_j t^k_l (c^iac^{j+1})(c^kb^l) = \sum t^k_l (t^i_j + t^i_{j-1})(c^iac^j)(c^kb^l),$$

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where \( t_{i-1} = 0 \) and \( 0 \leq i, j, k, l \leq m - 1 \). For brevity we shall denote \( t^k_i (t^j_j + t^i_{j-1}) \) simply by \( t^k_{ij} \); then

\[
t(a) *_{1+e} t(b) - t(ab) = \sum t^k_{ij} ((c^i a c^j b + (c^i a, c^j, b c^l)) - \sum t^k_{ij} ((c^i a)(b c^l) - (c^i, a, b)c^l + (c^i a, b, c^l))
\]

(2)

\[
= \sum \sum t^k_{ij} (c^i a c^j b)(b c^l) + \sum t^k_{ij} (c^i a)c^j + \sum t^k_{ij} (a, b, c^l) c^l.
\]

We need (1) to be true for any alternative algebra, in particular for an associative one, for which all the associators in (2) are zeroes and (1) is equivalent to

(3)

\[
\sum \sum t^k_{ij} c^i a c^j b c^l - \sum t^k_{ij} c^i a b c^l = 0.
\]

For the cases \( n = 0 \) and \( n \geq 1 \) we get that

(4)

\[
t^0_{i0} = t^i_i,
\]

(5)

\[
\sum_{j+k=n} t^k_{ij} = 0,
\]

for any \( i \) and \( l \). It is easy to see that (4) yields

(6)

\[
t^i_i = t^i_i = t^0_i.
\]

But then

\[
\sum_{j+k=n} t^k_{ij} = t^i_i \sum_{j+k=n} (t^k_{ij} + t^k_{j-1}),
\]

and therefore (5) yields

(7)

\[
\sum_{j+k=n} t^k_{ij} = - \sum_{j+k=n-1} t^k_{ij}.
\]

Consequently, since \( \sum_{j+k=1} t^k_{ij} = t^0_{i0} = 1 \) and by induction on \( n \),

(7)

\[
\sum_{j+k=n} t^k_{ij} = (-1)^n.
\]

Now we assume (6) and (7), which implies (3). Going back to the alternative case we note that (4) and (5) also imply

\[
\sum \sum t^k_{ij} (c^i a c^j b)(b c^l) - \sum t^k_{ij} (c^i a)(b c^l) = 0.
\]

So in the expression (2) only sums with the associators are left. It is easy to see that in any alternative algebra the Moufang identities imply

\[
(a, b, c^{k+l}) = -(a, c^l, c^b b) + b(a, c^l, c^k) + c^l(a, b, c^k) = (a, b, c^l) c^k + c^l(a, b, c^k).
\]
Using this identity we get

\[ t(a) * t(b) - t(ab) = \sum t_{ij}^k (-a(b,c^{i+j})c^{k+i} + (a,b,c^{i+j+k})c^j) \]

\[ - \sum t_{ij}^l (a,b,c^j)c^i + \sum t_{ij}^l (a,b,c^i)c^j \]

\[ = \sum T_{pq}(a,b,c^p)c^q, \]

where

\[ T_{pq} = t_q^p - t_p^q - \sum_{l+j=p, k+i=q} t_{ij}^{kl} + \sum_{l+j+k=p, i=q} t_{ij}^{kl}. \]

Let us simplify this expression for \( T_{pq} \). Clearly by (7)

\[ \sum_{l+j+k=p} t_{ij}^{kl} = \sum_{k+l=p-j} t_{ij}^k \sum_{j} t_{0j}^q = \sum_{j} (-1)^{p-j} t_{0j}^q \]

\[ = \sum_{j} (-1)^{p-j} (t_j^q + t_{j-1}^q) = t_p^q, \]

and so

\[ T_{pq} = t_q^p - \sum_{l+j=p, k+i=q} t_{ij}^{kl}. \]

Now we use induction on \( d = p + q \) to prove the existence of \( t_d^0 \) and \( t_d^0 \) such that \( T_{pq} = 0 \) for any \( p + q \leq d \). Note that although we do it for any \( p \) and \( q \), we need \( T_{pq} = 0 \) only for \( p \neq 0 \), because otherwise in (8) we have \( (a,b,c^p)c^q = 0 \). First, we consider the case \( q = 0 \) and find \( t_d^0 \):

\[ t_d^0 + t_d^0 = \sum_{l+j=d} t_j^l - \sum_{l+j=d, i \neq d, j \neq d} t_j^i \]

\[ = (-1)^d - \sum_{l+j=d, i \neq d, j \neq d} t_j^i. \]

So, since \( T_{pq} = 0 \),

\[ t_d^0 = \sum_{l+j=d} t_j^0 (t_j^0 + t_{j-1}^0) = \sum_{l+j=d, l \neq d, j \neq d} t_j^0 (t_j^0 + t_{j-1}^0) + t_{d-1}^0 + 2t_d^0 \]

\[ = \sum_{l+j=d, l \neq d, j \neq d} t_j^0 (t_j^0 + t_{d-1}^0) + 2 (-1)^d - \sum_{i+j=d, i \neq d, j \neq d} t_j^i t_j^0 - t_0^d, \]

and we get

\[ t_d^0 = \frac{1}{3} \left( \sum_{l+j=d, l \neq d, j \neq d} t_j^0 + t_{d-1}^0 + 2 (-1)^d - \sum_{i+j=d, i \neq d, j \neq d} t_j^i t_j^0 \right). \]

To complete the proof we must show that \( T_{qp} = 0 \) for any \( p, q \) such that \( p + q = d \) based on the assumption that this is true for \( p + q \leq d - 1 \). If \( p \neq 0 \) and \( q \neq 0 \), then

\[ T_{pq} = t_q^p - \sum_{k+i=q} t_{0k}^{ji} \sum_{l+j=p} t_{ij}^0 = t_q^p - t_0^p t_q^0 = 0. \]
The case \( q = 0 \) follows from (9) and for \( p = 0 \) we have
\[
T_{0d} = r_d^0 - \sum_{k+i=d} t_{0i}^k = t_d^0 - \sum_{k+i=d} \left( \sum_{l+j=k} t_{lj}^0 \right) t_i^j
\]
\[
= r_d^0 - \sum_{j \leq d} t_{0j}^0 \sum_{i+l=d-j} t_i^j = t_d^0 - \sum_{j \leq d} (t_{j}^0 + t_{j-1}^0)(-1)^{d-j}
\]
\[
= r_d^0 - \left( \sum_{j \leq d} t_{j}^0 (-1)^{d-j} - \sum_{j < d} (t_{j}^0)(-1)^{d-j} \right) = 0.
\]
This proves the theorem.

\[\square\]

**Corollary 1.** Let \( A, c \) and \( t(x) \) be the same as in the Theorem 1; then the mapping \( t : x \to t(x) \) is an isomorphism of \( A \) onto \( A^{(1+c)} \).

**Proof.** Theorem 1 states that \( t \) is a homomorphism. To prove that \( t \) is surjective, we have to find a polynomial \( T = t^{-1} \) such that \( t(T(a)) = a \). Let us use induction on \( m = \deg(c) \). For \( m = 1 \) the statement is obvious: \( T = 1 \). Assume that \( T_1 \) is the required polynomial for the case of \( \deg(c) = m - 1 \); then for \( \deg(c) = m \) we have \( t(T_1(a)) = a + R(a) \), where \( R \) is a homogeneous polynomial linear in \( a \) and of degree \( m - 1 \) in \( c \). We set
\[
T(a) = T_1(a) - T_1(R(a)).
\]
Then \( R(R(a)) = 0 \), since \( R(R(a)) \) is of degree \( 2(m - 1) \geq m \) in \( c \), and we have
\[
t(T(a)) = a + R(a) - R(a) - R(R(a)) = a - R(R(a)) = a.
\]
This completes the proof.

\[\square\]

Let \( x_1, \ldots, x_k \) be some generators of a free alternative algebra \( A_0 \), and \( f \) a polynomial in \( x_1, \ldots, x_k \). Since \( A_0 \) is free, for any \( s \in A_0 \) there is a homomorphism \( \phi_s : A_0 \to A_0^{(s)} \) such that \( x_i \mapsto x_i \). We denote the image of \( f \) by \( f^{(s)} \).

**Theorem 2.** Any alternative algebra \( A \) over a field of characteristics \( \neq 3 \) with a polynomial identity \( f = 0 \) satisfies also \( f^{(s)} = 0 \) for any \( s \in A \).

**Proof.** The variety of alternative algebras is homogeneous ([3], page 8); therefore we need to consider only the case of homogeneous polynomial \( f \). Let \( A_0 \) be a free alternative algebra on \( k+1 \) generators \( x_1, \ldots, x_k, c \). By Theorem 1 and the corollary,
\[
f^{(1+c)}(x_1, \ldots, x_k) = t(f(t^{-1}x_1, \ldots, t^{-1}x_k)) + c_m,
\]
where \( c_m \in \text{Id}(c)^m \) and \( m = \deg(f) \). Consider the homogeneous component of degree \( m - 1 \) in \( c \):
\[
f^{(c)}(x_1, \ldots, x_k) = \Delta_c^{m-1}t(f(t^{-1}x_1, \ldots, t^{-1}x_k)) \in \text{Id}(f).
\]
Here \( \Delta_c^{m-1} \) is the linearization operator (see [3]), and \( \text{Id}(f) \) denotes the ideal generated by all values of \( f \). Since \( A_0 \) is free and \( x_1, \ldots, x_k, c \) are its generators, (10) holds for any algebra \( A \) and for any elements \( x_1, \ldots, x_k, c \) in \( A \). This proves the theorem.

\[\square\]
For algebras over a field of characteristics 3 Theorem 2 is false. Consider a commutative but not associative alternative algebra; it satisfies \([x, y] = 0\). On the other hand,

\[
[x, y]^{(c)} = (xc)y - (yc)x = (x, c, y).
\]

So, generally speaking, \([x, y]^{(c)} \neq 0\).

References


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